ALGEBRA

AN

ELEMENTARY TEXT-BOOK

Uniform with Part I

PART II

Completing the Work and containing an Index to both Parts.

640 *pp., post 8vo, price* 12*s.* 6*d.*

ALGEBRA

AN ELEMENTARY TEXT-BOOK

FOR THE

HIGHER CLASSES OF SECONDARY SCHOOLS

AND FOR COLLEGES

BY

G. CHRYSTAL, M.A., LL.D.

HONORARY FELLOW OF CORPUS CHRISTI COLLEGE, CAMBRIDGE;

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF EDINBURGH

PART I.

FIFTH EDITION

LONDON

ADAM AND CHARLES BLACK

MCMIV

"I should rejoice to see . . . morphology introduced into the elements of Algebra."—SYLVESTER

Published July 1886
Reprinted, with corrections and additions, 1889
New impressions 1893 *and* 1898
Reprinted with corrections and additions 1904

PREFACE TO THE FIFTH EDITION.

In this Edition considerable alterations have been made in chapter xii. In particular, the proof of the theorem that every integral equation has a root has been amplified, and also illustrated by graphical considerations.

An Appendix has been added dealing with the general algebraic solution of Cubic and Biquadratic Equations; with the reducibility of equations generally; and with the possibility of solution by means of square roots. As the theorems established have interesting applications in Elementary Geometry, it is believed that they may find an appropriate place in an Elementary work on Algebra.

G. CHRYSTAL.

29th June 1904.

PREFACE TO THE SECOND EDITION.

The comparatively rapid sale of an edition of over two thousand copies of this volume has shown that it has, to some extent at least, filled a vacant place in our educational system. The letters which I have received from many parts of the United Kingdom, and from America, containing words of encouragement and of useful criticism, have also strengthened me in the hope that my labour has not been

in vain. It would be impossible to name here all the friends who have thus favoured me; and I take this opportunity of offering them collectively my warmest thanks.

The present edition has been thoroughly revised and corrected. The first chapter has been somewhat simplified; and, partly owing to experience with my own pupils, partly in consequence of some acute criticism sent to me by Mr. Levett of Manchester, the chapters on Indices have been recast, and, I think, greatly improved. In the verification and correction of the results of the exercises I have been indebted in a special degree to the Rev. John Wilson, Mathematical Tutor in Edinburgh.

The only addition of any consequence is a sketch of Horner's Method, inserted in chapter xv. I had originally intended to place this in Part II.; but, acting on a suggestion of Mr. Hayward's, I have now added it to Part I.

To help beginners, I have given, after the table of contents, an index of the principal technical terms used in the volume. This index will enable the student to turn up a passage where the "hard word" is either defined or otherwise made plain.

G. CHRYSTAL.

EDINBURGH, 11*th October* 1889.

PREFACE TO THE FIRST EDITION.

The work on Algebra of which this volume forms the first part, is so far elementary that it begins at the beginning of the subject. It is not, however, intended for the use of absolute beginners.

The teaching of Algebra in the earlier stages ought to consist in a gradual generalisation of Arithmetic; in other words, Algebra ought, in the first instance, to be taught as *Arithmetica Universalis* in the strictest sense. I suppose that the student has gone in this way the length of, say, the solution of problems by means of simple or perhaps even quadratic equations, and that he is more or less familiar with the construction of literal formulæ, such, for example, as that for the amount of a sum of money during a given term at simple interest.

Then it becomes necessary, if Algebra is to be anything more than a mere bundle of unconnected rules, to lay down generally the three fundamental laws of the subject, and to proceed deductively—in short, to introduce the idea of *Algebraic Form*, which is the foundation of all the modern developments of Algebra and the secret of analytical geometry, the most beautiful of all its applications. Such is the course followed from the beginning in this work.

As mathematical education stands at present in this country, the first part might be used in the higher classes of our secondary schools and in the lower courses of our colleges and universities. It will be seen on looking through the pages that the only knowledge required outside of Algebra proper is familiarity with the definition of the trigonometrical functions and a knowledge of their fundamental addition-theorem.

The first object I have set before me is to develop Algebra as a science, and thereby to increase its usefulness as an educational discipline. I have also endeavoured so to lay the foundations that nothing shall have to be unlearned and as little as possible added when the student comes to the higher parts of the subject. The neglect of this consideration I have found to be one of the most important of the many defects of the English text-books hitherto in vogue. Where immediate practical application comes in question, I have striven to adapt the matter to that end as far as the main general educational purpose would allow. I have also endeavoured, so far as possible, to give complete information on every subject taken up, or, in default of that, to indicate the proper sources; so that the book should serve the student both as a manual and as a book of reference. The introduction here and there of historical notes is intended partly to serve the purpose just mentioned, and partly to familiarise the student with the great names of the science, and to open for him a vista beyond the boards of an elementary text-book.

As examples of the special features of this book, I may ask the attention of teachers to chapters iv. and v. With respect to the opening chapter, which the beginner will

doubtless find the hardest in the book, I should mention that it was written as a suggestion to the teacher how to connect the general laws of Algebra with the former experience of the pupil. In writing this chapter I had to remember that I was engaged in writing, not a book on the philosophical nature of the first principles of Algebra, but the first chapter of a book on their consequences. Another peculiarity of the work is the large amount of illustrative matter, which I thought necessary to prevent the vagueness which dims the learner's vision of pure theory; this has swollen the book to dimensions and corresponding price that require some apology. The chapters on the theory of the complex variable and on the equivalence of systems of equations, the free use of graphical illustrations, and the elementary discussion of problems on maxima and minima, although new features in an English text-book, stand so little in need of apology with the scientific public that I offer none.

The order of the matter, the character of the illustrations, and the method of exposition generally, are the result of some ten years' experience as a university teacher. I have adopted now this, now that deviation from accepted English usages solely at the dictation of experience. It was only after my own ideas had been to a considerable extent thus fixed that I did what possibly I ought to have done sooner, viz., consulted foreign elementary treatises. I then found that wherever there had been free consideration of the subject the results had been much the same. I thus derived moral support, and obtained numberless hints on matters of detail, the exact sources of which it would be difficult to indicate. I may mention, however, as specimens

of the class of treatises referred to, the elementary text-books of Baltzer in German and Collin in French. Among the treatises to which I am indebted in the matter of theory and logic, I should mention the works of De Morgan, Peacock, Lipschitz, and Serret. Many of the exercises have been either taken from my own class examination papers or constructed expressly to illustrate some theoretical point discussed in the text. For the rest I am heavily indebted to the examination papers of the various colleges in Cambridge. I had originally intended to indicate in all cases the sources, but soon I found recurrences which rendered this difficult, if not impossible.

The order in which the matter is arranged will doubtless seem strange to many teachers, but a little reflection will, I think, convince them that it could easily be justified. There is, however, no necessity that, at a first reading, the order of the chapters should be exactly adhered to. I think that, in a final reading, the order I have given should be followed, as it seems to me to be the natural order into which the subjects fall after they have been fully comprehended in their relation to the fundamental laws of Algebra.

With respect to the very large number of Exercises, I should mention that they have been given for the convenience of the teacher, in order that he might have, year by year, in using the book, a sufficient variety to prevent mere rote-work on the part of his pupils. I should much deprecate the idea that any one pupil is to work all the exercises at the first or at any reading. We do too much of that kind of work in this country.

I have to acknowledge personal obligations to Professor

TAIT, to Dr. THOMAS MUIR, and to my assistant, Mr. R. E. ALLARDICE, for criticism and suggestions regarding the theoretical part of the work; to these gentlemen and to Messrs. MACKAY and A. Y. FRASER for proof reading, and for much assistance in the tedious work of verifying the answers to exercises. In this latter part of the work I am also indebted to my pupil, Mr. J. MACKENZIE, and to my old friend and former tutor, Dr. DAVID RENNET of Aberdeen.

Notwithstanding the kind assistance of my friends and the care I have taken myself, there must remain many errors both in the text and in the answers to the exercises, notification of which either to my publishers or to myself will be gratefully received.

G. CHRYSTAL.

EDINBURGH, 26*th June* 1886.

CONTENTS.

CHAPTER I.

FUNDAMENTAL LAWS AND PROCESSES OF ALGEBRA.

CHAPTER II.

MONOMIALS—LAWS OF INDICES—DEGREE.

CHAPTER III.

THEORY OF QUOTIENTS—FIRST PRINCIPLES OF THEORY OF NUMBERS.

CHAPTER IV.

DISTRIBUTION OF PRODUCTS—ELEMENTS OF THE THEORY OF RATIONAL INTEGRAL FUNCTIONS.

CHAPTER V.

TRANSFORMATION OF THE QUOTIENT OF TWO INTEGRAL FUNCTIONS.

CHAPTER VI.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

CHAPTER VII.

FACTORISATION OF INTEGRAL FUNCTIONS.

CHAPTER VIII.

RATIONAL FRACTIONS.

CHAPTER IX.

FURTHER APPLICATIONS TO THE THEORY OF NUMBERS.

CHAPTER X.

IRRATIONAL FUNCTIONS.

CHAPTER XI.

ARITHMETICAL THEORY OF SURDS.

CHAPTER XII.

COMPLEX NUMBERS.

CHAPTER XIII.

RATIO AND PROPORTION.

CHAPTER XIV.

ON CONDITIONAL EQUATIONS IN GENERAL.

CHAPTER XV.

VARIATION OF A FUNCTION.

CHAPTER XVI.

EQUATIONS AND FUNCTIONS OF FIRST DEGREE.

CHAPTER XVII.

EQUATIONS OF THE SECOND DEGREE.

CHAPTER XVIII.

GENERAL THEORY OF INTEGRAL FUNCTIONS.

CHAPTER XIX.

SOLUTION OF PROBLEMS BY MEANS OF EQUATIONS.

CHAPTER XX.

ARITHMETIC, GEOMETRIC, AND ALLIED SERIES.

CHAPTER XXI.

LOGARITHMS.

CHAPTER XXII.

THEORY OF INTEREST AND ANNUITIES.

APPENDIX.

INDEX OF PRINCIPAL TECHNICAL TERMS USED IN PART I.

CHAPTER I.

The Fundamental Laws and Processes of Algebra as exhibited in ordinary Arithmetic.

§ 1.] The student is already familiar with the distinction between abstract and concrete arithmetic. The former is concerned with those laws of, and operations with, numbers that are independent of the things numbered; the latter is taken up with applications of the former to the numeration of various classes of things.

Confining ourselves for the present to abstract arithmetic, let us consider the following series of equalities:—

$$\frac{2623}{61}+\frac{1023}{3}=\frac{2623\times 3+1023\times 61}{61\times 3}$$
$$=\frac{70272}{183}=384.$$

The first step is merely the assertion of the equivalence of two different sets of operations with the same numbers. The second and third steps, though doubtless based on certain simple laws from which also the first is a consequence, nevertheless require for their direct execution the application of certain rules, of a kind to which the name *arithmetical* is appropriated.

We have thus shadowed forth two great branches of the higher mathematics:—one, algebra, strictly so called, that is, the theory of operation with numbers, or, more generally speaking, with quantities; the other, the higher arithmetic, or theory of numbers. These two sciences are identical as to their fundamental laws, but differ widely in their derived processes. As is usual in elementary text-books, the elements of both will be treated in this work.

§ 2.] Ordinary algebra is simply the general theory of those operations with quantity of which the operations of ordinary abstract arithmetic are a particular case.

The fundamental laws of this algebra are therefore to be sought for in ordinary arithmetic.

However various and complex the operations of arithmetic may seem, it appears on consideration that they are merely the result of the application of a very small number of fundamental principles. To make this plain we return for a little to the very elements of arithmetic.

ADDITION,

AND THE GENERAL LAWS CONNECTED THEREWITH.

§ 3.] When a group of things, no matter how unlike, is considered merely with reference to the number of individuals it contains, it may be represented by another group, the individuals of which are all alike, provided only there be as many individuals in the representative as in the original group. The members of our representative group may be merely marks (1's say) on a piece of paper. The process of counting a group may therefore be conceived as the successive placing of 1's in our representative group, until we have as many 1's as there are individuals in the group to be numbered. This process of adding a 1 is represented by writing $+1$. We may thus have

$$+1, \quad +1+1, \quad +1+1+1, \quad +1+1+1+1, \text{ \&c.},$$

as representative groups or "numbers." As the student is of course aware, these symbols in ordinary arithmetic are abbreviated into

$$1, \quad 2, \quad 3, \quad 4, \text{ \&c.}$$

Hence using the symbol " = " to stand for "the same as," or "replaceable by," or "equal to," we have, as definitions of 1, 2, 3, 4, &c.,

$$\begin{aligned} 1 &= +1, \\ 2 &= +1+1, \\ 3 &= +1+1+1, \\ 4 &= +1+1+1+1, \text{ \&c.} \end{aligned}$$

And there is a further arrangement for abridging the representation of large numbers, which the student is familiar with as the decimal notation. With numerical notation we are not further concerned at present, but there is a view of the above equalities which is important. After the group $+1+1+1$ has been finished it may be viewed as representing a single idea to the mind, viz. the number "three." In other words, we may look at $+1+1+1$ as a series of successive additions, or we may think of it as a whole. When it is necessary for any purpose to emphasise the latter view, we enclose $+1+1+1$ in a bracket, thus $(+1+1+1)$; and it will be observed that precisely the same result is attained by writing the symbol 3 in place of $+1+1+1$, for in the symbol 3 all trace of the formation of the number by successive addition is lost. We might therefore understand the equality or equation

$$3 = +1+1+1$$

to mean

$$(+1+1+1) = +1+1+1,$$

and then the equation is a case of the algebraical LAW OF ASSOCIATION.

The full meaning of this law will be best understood by considering the case of two groups of individuals, say one of three and another of four. If we wish to find the number of a group made up by combining the two, we may adopt the child's process of counting through them in succession, thus,

$$+1+1+1 \mid +1+1+1+1 = 7.$$

But by the law of association we may write for $+1+1+1$

$$(+1+1+1),$$

and for $+1+1+1+1$

$$(+1+1+1+1),$$

and we have

$$+(+1+1+1)+(+1+1+1+1) = 7,$$

or

$$+3+4 = 7.$$

It will be observed that we have added a $+$ in each case before the bracket, and it may be asked how this is justified. The answer is simply that setting down a representative group of three individuals is an operation of exactly the same nature as

setting down a group of one. The law of association for addition worded in this way for the simple case before us would be this: To set down a representative group of three individuals is the same as to set down in succession three representative individuals.

The principle of association may be carried further. The representative group $+3+4$ may itself enter either as a whole or by its parts into some further enumeration: thus,

$$+6+(+3+4)=+6+3+4$$

is an example of the law of association which the student will have no difficulty in interpreting in the manner already indicated. The ultimate proof of the equality may be regarded as resting on a decomposition of all the symbols into a succession of units. There is, of course, no limit to the complication of associations. Thus we have

$$\begin{aligned}&+[(+9+8)+\{+6+(+5+3)\}]+\{+6+(+3+5)\}\\ &\quad=+(+9+8)+\{+6+(+5+3)\}+6+(+3+5),\\ &\quad=+9+8+6+(+5+3)+6+3+5,\\ &\quad=+9+8+6+5+3+6+3+5,\end{aligned}$$

each single removal of a bracket being an assertion of the law of association. The student will remark the use of brackets of different forms to indicate clearly the different associations.

§ 4.] It follows from the definitions

$$3=+1+1+1, \quad 2=+1+1,$$

that

$$+3+2=+2+3;$$

and by a similar proof we might show that

$$+3+4+6=+3+6+4=+4+3+6, \text{ \&c.};$$

in other words, *the order in which a series of additions is arranged is indifferent.*

This is the algebraical LAW OF COMMUTATION, and it will be observed that its application is unrestricted in arithmetical operations where additions alone are concerned. The statement of this law at once suggests a principle of great importance in algebra, namely, the attachment of the "symbol of operation" or "operator" to the number, or, more generally speaking, "subject" or "operand," on which it acts. Thus in the above equations

the + before the 3 is supposed to accompany the 3 when it is transferred from one part of the chain of additions to another.

The operands in $+3$, $+4$, and $+6$ are already complex; and it may be shown by a further application of the reasoning used in the beginning of this article that the operand may be complex to any degree without interfering with the validity of the commutative law; for example,

$$\begin{aligned} &+\{+3+(+2+3)\}+(+6+8) \\ &=+(+6+8)+\{+3+(+2+3)\}, \end{aligned}$$

of which a proof might also be given by first dissociating, then commutating the individual terms $+6$, $+8$, $+3$, &c., and then reassociating.

The Law of Commutation, thus suggested by arithmetical considerations, is now laid down as a general law of algebra; and forms a part of the definition of the algebraic symbol + .*

SUBTRACTION.

§ 5.] For algebraical purposes the most convenient course is to define subtraction as the inverse of addition; or, as is more convenient for elementary exposition, we lay down that addition and subtraction are inverse to each other.† By this we mean that, whatever the interpretation of the operation $+b$ may be, the operation $-b$ annuls the effect of $+b$; and *vice versa*.

Thus, − is defined relatively to + by the equation

$$+a-b+b=+a \qquad (1),$$

or

$$+a+b-b=+a \qquad (2).$$

These might also be written ††

* See the general remarks in § 27.

† Here we virtually assume that if $x+a=y+a$, then $x=y$. See Hankel, *Vorlesungen ü. d. Complexen Zahlen* (Leipzig, 1867), p. 19.

†† It may conduce to clearness in following some of the above discussions to remember that the primary view of a chain of operations written in any order is that the operations are to be carried out successively from left to right; for example, if we think merely of the last addition, $+2+3+5+6$ in more fully expressive symbols means $+(+2+3+5)+6$, that is, $+10+6$; $+a+b+c$ means $+(+a+b)+c$; $+a-b+c$ means $+(+a-b)+c$; and so on. We may here remind the reader that, in ordinary practice, when + occurs before the first member of a chain of additions and subtractions, it is usually omitted for brevity.

$$+(+a-b)+b=+a \qquad (1'),$$
$$+(+a+b)-b=+a \qquad (2').$$

From a quantitative point of view we might put the matter thus: the question, What is the result of subtracting b from a? is regarded as the same as the question, What must be added to $+b$ to produce $+a$? and the quantity which is the answer to this question is symbolised by $+a-b$. Starting with the definition involved in (1) and (2), *and putting no restriction upon the operands a and b*, or, what is the same thing from a quantitative point of view, *assuming that the quantity $+a-b$ always exists*, we may show that the laws of commutation and association hold for chains of operations whose successive links are additions and subtractions. We, of course, assume the commutative law for addition, having already laid it down as one of our fundamental laws.

§ 6.] Since $+a-c+c=+a$ by the definition of the mutual relation between addition and subtraction, we have

$$a+b-c=a-c+c+b-c;$$
$$=a-c+b+c-c,$$
by law of commutation for addition;
$$=a-c+b \qquad (1),$$
by definition of subtraction.

Also $$a-b-c=a-c+c-b-c,$$
by definition;
$$=a-c-b+c-c,$$
by case (1);
$$=a-c-b \qquad (2),$$
by definition

Equations (1) and (2) may be regarded as extending the law of commutation to the sign $-$.* We can now state this law fully as follows:—

$$\pm a \pm b = \pm b \pm a;$$

* It might be objected here that it has not been shown that $-c$ may come into the first place in the chain of operations. The answer to this would be that $+a-c-b$ may either be a complete chain in itself or merely the latter part of a longer chain, say $p+a-c-b$. In the second case our proof would show that $p+a-c-b=p-c+a-b$; and the nature of algebraic generality

or, in words, *In any chain of additions and subtractions the different members may be written in any order, each with its proper sign attached.*

Here the full significance of the attachment of the operator to the operand appears. Thus in the following instance the quantities change places, carrying their signs of operation with them in accordance with the commutative law:—

$$\begin{aligned} +3-2+1-1 &= +3+1-1-2, \\ &= +3-1+1-2, \\ &= -2-1+1+3. \end{aligned}$$

§ 7.] By the definition of the mutual relation between addition and subtraction, we have

$$\begin{aligned} a+(+b-c) &= +a+(+b-c)+c-c, \\ &= a+b-c \qquad (1). \end{aligned}$$

Again, by the definition,

$$\begin{aligned} p+b-c+c-b &= p+b-b, \\ &= p. \end{aligned}$$

Hence

$$\begin{aligned} a-(+b-c) &= a-(+b-c)+b-c+c-b; \\ &= a-(+b-c)+(+b-c)+c-b, \\ &\qquad \text{by case (1)}; \\ &= a+c-b, \\ &\qquad \text{by the definition}; \\ &= a-b+c \qquad (2), \\ &\qquad \text{by the law of commutation already established.} \end{aligned}$$

§ 8.] The results in last paragraph, taken along with those of § 3 above, may be looked upon as establishing the law of association for addition and subtraction. This law may be symbolised as follows:—

$$\pm(\pm a \pm b \pm c \pm \&c.) = \pm(\pm a) \pm (\pm b) \pm (\pm c) \pm \&c.,$$

with the following law of signs,

$$\begin{aligned} +(+a) &= +a, & -(+a) &= -a, \\ +(-a) &= -a, & -(-a) &= +a. \end{aligned}$$

---

requires that $+a-c-b$ should not have any property in composition which it has not *per se*. As to all questions of this kind see § 27.

The same may be stated in words as follows:—*If any number of quantities affected with the signs + or − occur in a bracket, the bracket may be removed, all the signs remaining the same if + precede the bracket, each + being changed into − and each − into + if − precede the bracket.*

In the above symbolical statement double signs ($\pm$) have been used for compactness. The student will observe that with three letters $2 \times 2 \times 2 \times 2$, that is, 16, cases are included. Thus the law gives

$$+(+a+b+c) = +a+b+c,$$
$$+(-a+b+c) = -a+b+c,$$
$$-(-a+b+c) = +a-b-c, \text{ \&c.}$$

§ 9.] It will not have escaped the student that, in the assumption that $+a-b$ is a quantity that always exists, we have already transcended the limits of ordinary arithmetic. He will therefore be the less surprised to find that many of the cases included under the laws of commutation and association exhibit operations that are not intelligible in the ordinary arithmetical sense.

If $$a = 3 \text{ and } b = 2,$$

then by the law of association and by the definition of subtraction

$$\begin{aligned} +3-2 &= +1+2-2, \\ &= +1, \end{aligned}$$

in accordance with ordinary arithmetical notions.

On the other hand, if

$$a = 2 \text{ and } b = 3,$$

then by the laws of commutation and association and by the definition of subtraction

$$\begin{aligned} +2-3 &= +2-(+2+1), \\ &= +2-2-1, \\ &= -1+2-2, \\ &= -1. \end{aligned}$$

Here we have a question asked to which there is no ordinary arithmetical answer, and an answer arrived at which has no meaning in ordinary arithmetic.

Such an operation as $+2-3$, or its algebraical equivalent, -1, is to be expected as soon as we begin to reason about operations according to general laws without regard to the application or interpretation of the results to be arrived at. It must be remembered that the result of a series of operations may be looked on either as an end in itself, say the number of individuals in a group, or it may be looked upon merely as an operand destined to take place in further operations. In the latter case, if additions and subtractions be in question, it must have either the $+$ or the $-$ sign, and either is as likely to occur and is as reasonably to be expected as the other. Thus, as the results of any partial operation, $+1$ and -1 mean respectively 1 to be added and 1 to be subtracted.

The fact that the operations may end in results that have no direct interpretation as ordinary arithmetical quantities need not disturb the student. He must remember that algebra is the *general* theory of those operations with quantity of which ordinary arithmetical operations are particular cases. He may be assured from the way in which the general laws of algebra are established that, when algebraical results admit of arithmetical meanings, these results will be arithmetically right, even when some of the steps by which they have been arrived at may not be arithmetically interpretable. On the other hand, when the end results are not arithmetically intelligible, it is merely in the first instance a question of the consistency of algebra with itself. As to what the application of such purely algebraical results may be, that is simply a question of the various uses of algebra; some of these will be indicated in the course of this treatise, and others will be met with in abundance by the student in the course of his mathematical studies. It will be sufficient at this stage to give one example of the advantage that the introduction of algebraic generality gives in arithmetical operations. $+a-b$ asks the question what must be added to $+b$ to give $+a$. If $a=3$ and $b=2$, the answer is 1; if $a=2$ and $b=3$, then, arithmetically speaking, there is no answer, because 3 is already greater than 2. But if we regard $+a-b$ as asking what must be added to or subtracted

from $+b$ to get $+a$, then the evaluation of $+a-b$ in any case by the laws of algebra will give a result whose sign will indicate whether addition or subtraction must be resorted to, and to what extent; for example, if $a=3$ and $b=2$, the result is $+1$, which means that 1 must be added; if $a=2$ and $b=3$, the result is -1, which means that 1 must be subtracted.

§ 10.] The application of the commutative and associative laws for addition and subtraction leads us to a useful practical rule for reducing to its simplest value an expression consisting of a chain of additions and subtractions.

We have, for example,

$$\begin{aligned} &+a-b+c+d-e-f+g \\ &\qquad = +a+c+d+g-b-e-f, \\ &\qquad = +(a+c+d+g)-(b+e+f), \\ &\qquad = +\ \{+(a+c+d+g)-(b+e+f)\} \qquad (1), \\ &\qquad = -\ \{+(b+e+f)-(a+c+d+g)\} \qquad (2). \end{aligned}$$

If $a+c+d+g$ be numerically greater than $b+e+f$, (1) is the most convenient form; if $a+c+d+g$ be numerically less than $b+e+f$, (2) is the most convenient. The two taken together lead to the following rule for evaluating a chain of additions and subtractions:—*

Add all the quantities affected with the sign $+$, *also all those affected with the sign* $-$; *take the difference of the two sums and affix the sign of the greater.*

Numerical example:—

$$\begin{aligned} &+3-5+6+8-9-10+2 \\ &\qquad = +(3+6+8+2)-(5+9+10), \\ &\qquad = +19-24, \\ &\qquad = -(24-19) = -5. \end{aligned}$$

§ 11.] The special case $+a-a$ deserves close attention. A special symbol, namely 0, is used to denote it. The operational definition of 0 is therefore given by the equations

$$+a-a = -a+a = 0.$$

In accordance with this we have, of course, the results,

* Such a chain is usually spoken of as an "algebraical sum."

$$b + 0 = b = b - 0,$$

and
$$+0 = -0,$$

as the student may prove by applying the laws of commutation and association along with the definition of 0.

§ 12.] It will be observed that 0, as operationally defined, is to this extent indefinite that the a used in the above definition may have any value whatever.

It remains to justify the use of the 0 of the ordinary numerical notation in the new meaning. This is at once done when we notice that in a purely quantitative sense 0 stands for the limit of the difference of two quantities that have been made to differ by as little as we please.

Thus, if we consider $a + x$ and a,

$$+(a + x) - a = +a - a + x = x.$$

If we now cause the x to become smaller than any assignable quantity, the above equation becomes an assertion of the identity of the two meanings of 0.

MULTIPLICATION.

§ 13.] The primary definition of multiplication is as an abbreviation of addition. Thus $+a+a$, $+a+a+a$, $+a+a+a+a$, &c., are abbreviated into $+a \times 2$, $+a \times 3$, $+a \times 4$, &c.; and, in accordance with this notation, $+a$ is also represented by $+a \times 1$. $a \times 2$ is called the product of a by 2, or of a into 2; a is also called the multiplicand and 2 the multiplier. Instead of the sign $\times$, a dot, or mere apposition, is often used where no ambiguity can arise. Thus $a \times 2$, $a \,.\, 2$, and $a2$ all denote the same thing.

§ 14.] So long as a and b represent integral numbers, as is supposed in the primary definition of multiplication, it is easy to prove that

$$a \times b = b \times a;$$

or, adopting the principle of attachment of operator and operand, with full symbolism (see above, § 4),

$$\times a \times b = \times b \times a.$$

The same may be established for any number of integers, for example,

$$\times a \times b \times c = \times a \times c \times b = \times b \times c \times a, \text{ \&c.}$$

In other words, *The order of operations in a chain of multiplication is indifferent.*

This is the COMMUTATIVE LAW for multiplication.

§ 15.] We may introduce the use of brackets and the idea of association in exactly the same way as we followed in the case of addition. Thus in $\times a \times (\times b \times c)$ we are directed to multiply a by the product of b by c. The LAW OF ASSOCIATION asserts that this is the same as multiplying a by b, and then multiplying this product by c. Thus

$$\times a \times (\times b \times c) = \times a \times b \times c.$$

The like holds for a bracket containing any number of factors. In the case where a, b, c, &c., are integers, a proof of the truth of this law might be given resting on the definition of multiplication and on the laws of commutation and association for addition.

§ 16.] Even in arithmetic the operation of multiplication is extended to cases which cannot by any stretch of language be brought under the original definition, and it becomes important to inquire what is common to the different operations thus comprehended under one symbol. The answer to this question, which has at different times greatly perplexed inquirers into the first principles of algebra, is simply that what is common is the formal laws of operation which we are now establishing, namely, the commutative and associative laws, and another presently to be mentioned. These alone define the fundamental operations of addition, multiplication, and division, and anything further that appears in any particular case (for example, the statement that $\frac{2}{3} \times \frac{1}{2}$ is $\frac{1}{2}$ of $\frac{2}{3}$) is merely a matter of some interpretation, arithmetical or other, that is given to a symbolical result demonstrably in accordance with the laws of symbolical operation.

Acting on this principle we now lay down the laws of commutation and association as holding for the operation of multiplication, and, indeed, as in part defining it.

§ 17.] The consideration of composite multipliers or composite multiplicands introduces the last of the three great laws of algebra.

It is easy enough, if we confine ourselves to the primary definition of multiplication, to prove that

$$+a \times (+b+c) = +a \times b + a \times c,$$
$$+a \times (+b-c) = +a \times b - a \times c,$$
$$(+a-b) \times (+c-d) = +a \times c - a \times d - b \times c + b \times d.$$

These suggest the following, which is called the DISTRIBUTIVE LAW :—

The product of two expressions, each of which consists of a chain of additions and subtractions, is equal to the chain of additions and subtractions obtained by multiplying each constituent of the first expression by each constituent of the second, setting down all the partial products thus obtained, and prefixing the + sign if the two constituents previously had like signs, the − sign if the constituents previously had unlike signs.

Symbolically, thus :—

$$(\pm a \pm b) \times (\pm c \pm d)$$
$$= (\pm a) \times (\pm c) + (\pm a) \times (\pm d) + (\pm b) \times (\pm c) + (\pm b) \times (\pm d),$$

with the following law of signs :—

$$(+a) \times (+c) = +ac, \qquad (+a) \times (-c) = -ac,$$
$$(-a) \times (+c) = -ac, \qquad (-a) \times (-c) = +ac.$$

There are sixteen different cases included in the above equation, as will be seen by taking every combination of one or other of the double signs before each letter.

Thus $$(+a-b)(+c+d)$$
$$= +ac + ad - bc - bd\ ;$$
$$(-a-b)(-c+d)$$
$$= +ac - ad + bc - bd\ ;$$

and so on.

There may, of course, be as many constituents in each bracket as we please. If, for example, there be m in one

bracket and n in the other, there will be mn partial products and 2^{m+n} different arrangements of the signs.

Thus
$$(+a-b+c)(-d+e)$$
$$=-ad+bd-cd+ae-be+ce\,;$$
and so on.

The distributive law, suggested, as we have seen, by the primary definition of multiplication, is now laid down as a law of algebra. It forms the connecting link between addition and multiplication, and, along with the commutative and associative laws, completes the definition of both these operations.

§ 18.] By means of the distributive law we can prove another property of 0. For, if b be any definite quantity, subject without restriction to the laws of algebra, we have

$$+ba-ba=+b\times(+a-a)=(+a-a)\times(+b),$$
$$=-b\times(+a-a)=(+a-a)\times(-b),$$

whence $\quad 0=(+b)\times 0=0\times(+b)=(-b)\times 0=0\times(-b)\,;$

or briefly $\quad b\times 0=0\times b=0.$

DIVISION.

§ 19.] Division for the purposes of algebra is best defined as the inverse operation to multiplication: that is to say, the mutual relation of the symbols $\times$ and $\div$ is defined by

$$\times a\div b\times b=\times a \qquad (1),$$

or*

$$\times a\times b\div b=\times a \qquad (2).$$

From a quantitative point of view, this amounts to defining the quotient of a by b, that is, $a\div b$, as that quantity which, when multiplied by b, gives a.

In $a\div b$, a is called the dividend and b the divisor. Sometimes a is called the antecedent and b the consequent of the quotient.

Another notation for a quotient is very often used, namely, $\frac{a}{b}$ or a/b. As this is the notation of fractions, and therefore has a meaning already attached to it in the case where a and b are integers, it is incumbent upon us to justify its use in another

* See second footnote, p. 5.

meaning. To do this we have simply to remark that b times $\frac{a}{b}$, that is, b times a of the bth parts of unity, is evidently a times unity, that is, a; also, by the definition of $a \div b$, b times $a \div b$ is a. Hence we conclude that $\frac{a}{b}$ is operationally equivalent to $a \div b$ in the case where a and b are integers. No further justification is necessary, for when either a or b, or both, are not integers, $\frac{a}{b}$ loses its meaning as primarily defined, and there is no obstacle to regarding it as an alternative notation for $a \div b$.

In the above definition we have not written the signs + or − before a and b, but they were omitted simply for brevity, and one or other must be understood before each letter. We shall continue to omit them until the question as to their manipulation arises.

§ 20.] Since division is fully defined as the inverse of multiplication, we ought to be able to deduce all its laws from the definition and the laws of multiplication.

We have *

$$\times a \times b \div c = \times a \div c \times c \times b \div c,$$
by definition;
$$= \times a \div c \times b \times c \div c,$$
by law of commutation for multiplication;
$$= \times a \div c \times b \qquad (1),$$
by definition.

Again,
$$\times a \div b \div c = \times a \div c \times c \div b \div c,$$
by definition;
$$= \times a \div c \div b \times c \div c,$$
by case (1);
$$= \times a \div c \div b \qquad (2),$$
by definition.

In this way we establish the law of commutation for division.

* Here again the remark made in the third note at the foot of p. 5 applies, namely, $a \times b \div c$ primarily means, if we think only of the last operation, the same as $(a \times b) \div c$; $a \div b \times c$ the same as $(a \div b) \times c$; and so on.

As in the case of $+a$, when $\times a$ comes first in a chain of operations, $\times$ is in practice usually omitted for brevity.

Taking multiplication and division together and attaching the symbol of operation to the operand, we may now give the full statement of this law as follows :—

In any chain of multiplications and divisions the order of the constituents is indifferent, provided the proper sign be attached to each constituent and move with it.

Or, in symbols, for two constituents,

$$\stackrel{\times}{\div}a\stackrel{\times}{\div}b = \stackrel{\times}{\div}b\stackrel{\times}{\div}a,$$

there being 4 cases here included, for example,

$$\div a \times b = \times b \div a,$$
$$\div a \div b = \div b \div a, \text{ and so on.}$$

§ 21.] By the definition of the mutual relation between multiplication and division, we have

$$\begin{aligned} \times a \times (\times b \div c) &= \times a \times (\times b \div c) \times c \div c, \\ &= \times a \times b \div c \qquad (1). \end{aligned}$$

Again, since

$$\begin{aligned} \times p \times b \div c \times c \div b &= \times p \times b \div b, \\ &= \times p, \end{aligned}$$

therefore

$$\begin{aligned} \times a \div (\times b \div c) &= \times a \div (\times b \div c) \times b \div c \times c \div b; \\ &= \times a \div (\times b \div c) \times (\times b \div c) \times c \div b, \\ &\qquad \text{by case (1);} \\ &= \times a \times c \div b, \\ &\qquad \text{by definition;} \\ &= \times a \div b \times c \qquad (2), \\ &\qquad \text{by the law of commutation already established.} \end{aligned}$$

These are instances of the law of association for division and multiplication combined, which we may now state as follows :—

When a bracket contains a chain of multiplications and divisions, the bracket may be removed, every sign being unchanged if × *precede the bracket, and every sign being reversed if* ÷ *precede the bracket.*

Or, in symbols, for two constituents,

$$\stackrel{\times}{\div}(\stackrel{\times}{\div}a\stackrel{\times}{\div}b) = \stackrel{\times}{\div}(\stackrel{\times}{\div}a)\stackrel{\times}{\div}(\stackrel{\times}{\div}b),$$

with the following law of signs :—

$$\times(\times a) = \times a, \qquad \times(\div a) = \div a,$$
$$\div(\times a) = \div a, \qquad \div(\div a) = \times a.$$

In the above equation eight cases are included, for example,

$$\times(\div a \times b) = \div a \times b,$$
$$\div(\div a \times b) = \times a \div b,$$
$$\div(\div a \div b) = \times a \times b,$$

and so on.

§ 22.] Just as in subtraction we denote the special case $+a-a$ by a separate symbol 0, so in division we denote $\times a \div a$ by a separate symbol 1. From this point of view, 1 has a purely operational meaning, and we can prove for it the following laws analogous to those established for 0 in § 11 :—

$$\times a \div a = \div a \times a = 1,$$
$$b \times 1 = b = b \div 1,$$
$$\times 1 = \div 1.$$

Like 0, 1 has both a quantitative and a purely operational meaning. Quantitatively we may look on it as the limit of the quotient of two quantities that differ from each other by a quantity which is as small a fraction as we please of either. For example, consider $a+x$ and a, then the equation

$$(a+x) \div a = a \div a + x \div a$$
$$= 1 + x \div a$$

becomes, when x is made as small a fraction of a as we please, an assertion of the compatibility of the two meanings of 1.

It should be noted that, owing to the one-sidedness of the law of distribution (that is, owing to the fact that in ordinary algebra $b + (\times a \div c) = \times(b+a) \div (b+c)$ is not a legitimate transformation), there is no analogue for 1 to the equation

$$b \times 0 = 0,$$

which is true in the case of 0.

§ 23.] If the student will now compare the laws of commutation and association for addition and subtraction on the one hand and for multiplication and division on the other, he will find them to be *formally identical*. It follows, therefore, that so far as these

laws are concerned there is virtually no distinction between addition and subtraction on the one hand and multiplication and division on the other, except the accident that we use the signs $+$ and $-$ in the one case and $\times$ and $\div$ in the other,—a conclusion at first sight a little startling. This duality ceases wherever the law of distribution is concerned.

§ 24.] We have already been led to consider such expressions as $+(+2)$ and $+(-2)$, and to see that $+a$ may, according to the value given to a, be made to stand for $+(+2)$, that is, $+2$, or $+(-2)$, that is, -2. The mere fact that a particular sign, say $+$, stands before a certain letter, indicates nothing as to its reduced or ultimate value; the sign $+$ merely indicates what has to be done with the letter when it enters into operation.

In what precedes as to division, and in fact in all our general formulæ, we may therefore suppose the letters involved to stand for positive or negative quantities at pleasure, without affecting the truth of our statement in the least.

For example, by the law of distribution,

$$(a-b)(c+d) = ac + ad - bc - bd\,;$$

here we may, if we like, suppose d to stand for $-d'$.

We thus have

$$(a-b)\{c+(-d')\} = ac + a(-d') - bc - b(-d'),$$

which gives, when we reduce by means of the law of signs proper to the case,

$$(a-b)(c-d') = ac - ad' - bc + bd',$$

which is true, being in fact merely another case of the law of distribution, which we have reproduced by a substitution from the former case. This *principle of substitution* is one of the most important elements in the science; it is this that gives to algebraic calculation its immense power and almost endless capability of development.

§ 25.] We have now to consider the effect of explicit signs attached to the constituents of a quotient. As this is closely bound up with the operation of the distributive law for division, it will be best to take the two together.

The full symbolical statement of this law for a dividend having two constituents is as follows:—

$$(\pm a \pm b) \div (\pm c) = (\pm a) \div (\pm c) + (\pm b) \div (\pm c),$$

with the following law of signs,

$$(+a) \div (+c) = +a \div c, \qquad (+a) \div (-c) = -a \div c,$$
$$(-a) \div (+c) = -a \div c, \qquad (-a) \div (-c) = +a \div c.$$

Or briefly in words—

In division the dividend may be distributed, the signs of the partial quotients following the same law as in multiplication.

The above equation includes of course eight cases. It will be sufficient to give the formal proof of the correctness of the law for one of them, say

$$(+a-b) \div (-c) = -a \div c + b \div c.$$

By the law of distribution for multiplication, we have

$$\begin{aligned}(-a \div c + b \div c) \times (-c) &= +(a \div c) \times c - (b \div c) \times c;\\ &= +a - b,\end{aligned}$$

by the definition.

Hence $\qquad (+a-b) \div (-c) = (-a \div c + b \div c) \times (-c) \div (-c);$

$$= -a \div c + b \div c,$$

by the definition.

§ 26.] The law of distribution has only a limited application to division, for although, as just proved, the dividend may be distributed, the same is not true of the divisor. Thus it is not true in general that

$$a \div (b+c) = a \div b + a \div c,$$

or that $\qquad a \div (b-c) = a \div b - a \div c,$

as the student may readily satisfy himself in a variety of ways.

§ 27.] As we have now completed our discussion of the fundamental laws of ordinary algebra, it may be well to insist once more upon the exact position which they hold in the science. To speak, as is sometimes done, of the proof of these laws in all their generality is an abuse of terms. They are simply laid down as the canons of the science. The best evidence that this is their real position is the fact that algebras are

in use whose fundamental laws differ from those of ordinary algebra. In the algebra of quaternions, for example, the law of commutation for multiplication and division does not hold generally.

What we have been mainly concerned with in the present chapter is, 1st, to see that the laws of ordinary algebra shall be self-consistent, and, 2nd, to take care that the operations they lead to shall contain those of ordinary arithmetic as particular cases.

In so far as the abstract science of ordinary algebra is concerned, the definitions of the letters and symbols used are simply the general laws laid down for their use. When we come to the application of the formulæ of ordinary algebra to any particular purpose, such as the calculation of areas, for example, we have in the first instance to see that the meanings we attach to the symbols are in accordance with the fundamental laws above stated. When this is established, the formulæ of algebra become mere machines for the saving of mental labour.

§ 28.] We now collect, for the reader's convenience, the general laws of ordinary algebra.

Definitions connecting the Direct and Inverse Operations.

Addition and subtraction—	Multiplication and division—
$+a-b+b=+a,$	$\times a\div b\times b=\times a,$
$+a+b-b=+a.$	$\times a\times b\div b=\times a.$

Law of Association.

For addition and subtraction—	For multiplication and division—
$\pm(\pm a\pm b)=\pm(\pm a)\pm(\pm b),$	$\overset{\times}{\div}(\overset{\times}{\div}a\overset{\times}{\div}b)=\overset{\times}{\div}(\overset{\times}{\div}a)\overset{\times}{\div}(\overset{\times}{\div}b),$

with the following law of signs:—

The concurrence of like signs gives the direct sign;

The concurrence of unlike signs the inverse sign.

Thus—

$$+(+a) = +a, \quad +(-a) = -a, \qquad \times(\times a) = \times a, \quad \times(\div a) = \div a,$$
$$-(-a) = +a, \quad -(+a) = -a. \qquad \div(\div a) = \times a, \quad \div(\times a) = \div a.$$

Law of Commutation.

For addition and subtraction—

$$\pm a \pm b = \pm b \pm a,$$

For multiplication and division—

$$\mathop{\times}\limits_{\div} a \mathop{\times}\limits_{\div} b = \mathop{\times}\limits_{\div} b \mathop{\times}\limits_{\div} a,$$

the operand always carrying its own sign of operation with it.

Properties of 0 *and* 1.

$$0 = +a - a,$$
$$\pm b + 0 = \pm b - 0 = \pm b,$$
$$+0 = -0.$$

$$1 = \times a \div a,$$
$$\mathop{\times}\limits_{\div} b \times 1 = \mathop{\times}\limits_{\div} b \div 1 = \mathop{\times}\limits_{\div} b,$$
$$\times 1 = \div 1.$$

Law of Distribution.

For multiplication—

$$(\pm a \pm b) \times (\pm c \pm d) = +(\pm a) \times (\pm c) + (\pm a) \times (\pm d)$$
$$+ (\pm b) \times (\pm c) + (\pm b) \times (\pm d),$$

with the following law of signs:—

If a partial product has constituents with like signs, it must have the sign $+$;

If the constituents have unlike signs, it must have the sign $-$.

Thus—

$$+(+a) \times (+c) = +a \times c, \quad +(+a) \times (-c) = -a \times c,$$
$$+(-a) \times (-c) = +a \times c, \quad +(-a) \times (+c) = -a \times c.$$

Property of 0.

$$0 \times b = b \times 0 = 0.$$

For division—

$$(\pm a \pm b) \div (\pm c) = +(\pm a) \div (\pm c) + (\pm b) \div (\pm c),$$

with the following law of signs:—

If the dividend and divisor of a partial quotient have like signs, the partial quotient must have the sign $+$;

If they have unlike signs, the partial quotient must have the sign $-$.

Thus—

$$+(+a)\div(+c) = +a\div c, \quad +(+a)\div(-c) = -a\div c,$$
$$+(-a)\div(-c) = +a\div c, \quad +(-a)\div(+c) = -a\div c.$$

N.B.—The divisor cannot be distributed.

Property of 0.

$$0\div b = 0.$$

N.B.—Nothing is said regarding $b\div 0$. This case will be discussed later on.

The reader should here mark the exact signification of the sign $=$ as hitherto used. It means "is transformable into by applying the laws of algebra, without any assumption regarding the operands involved."

Any "equation" which is true in this sense is called an "Identical Equation," or an "Identity"; and must, in the first instance at least, be carefully distinguished from an equation the one side of which can be transformed into the other by means of the laws of algebra *only when the operands involved have particular values or satisfy some particular condition.*

Some writers constantly use the sign $\equiv$ for the former kind of equation, and the sign $=$ for the latter. There is much to be said for this practice, and teachers will find it useful with beginners. We have, however, for a variety of reasons, adhered, in general, to the old usage; and have only introduced the sign $\equiv$ occasionally in order to emphasise the distinction in cases where confusion might be feared.

Exercises I.

[In working this set of examples the student is expected to avoid quoting derived formulæ that he may happen to recollect, and to refer every step to the fundamental principles discussed in the above chapter.]

(1.) Point out in what sense the usual arrangement of the multiplication of 365 by 492 is an instance of the law of distribution.

(2.) I have a multiplying machine, but the most it can do at one time is to multiply a number of 10 digits by another number of 10 digits. Explain how I can use my machine to multiply 13693456783231 by 46381239245932.

(3.) To divide 5004 by 12 is the same as to divide 5004 by 3, and then divide the quotient thus obtained by 4. Of what law of algebra is this an instance?

(4.) If the remainder on dividing N by a be R, and the quotient P, and if we divide P by b and find a remainder S, show that the remainder on dividing N by ab will be $a\mathrm{S}+\mathrm{R}$.

Illustrate with $5015 \div 12$.

(5.) Show how to multiply two numbers of 10 digits each so as to obtain merely the number of digits in the product, and the first three digits on the left of the product.

Illustrate by finding the number of digits, and the first three left-hand digits in the following:—

1st. $365989345678932567 8 \times 342973489379265$;

2nd. 2^{64}.

(6.) Express in the simplest form—

$$-(-(-(-(\ . \ . \ . \ (-1) \ . \ . \ . \ )))),$$

1st. Where there are $2n$ brackets;

2nd. Where there are $2n+1$ brackets; n being any whole number whatever.

(7.) Simplify and condense as much as possible—

$$2a - \{3a - [a - (b-a)]\}.$$

(8.) Simplify—

$$\text{1st.} \quad 3\{4 - 5[6 - 7(8 - 9\cdot\overline{10 - 11})]\},$$

$$\text{2nd.} \quad \tfrac{1}{3}\{\tfrac{1}{4} - \tfrac{1}{5}[\tfrac{1}{6} - \tfrac{1}{7}(\tfrac{1}{8} - \tfrac{1}{9}\cdot\overline{\tfrac{1}{10} - \tfrac{1}{11}})]\}.$$

(9.) Simplify—

$$1 - (2 - (3 - (4 - \ . \ . \ . \ (9 - (10 - 11)) \ . \ . \ . \ ))).$$

(10.) Distribute the following products:—1st. $(a+b)\times(a+b)$; 2nd $(a-b)\times(a+b)$; 3rd. $(3a-6b)\times(3a+6b)$; 4th. $(\tfrac{1}{3}a - \tfrac{1}{6}b)\times(\tfrac{1}{3}a+\tfrac{1}{6}b)$.

(11.) Simplify, by expanding and condensing as much as possible—

$$\{(m+1)a+(n+1)b\}\{(m-1)a+(n-1)b\}$$
$$+\{(m+1)a-(n+1)b\}\{(m-1)a-(n-1)b\}.$$

(12.) Simplify—

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right).$$

(13.) Simplify—

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)-\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\left(z-\frac{1}{z}\right).$$

(14.) Expand and condense as much as possible—

$$(\tfrac{1}{2}x+\tfrac{1}{4}y+\tfrac{1}{8}z)(\tfrac{1}{2}x-\tfrac{1}{4}y+\tfrac{1}{8}z).$$

Historical Note.—The separation and classification of the fundamental laws of algebra has been a slow process, extending over more than 2000 years. It is most likely that the first ideas of algebraic identity were of geometrical origin. In the second book of Euclid's *Elements* (about 300 B.C.), for example, we have a series of propositions which may be read as algebraical identities, the operands being lines and rectangles. In the extant works of the great Greek algebraist Diophantos (350 ?) we find what has been called a syncopated algebra. He uses contractions for the names of the powers of the variables ; has a symbol ⩚ to denote subtraction ; and even enunciates the abstract law for the multiplication of positive and negative numbers ; but has no idea of independent negative quantity. The Arabian mathematicians, as regards symbolism, stand on much the same platform ; and the same is true of the great Italian mathematicians Ferro, Tartaglia, Cardano, Ferrari, whose time falls in the first half of the sixteenth century. In point of method the Indian mathematicians Aryabhatta (476), Brahmagupta (598), Bhaskara (1114), stand somewhat higher, but their works had no direct influence on Western science.

Algebra in the modern sense begins to take shape in the works of Regiomontanus (1436-1476), Rudolff (about 1520), Stifel (1487-1567), and more particularly Viète (1540-1603) and Harriot (1560-1621). The introduction of the various signs of operation now in use may be dated, with more or less certainty, as follows: $\frac{a}{b}$ and apposition to indicate multiplication, as old as the use of the Arabic numerals in Europe ; $+$ and $-$, Rudolff 1525, and Stifel 1544 ; $=$, Recorde 1557 ; vinculum, Viète 1591 ; brackets, first by Girard 1629, but not in familiar use till the eighteenth century ; $< >$, Harriot's *Praxis*, published 1631 ; $\times$, Oughtred, and $\div$, Pell, about 1631.

It was not until the *Geometry* of Descartes appeared (in 1637) that the important idea of using a single letter to denote a quantity which might be either positive or negative became familiar to mathematicians.

The establishment of the three great laws of operation was left for the present century. The chief contributors thereto were Peacock, De Morgan, D. F. Gregory, Hankel, and others, working professedly at the philosophy of the first principles ; and Hamilton, Grassmann, Peirce, and their followers, who threw a flood of light on the subject by conceiving algebras whose laws differ from those of ordinary algebra. To these should be added Argand, Cauchy, Gauss, and others, who developed the theory of imaginaries in ordinary algebra.

CHAPTER II.

Monomials—Laws of Indices—Degree.

THEORY OF INDICES.

§ 1.] The product of a number of letters, or it may be numbers, each being supposed simple, so that multiplication merely and neither addition nor subtraction nor division occurs, is called an *integral term*, or more fully a *rational integral monomial* (that is, one-termed) *algebraical function*, for example, $a \times 3 \times 6 \times x \times a \times x \times x \times y \times b \times b$.

By the law of commutation we may arrange the constituents of this monomial in any order we please. It is usual and convenient to arrange and associate together all the factors that are mere numbers and all the factors that consist of the same letter; thus the above monomial would be written

$$(3 \times 6) \times (a \times a) \times (b \times b) \times (x \times x \times x) \times y.$$

3×6 can of course be replaced by 18, and a further contraction is rendered possible by the introduction of indices or exponents. Thus $a \times a$ is written a^2, and is read "a square," or "a to the second power." Similarly $b \times b$ is replaced by b^2, and $x \times x \times x$ by x^3, which is read "x cube," or "x to the third power." We are thus led to introduce the abbreviation x^n for $x \times x \times x \times \ldots$ where there are n factors, n being called the *index* or *exponent*,* while x^n is called the nth power of x, or x to the nth power.

§ 2.] It will be observed that, in order that the above definition may have any meaning, the exponent n must be a positive

---

* In accordance with this definition x^1 of course means simply x, and is usually so written.

integral number. Confining ourselves for the present to this case, we can deduce the following "laws of indices."

I. (α) $$a^m \times a^n = a^{m+n},$$

and generally $$a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}$$

(β) $$\frac{a^m}{a^n} = a^{m-n} \text{ if } m > n,$$

$$= \frac{1}{a^{n-m}} \text{ if } m < n.$$

II. $$(a^m)^n = a^{mn} = (a^n)^m.$$

III. (α) $$(ab)^m = a^m b^m,$$

and generally $$(abc \ldots)^m = a^m b^m c^m \ldots$$

(β) $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

To prove I. (α), we have, by the definition of an index,

$a^m \times a^n = (a \times a \times a \ldots m \text{ factors}) \times (a \times a \times a \ldots n \text{ factors}),$
$= a \times a \times a \ldots \overline{m+n}$ factors, by the law of association,
$= a^{m+n}$, by the definition of an index.

Having proved the law for two factors, we can easily extend it to the case of three or more,

for $a^m \times a^n \times a^p = (a^m \times a^n) \times a^p$, by law of association,
$= a^{m+n} \times a^p$, by case already proved,
$= a^{(m+n)+p}$, by case already proved,
$= a^{m+n+p}$;

and so on for any number of factors.

In words this law runs thus: The product of any number of powers of one and the same letter is equal to a power of that letter whose exponent is the sum of the exponents of these powers.

To prove I. (β),

$\frac{a^m}{a^n} = (a \times a \times \ldots m \text{ factors}) \div (a \times a \times \ldots n \text{ factors}),$
by definition of an index,

$= a \times a \times a \ . \ . \ . \ m$ factors $\div a \div a \div \ . \ . \ . \ n$ divisions,
by law of association.

Now if $m > n$ we may arrange these as follows :—

$$\frac{a^m}{a^n} = (a \times a \times \ . \ . \ . \ \overline{m-n} \text{ factors}) \times (a \div a) \times (a \div a) \ . \ . \ . \ n \text{ factors},$$
by laws of commutation and association,

$$= a \times a \times \ . \ . \ . \ \overline{m-n} \text{ factors, by the properties of division,}$$
$$= a^{m-n}.$$

If $m < n$, the rearrangement of the factors may be effected thus :—

$$\frac{a^m}{a^n} = \div (a \times a \times \ldots \overline{n-m} \text{ factors}) \times (a \div a) \times (a \div a) \ldots m \text{ factors}$$
$$= \div a^{n-m},$$
$$= \frac{1}{a^{n-m}}.$$

It is important to notice that I. (β) can be deduced from I. (a) without any further direct appeal to the definition of an index. Thus, if $m > n$, so that $m - n$ is positive,

$$a^{m-n} \times a^n = a^{(m-n)+n} \text{ by I. } (a),$$
$$= a^m.$$

Hence

$$a^{m-n} \times a^n \div a^n = a^m \div a^n.$$

Therefore, by the definition of $\times$ and $\div$,

$$a^{m-n} = a^m \div a^n.$$

Again, if $m < n$, so that $n - m$ is positive,

$$a^m \times a^{n-m} = a^{m+(n-m)}, \text{ by I. } (a),$$
$$= a^n, \text{ by the laws of } + \text{ and } -$$

Hence

$$a^m \times a^{n-m} \div a^{n-m} = a^n \div a^{n-m}.$$

Therefore, by the definition of $\times$ and $\div$,

$$a^m = a^n \div a^{n-m}.$$

Hence, by the laws of $\times$ and $\div$,

$$a^m \div a^n = a^n \div a^{n-m} \div a^n,$$
$$= (a^n \div a^n) \div a^{n-m},$$
$$= 1 \div a^{n-m}.$$

To prove II.,

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times \ldots n \text{ factors, by definition,}\\ &= (a \times a \times \ldots m \text{ factors}) \times (a \times a \times \ldots m \text{ factors})\\ &\qquad \times \ldots, n \text{ sets, by definition,}\\ &= a \times a \times \ldots mn \text{ factors, by law of association,}\\ &= a^{mn}, \text{ by definition.}\end{aligned}$$

To prove III. (α),

$$\begin{aligned}(ab)^m &= (ab) \times (ab) \times \ldots m \text{ factors, by definition,}\\ &= (a \times a \times \ldots m \text{ factors}) \times (b \times b \times \ldots m \text{ factors}),\\ &\qquad \text{by laws of commutation and association,}\\ &= a^m b^m, \text{ by definition.}\end{aligned}$$

Again,

$$\begin{aligned}(abc)^m &= \{(ab)c\}^m,\\ &= (ab)^m c^m, \text{ by last case,}\\ &= (a^m b^m)c^m, \text{ by last case,}\\ &= a^m b^m c^m, \text{ and so on.}\end{aligned}$$

Hence the mth power of the product of any number of letters is equal to the product of the mth powers of these letters.

To prove III. (β),

$$\begin{aligned}\left(\frac{a}{b}\right)^m &= (a \div b) \times (a \div b) \times \ldots m \text{ factors, by definition,}\\ &= (a \times a \times \ldots m \text{ factors}) \div (b \times b \times \ldots m \text{ factors}),\\ &\qquad \text{by commutation and association,}\\ &= a^m \div b^m,\\ &= \frac{a^m}{b^m}.\end{aligned}$$

In words: The mth power of the quotient of two letters is the quotient of the mth powers of these letters.

The second branch of III. may be derived from the first without further use of the definition of an index. Thus

$$\begin{aligned}\left(\frac{a}{b}\right)^m \times b^m &= \left(\frac{a}{b} \times b\right)^m, \text{ by III. } (\alpha),\\ &= a^m, \text{ by definition of } \times \text{ and } \div.\end{aligned}$$

Hence

$$\left(\frac{a}{b}\right)^m \times b^m \div b^m = a^m \div b^m,$$

that is,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

§ 3.] In so far as positive integral indices are concerned, the above laws are a deduction from the definition and from the laws of algebra. The use of indices is not confined to this case, however, and the above are *laid down* as the laws of indices generally. The laws of indices regarded in this way become in reality part of the general laws of algebra, and might have been enumerated in the Synoptic Table already given. In this respect, they are subject to the remarks in chap. i., § 27. The question of the *meaning* of fractional and negative indices is deferred till a later chapter, but the student will have no difficulty in working the exercises given below. All he has to do is to use the above laws whenever it is necessary, without regard to any restriction on the value of the indices.

§ 4.] The following examples are worked to familiarise the student with the meaning and use of the laws of indices. At first he should be careful to refer each step to the proper law, and to see that he takes no step which is not sanctioned by some one of the laws of indices, or by one of the fundamental laws of algebra.

Example 1.

$(a^3b^2c^5)\times(a^5b^6c^{11})\div(a^4b^3c^{15})$

$=a^3a^5b^2b^6c^5c^{11}\div a^4\div b^3\div c^{15}$, by commutation and association,

$=a^{3+5}b^{2+6}c^{5+11}\div a^4\div b^3\div c^{15}$, by law of indices, I. (α),

$=(a^{3+5}\div a^4)\times(b^{2+6}\div b^3)\times(c^{5+11}\div c^{15})$, by commutation and association.

$=a^{3+5-4}\times b^{2+6-3}\times c^{5+11-15}$, by law of indices, I. (β),

$=a^4b^5c.$

Example 2.

$(15x^2y^3z^5)^2\times\left(\dfrac{x^2}{12y^4z^5}\right)^2$

$=15^2(x^2)^2(y^3)^2(z^5)^2\times\dfrac{(x^2)^2}{(12y^4z^5)^2}$, by laws of indices, III. (α) and III. (β),

$=\dfrac{(3\times5)^2x^4y^6z^{10}x^4}{(3\times4)^2(y^4)^2(z^5)^2}$, by II. and III. ($\alpha$),

$=\dfrac{3^2\times5^2x^8y^6z^{10}}{3^2\times4^2y^8z^{10}}$, by I. ($\alpha$) and II.,

$=3^2\div3^2\times5^2\div4^2\times x^8\times y^6\div y^8\times z^{10}\div z^{10}$,

$=5^2\div4^2\times x^8\div y^2$,

$=\left(\dfrac{5}{4}\right)^2\dfrac{x^8}{y^2}.$

THEORY OF DEGREE.

§ 5.] The result of multiplying or dividing any number of letters or numbers one by another, addition and subtraction being excluded, for example, $3 \times a \times x \times b \div c \div y \times d$, is called a (rational) *monomial algebraical function* of the numbers and letters involved, or simply a *term*. If the monomial either does not contain or can be so reduced as not to contain the operation of division, it is said to be *integral*; if it cannot be reduced so as to become entirely free of division, it is said to be *fractional*. In drawing this distinction, division by mere numbers is usually disregarded, and even division by certain specified letters may be disregarded, as will be explained presently.

§ 6.] The number of times that any particular letter occurs by way of multiplication in an integral monomial is called the *degree* (or dimension) of the monomial in that particular letter; and the degree of the monomial in any specified letters is the sum of its degrees in each of these letters. For example, the degree of $6 \times a \times a \times x \times x \times x \times y \times y$, that is, of $6a^2x^3y^2$, in a is 2, in x 3, in y 2, and the degree in x and y is 5, and in a, x, and y 7.

In other words, the degree is the sum of the indices of the named letters. The choice of the letters which are to be taken into account in reckoning the degree is quite arbitrary; one choice being made for one purpose, another for another. When certain letters have been selected, however, for this purpose, it is usual to call them the *variables*, and to call the other letters, including mere numbers, *constants*. The monomial is usually arranged so that all the constants come first and the variables last; thus, x and y being the variables, we write $32a^2bcx^3y^2$; and the part $32a^2bc$ is called the *coefficient*.

In considering whether a monomial is integral or not, division by *constants* is not taken into account.

§ 7.] The notion of degree is an exceedingly important one, and the student must at once make himself perfectly familiar with it. He will find as he goes on that it takes to a large extent in algebra the same place as numerical magnitude in arithmetic.

The following theorems are particular cases of more general ones to be proved by and by.

The degree of the product of two or more monomials is the sum of their respective degrees.

If the quotient of two monomials be integral, its degree is the excess of the degree of the dividend over that of the divisor.

For let
$$A = cx^l y^m z^n u^p \ . \ . \ .$$
$$A' = c'x^{l'} y^{m'} z^{n'} u^{p'} \ . \ . \ .$$
where c and c' are the coefficients, $x, y, z, u \ . \ . \ .$ the variables, and $l, m, n, p \ldots, l', m', n', p' \ldots$ are of course positive integral numbers. Then the degree, d, of A is given by $d = l + m + n + p + . \ . \ .$, and the degree, d', of A′ by $d' = l' + m' + n' + p' + . \ . \ .$

But
$$A \times A' = (cx^l y^m z^n u^p \ . \ . \ .) \times (c'x^{l'} y^{m'} z^{n'} u^{p'} \ . \ . \ .)$$
$$= (c \times c')x^{l+l'} y^{m+m'} z^{n+n'} u^{p+p'} \ . \ . \ .$$
the degree of which is $(l + l') + (m + m') + (n + n') + (p + p') \ . \ . \ .$, that is, $(l + m + n + p \ . \ . \ .) + (l' + m' + n' + p' + \ . \ . \ .)$, that is, $d + d'$, which proves the first proposition for two factors. The law of association enables us at once to extend it to any number of factors.

Again, let $Q = A \div A'$, and let Q be integral and its degree δ. Now we have, by the definition of division, $Q \times A' = A$. Hence, by last proposition, the degrees of A and A′ being d and d', as before, we have $d = \delta + d'$, and thence $\delta = d - d'$.

As an example, let $A = 6x^9y^5$, $A' = 7x^7y^3$, then $A \times A' = 42x^{16}y^8$, and $A \div A' = \frac{6}{7}x^2y^2$. The degree of $A \times A'$ is 24, that is, $14 + 10$; that of $A \div A'$ is 4, that is, $14 - 10$.

The student will probably convince himself most easily of the truth of the two propositions by considering particular cases such as these; but he should study the general proof as an exercise in abstract reasoning for on such reasoning he will have to rely more and more as he goes on.

Exercises II.

[Wherever it is possible in working the following examples, the student should verify the laws of degree, §§ 5-7.]

(1.) Simplify— $\dfrac{5^7 \times 12^4 \times 32^2 \times (3^2 \times 4^2 \times 5)^2}{(3 \times 15 \times 2^3)^{10}}$.

(2.) Which is greater, $(2^2)^{2^2}$ or $2^{2^{2^2}}$? Find the difference between them.

(3.) Simplify—

$$\frac{2^{2^{2^2}}}{2(2^2)^{2^2}}.$$

(4.) Simplify—
$$\frac{36^2a^7b^5c^3d^2}{81a^4b^3c^3}.$$

(5.) Express in its simplest form—

$$\left(\frac{a^4c^3y^2}{a^3b^2c^3x^2y^2}\right)^2 \times \left(\frac{a^3b^3x^3}{a^3b^2c^3x^3y^2}\right)^3 \times \left(\frac{b^2c^4x^2y}{a^4b^3x^3y^3}\right)^4.$$

(6.) Simplify—
$$\left(\frac{45a^3b^2c^4}{27a^2b^2c}\right)^2 \times \left(\frac{243a^4b^4c^4x}{180a^2bc}\right)^2.$$

(7.) Simplify—
$$(3xyz)^2 \times \frac{x^6y^9}{x^9y^6} \div \left(\frac{3x^2z^2}{y^2z^2}\right)^2.$$

(8.) Simplify—
$$\frac{(x^3y^2z^2)^7 \times (y^3z^2x^2)^7 \times (z^3x^2y^2)^7}{\left(\frac{x^2}{yz}\right)^7 \times \left(\frac{y^2}{zx}\right)^7 \times \left(\frac{z^2}{xy}\right)^7}.$$

(9.) Simplify—
$$\left(\frac{x^m}{x^n}\right)^{m+n} \times \left(\frac{x^n}{x^l}\right)^{n+l} \times \left(\frac{x^l}{x^m}\right)^{l+m}.$$

(10.) Simplify—
$$\left\{\frac{(a^{p-q})^q \times (a^{q-r})^{r-p}}{(a^{r+q})^{r-q}}\right\}^{1/r}.$$

(11.) Simplify—
$$\frac{(x^{a-b} \times x^{b-c})^a \times \left(\frac{x^a}{x^c}\right)}{(x^a \times x^c)^a \div (x^{a+c})^c}.$$

(12.) Simplify—
$$\left(\frac{x^p}{x^q}\right)^{p+q} \div \left(\frac{x^{p+q}}{x^{p-q}}\right)^{p^2/q}.$$

(13.) Simplify—

$$\left\{\left(\frac{x^l}{x^m}\right)^l \times \left(\frac{x^m}{x^l}\right)^m\right\} \div \{(x^l)^l \times (x^m)^m\} \times \{(x^m)^l \times (x^l)^m\}.$$

(14.) Prove that—

$$\frac{(yz)^{qr}(zx)^{rp}(xy)^{pq}}{(y^{q-1}z^{r-1})^p(z^{r-1}x^{p-1})^q(x^{p-1}y^{q-1})^r} = \frac{(xyz)^{p+q+r}}{x^py^qz^r}.$$

(15.) Distribute the product—

$$(x^{b-c} - x^{c-a} + x^{a-b})\left(\frac{1}{x^{b-c}} + \frac{1}{x^{c-a}} + \frac{1}{x^{a-b}}\right).$$

(16.) Distribute—
$$\left(a^p + \frac{1}{b^q}\right)\left(a^q + \frac{1}{b^p}\right)^2.$$

(17.) If $m = a^x$, $n = a^y$, $a^2 = (m^yn^x)^z$; show that $xyz = \mathbf{1}$.

CHAPTER III.

Fundamental Formulæ relating to Quotients or Fractions, with Applications to Arithmetical Fractions and to the Theory of Numbers.

OPERATIONS WITH FRACTIONS.

§ 1.] Before proceeding to cases where the fundamental laws are masked by the complexity of the operations involved, we shall consider in the light of our newly-acquired principles a few cases with most of which the student is already partly familiar. He is not in this chapter to look so much for new results as to exercise his reasoning faculty in tracing the operation of the fundamental laws of algebra. It will be well, however, that he should bear in mind that the letters used in the following formulæ may denote any operands subject to the laws of algebra; for example, mere numbers integral or fractional, single letters, or any functions of such, however complex.

§ 2.] Bearing in mind the equivalence of the notations $\frac{a}{b}$, a/b, and $a \div b$, the laws of association and commutation for multiplication and division, and finally the definition of a quotient, we have

$$\begin{aligned} \frac{pa}{pb} = (pa) \div (pb) &= p \times a \div p \div b, \\ &= a \div b \div p \times p, \\ &= a \div b; \end{aligned}$$

that is,

$$\frac{pa}{pb} = \frac{a}{b}.$$

Read forwards and backwards this equation gives us the important proposition that *we may divide or multiply the numerator and denominator of a fraction by the same quantity without altering its value.*

§ 3.] Using the principle just established, and the law of distribution for quotients, we have

$$\begin{aligned} \pm\frac{a}{b} \pm \frac{p}{q} &= \pm\frac{qa}{qb} \pm \frac{pb}{qb}, \\ &= \frac{\pm qa \pm pb}{qb}; \end{aligned}$$

that is, *To add or subtract two fractions, transform each by multiplying numerator and denominator so that both shall have the same denominator, add or subtract the numerators, and write underneath the common denominator.*

The rule obviously admits of extension to the addition in the algebraic sense (that is, either addition or subtraction) of any number of fractions whatever.

Take, for example, the case of three :—

$$\begin{aligned} \pm\frac{a}{b} \pm \frac{c}{d} \pm \frac{e}{f} &= \pm\frac{adf}{bdf} \pm \frac{cbf}{bdf} \pm \frac{ebd}{bdf}, \text{ by § 2,} \\ &= \frac{\pm adf \pm cbf \pm ebd}{bdf}, \text{by law of distribution.} \end{aligned}$$

The following case shows a modification of the process, which often leads to a simpler final result. Suppose $b = lc$, $q = lr$; then, taking a particular case out of the four possible arrangements of sign,

$$\begin{aligned} \frac{a}{b} - \frac{p}{q} &= \frac{a}{lc} - \frac{p}{lr}, \\ &= \frac{ar}{lcr} - \frac{pc}{lrc} \\ &= \frac{ar - pc}{lcr}. \end{aligned}$$

Here the common denominator lcr is simpler than bq, which is l^2cr.

The same result would of course be arrived at by following

the process given above, and simplifying the resulting fraction at the end of the operation, thus :—

$$\frac{a}{lc}-\frac{p}{lr}=\frac{alr-plc}{(lc)(lr)}, \text{ as above};$$
$$=\frac{(ar-pc)l}{l^2cr},$$

by using the law of distribution in the numerator, and the laws of association and commutation in the denominator;

$$=\frac{ar-cp}{lcr}, \text{ by § 2.}$$

§ 4.] The following are merely particular cases of the laws of association and commutation for multiplication and division:—

$$\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)=(a\div b)\times(c\div d),$$
$$=a\div b\times c\div d,$$
$$=a\times c\div b\div d,$$
$$=(ac)\div(bd),$$
$$=\frac{ac}{bd};$$

or, in words, *To multiply two fractions, multiply their numerators together for the numerator, and the denominators together for the denominator of the product.*

Again,

$$\left(\frac{a}{b}\right)\div\left(\frac{c}{d}\right)=(a\div b)\div(c\div d),$$
$$=a\div b\div c\times d,$$
$$=a\times d\div b\div c,$$
$$=(ad)\div(bc),$$
$$=\frac{ad}{bc};$$

also

$$=\left(\frac{a}{b}\right)\times\left(\frac{d}{c}\right),$$

by last case. In words: *To divide one fraction by another, invert the latter and then multiply.*

§ 5.] In last paragraph, and in § 2 above, we have for

simplicity omitted all explicit reference to sign. In reality we have not thereby restricted the generality of our conclusions, for by the principle of substitution (which is merely another name for the generality of algebraic formulæ) we may suppose the p, for example, of § 2 to stand for $-\omega$, say, and we then have

$$\frac{(-\omega)a}{(-\omega)b}=\frac{a}{b};$$

that is, taking account of the law of signs,

$$\frac{-\omega a}{-\omega b}=\frac{a}{b};$$

and so on.

Exercises III.

(1.) Express in its simplest form—

$$\frac{x^2}{x-y}+\frac{y^2}{y-x}.$$

(2.) Express in its simplest form—

$$\frac{a}{a-b}+\frac{b}{b-a}.$$

(3.) Simplify— $$\frac{P+Q}{P-Q}-\frac{P-Q}{P+Q},$$

where $P=x+y,\quad Q=x-y.$

(4.) Simplify— $$\frac{1-\dfrac{x(1-y)}{x+y}}{1+\dfrac{1-y}{x+y}}.$$

(5.) Simplify— $$\frac{\dfrac{1}{ab}-\dfrac{1}{ac}-\dfrac{1}{bc}}{\dfrac{a^2-(b-c)^2}{a}}.$$

(6.) Simplify— $$\left(a-\frac{b^2}{a+b}\right)\times\left(a+\frac{b^2}{a-b}\right).$$

(7.) Simplify— $$\frac{1}{x+y}+\frac{1}{x-y}-\frac{2x}{x^2+y^2}.$$

(8.) Simplify— $$\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{a}{b}+\frac{b}{a}\right)-\left(\frac{x}{y}-\frac{y}{x}\right)\left(\frac{a}{b}-\frac{b}{a}\right).$$

(9.) Simplify— $$\left(\frac{a}{b}-\frac{b}{a}\right)\div\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right).$$

(10.) Simplify—
$$\frac{a(a-b)-b(a+b)}{\dfrac{a}{a+b}-\dfrac{b}{a-b}}.$$

(11.) Simplify—
$$\frac{1-x}{1+x+x^2}-\frac{1+x}{1-x+x^2}.$$

(12.) Simplify—
$$\frac{1+\dfrac{x}{1+x}}{x+\dfrac{1}{1+x}}\div\frac{(x+1)^2-x^2}{x^2+x+1}.$$

(13.) Simplify—
$$\frac{a^2+b^2}{(a+b)^2}+\frac{\dfrac{2}{ab}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^3}.$$

(14.) Show that
$$\frac{x^4}{a^2b^2}+\frac{(x^2-a^2)^2}{a^2(a^2-b^2)}-\frac{(x^2-b^2)^2}{b^2(a^2-b^2)}$$
is independent of x.

(15.) Simplify—
$$\frac{a}{b-\dfrac{c}{d-\dfrac{e}{f}}}.$$

(16.) Simplify—
$$\frac{1}{a-2b-\dfrac{1}{a-2b-\dfrac{1}{a-2b}}}.$$

(17.) Simplify—
$$\frac{a+b}{a+b+\dfrac{1}{a-b+\dfrac{1}{a+b}}}.$$

APPLICATIONS TO THE THEORY OF NUMBERS.

§ 6.] In the applications that follow, the student should look somewhat closely at the meanings of some of the terms employed. This is necessary because, unfortunately, some of these terms, such as *integral*, *factor*, *divisible*, &c., are used in algebra generally in a sense very different from that which they bear in ordinary arithmetic and in the theory of numbers.

An *integer*, unless otherwise stated, means for the present a *positive* (or *negative*) *integral number*. The *ordinary* notion of greater and less in connection with such numbers, irrespective of their sign, is assumed as too simple to need definition.* When

* This is a very different thing from the algebraical notion of greater and less. See chap. xiii., § 1. It may not be superfluous to explain

an integer a can be produced by multiplying together two others, b and c, b and c are called *factors* of a, and a is said to be *exactly divisible* by b and by c, and to be a *multiple* of b or of c. Since the product of two integers, neither of which is unity, is an integer greater than either of the two, it is clear that *no integer is exactly divisible by another greater than itself.*

It is also obvious that every integer (other than unity) has at least two divisors, namely, unity and itself; if it has more, it is called a *composite integer,* if it has no more, a *prime integer.* For example, 1, 2, 3, 5, 7, 11, 13, . . . are all prime integers, whereas 4, 6, 8, 9, 10, 12, 14 are composite.

If an integer divide each of two others it is said to be a common factor or *common measure* of the two. If two integers have no *common measure* except unity they are said to be *prime to each other.* It is of course obvious that two integers, such as 6 and 35, which are *prime to each other* need not be themselves *prime integers.* We may also speak of a common measure of more than two integers, and of a group of more than two integers that are prime to each other, meaning, in the latter case, a set of integers no two of which have any common measure.

§ 7.] If we consider any composite integer N, and take in order all the primes that are less than it, any one of these either will or will not divide N. Let the first that divides N be a, then $N = aN_1$, where N_1 is an integer; if N_1 be also divisible by a we have $N_1 = aN_2$, and $N = a(aN_2) = a^2N_2$; and clearly, finally, say $N = a^\alpha N_\alpha$, where N_α is either 1 or no longer divisible by a. N_α (if not $=1$) is now either prime or is divisible by some prime $> a$ and $< N_\alpha$, and, *a fortiori*, $< N$, say b; we should on the last supposition have $N_\alpha = b^\beta N_\beta$, where $N_\beta < N_\alpha$, and so on. The process clearly must end with unity, so that we get

$$N = a^\alpha b^\beta \dots,$$

where $a, b, \dots$ are primes, and $\alpha, \beta, \dots$ positive integers. It

here the use of the inequality symbols $\ngtr$, $>$, $<$, $\ngtr$, $\nless$; they mean respectively "is not equal to," "is greater than," "is less than," "is not greater than," "is not less than." Instead of $\ngtr$, $\nless$ we may use $\leqslant, \geqslant$ which may be read "is equal to or less than," "is equal to or greater than."

is to be observed that a^α, b^β, . . . are powers of primes, and therefore, as we shall prove presently, prime to each other. *It is therefore always possible to resolve every composite integer into factors that are powers of primes*; and we shall presently show that this resolution can be effected in one way only.

§ 8.] *If a be divisible by c, then any integral multiple of a, say ma, is divisible by c; and, if a and b be each divisible by c, then the algebraic sum of any integral multiples of a and b, say $ma + nb$, is divisible by c.*

For by hypothesis $a = \alpha c$ and $b = \beta c$, where α and β are integers, hence $ma = m\alpha c = (m\alpha)c$, where $m\alpha$ is an integer, that is, ma is divisible by c. And $ma + nb = m\alpha c + n\beta c = (m\alpha + n\beta)c$, where $m\alpha + n\beta$ is an integer, that is, $ma + nb$ is divisible by c. The student should observe that, by virtue of the extension of the notion of divisibility by the introduction of negative integers, any of the numbers in the above proposition may be negative.

§ 9.] From the last article we can deduce a proposition which at once gives us the means of finding the greatest common measure of two integers, or of proving that they are prime to each other.

If $a = pb + c$, where a, b, c, p are all integers, then the G.C.M. of a and b is the G.C.M. of b and c.

To prove this it is necessary and it is sufficient to show—1st, that every divisor of b and c divides a and b, and, 2nd, that every divisor of a and b divides b and c.

Since $a = pb + c$, it follows from § 8 that every divisor of b and c divides a, that is, every divisor of b and c divides a and b.

Again, since $a = pb + c$, it follows that $c = a - pb$; hence, again by § 8, every divisor of a and b divides c, that is, every divisor of a and b divides b and c. Thus the two parts of the proof are furnished.

Let now a and b be two numbers whose G.C.M. is required; they will not be equal, for then the G.C.M. would be either of them. Let b denote the less, and divide a by b, the quotient being p and the remainder c, where of course $c < b$.* Next divide b by c, the quotient being q, the remainder d; then divide c by d, the quotient being r, the remainder e, and so on.

---

* For a formal definition of the remainder see § 11.

Since $a > b$, $b > c$, $c > d$, $d > e$, &c., it is clear that the remainders must diminish down to zero. We thus have the following series of equations:—

$$\begin{aligned} a &= pb + c \\ b &= qc + d \\ c &= rd + e \\ &\cdots\cdots \\ &\cdots\cdots \\ l &= vm + n \\ m &= wn. \end{aligned}$$

Hence the G.C.M. of a and b is the same as that of b and c, which is the same as that of c and d, that is, the same as that of d and e, and finally the same as that of m and n. But, since $m = wn$, the G.C.M. of m and n is n, for n is the greatest divisor of n itself. Hence the G.C.M. of a and b is the divisor corresponding to the remainder 0 in the chain of divisions above indicated.

If n be different from unity, then a and b have a G.C.M. in the ordinary sense.

If n be equal to unity, then they have no common divisor except unity, that is, they are prime to each other.

§ 10.] It should be noticed that the essence of the foregoing process for finding the G.C.M. of two integers is the substitution for the original pair, of successive pairs of continually decreasing integers, each pair having the same G.C.M. All that is necessary is that $p, q, r, \ldots$ be integers, and that $a, b, c, d, e, \ldots$ be in decreasing order of magnitude.

The process might therefore be varied in several ways. Taking advantage of the use of negative integers, we may sometimes abbreviate it by taking a negative instead of a positive remainder, when the former happens to be numerically less than the latter.

For example, take $a = 4323$, $b = 1595$,

we might take $4323 = 2 \times 1595 + 1133$

or $4323 = 3 \times 1595 - 462$;

the latter is to be preferred, because 462 is less than 1133. In practice the negative sign of 462 may be neglected in the rest of the operation, which may be arranged as follows, for the sake of comparison with the ordinary process already familiar to the student:—

```
1595)4323(3
     4785
     ----
      462)1595(3
          1386
          ----
           209)462(2
               418
               ---
                44)209(5
                   220
                   ---
                    11)44(4
                       44
                       --
```

G.C.M. = 11.

By means of the process for finding the G.C.M. we may prove the following proposition, of whose truth the student is in all probability already convinced by experience :—

If a and b be prime to each other, and h any integer, then any common factor of ah and b must divide h exactly.

For, since a and b are prime, we have by § 9,

$$\left.\begin{aligned} a &= pb + c \\ b &= qc + d \\ c &= rd + e \\ &\cdots\cdots \\ l &= vm + 1 \end{aligned}\right\} (1). \quad \text{Hence} \left\{\begin{aligned} ah &= pbh + ch \\ bh &= qch + dh \\ ch &= rdh + ch \\ &\cdots\cdots\cdots, \\ lh &= vmh + h \end{aligned}\right\} (2).$$

Now, since any common factor of ah and b is a common factor of ah and bh, it follows from the first of equations (2) that such a common factor divides ch exactly, and by the second that it also divides dh exactly, and so on ; and, finally, by the last of equations (2), that any common factor of ah and b divides h exactly.

In particular, since b is a factor of itself, we have

Cor 1. *If b divide ah exactly and be prime to a, it must divide h exactly.*

Cor. 2. *If a' be prime to a and to b and to c, &c., then it is prime to their product abc . . .*

For, if a' had any factor in common with abc . . ., that is, with $a(bc\ldots)$, then, since a' is prime to a, that factor, by the proposition above, must divide bc . . . exactly ; hence, since a'

is prime to b, the supposed factor must divide c . . . exactly, and so on. But in this way we exhaust all the factors of the product, since all are prime to a'. Hence no such factor can exist, that is, a' is prime to abc . . .

An easy extension of this is the following :—

Cor. 3. *If all the integers $a', b', c', \ldots$ be prime to all the integers $a, b, c, \ldots$, then the product $a'b'c' \ldots$ is prime to the product $abc \ldots$*

A particular case of which is

Cor. 4. *If a' be prime to a (and in particular if both be primes), then any integral power of a' is prime to any integral power of a.*

§ 11.] It is obvious that, if a and b be two integers, we can in an infinite number of ways put a into the form of $qb+r$, where q and r are integers, for, if we take q any integer whatever, and find r so that $a-qb=r$, then $a=qb+r$.

There are two important special cases, those, namely, where we restrict r to be numerically less than b, and either (1) positive or (2) negative. In each of these cases the resolution of a is always possible in one way only. For, in case 1, if qb be the greatest multiple of b which does not exceed a, then $a-qb=r$, where $r<b$; hence $a=qb+r$; and in case 2, if $q'b$ be the least multiple of b which is not less than a, then $a-q'b=-r'$, where $r'<b$. Also the resolution is unique; for suppose, in case 1, that there were two resolutions, another being $a=\chi b+\rho$, say; then $qb+r=\chi b+\rho$, therefore $r-\rho=(\chi-q)b$; hence $r-\rho$ is divisible by b; but, r and ρ being each positive, and each numerically $<b$, $r-\rho$ is numerically less than b, and therefore cannot be divisible by b. Hence there cannot be more than one resolution of the form 1. Similar reasoning applies to case 2.

r and r' are often spoken of as the least *positive* and *negative remainders* of a with respect to b. When the remainder is spoken of without qualification the least positive remainder is meant. If a more general term is required, corresponding to the removal of the restriction r numerically $<b$, the word *residue* is used.

It is obvious, from the definitions laid down in § 6, that *a is or is not exactly divisible by b according as the least remainder of a with respect to b does or does not vanish.*

The student will also prove without difficulty that *if the remainders of a and of a' with respect to b be the same, then $a - a'$ is divisible by b; and conversely.*

Cor. *If q be a fixed integer (sometimes spoken of as a modulus), then every other integer can be expressed in one or other of the forms*

$$bq,\ bq+1,\ bq+2,\ \ldots,\ bq+(q-1),$$

where b is an integer.

For, as we have seen, we can put any given integer a into the form $bq + r$, where $r \not> q$, and here r must have one of the values $0, 1, 2, \ldots, (q-1)$.

Example. Take $q=5$, then

$$\begin{array}{lllll} 0=0.5, & 1=0.5+1, & 2=0.5+2, & 3=0.5+3, & 4=0.5+4\,; \\ 5=1.5, & 6=1.5+1, & 7=1.5+2, & 8=1.5+3, & 9=1.5+4\,; \\ 10=2.5, & 11=2.5+1, & 12=2.5+2, & 13=2.5+3, & 14=2.5+4\,; \end{array}$$

and so on.

It should be noticed that, since $bq+(q-1)=(b+1)q-1$, $bq+(q-2)=(b+1)q-2$, &c., we might put every integer into one or other of the forms

$$bq,\ bq \pm 1,\ bq \pm 2,\ \ldots,\ \text{\&c.}$$

For example,

$$8=2.5-2, \quad 9=2.5-1, \quad 10=2.5, \quad 11=2.5+1, \quad 12=2.5+2.$$

The above principle, which may be called the periodicity of the integral numbers with respect to a given modulus, is of great importance in the theory of numbers.

§ 12.] When the quotient a/b cannot be expressed as an integer, it is said to be *fractional* or *essentially fractional*; if $a > b$, a/b is called in this case an *improper fraction*; if $a < b$, a *proper fraction.*

Hence *no true fraction, proper or improper, can be equal to an integer.*

Every improper fraction a/b can be expressed in the form $q + r/b$, where q is an integer and r/b a proper fraction. For, if r be the least positive remainder when a is divided by b, $a = qb + r$, and $a/b = (qb+r)/b = q + r/b$, where q and r are integers and $r < b$.

If two improper fractions a/b and a'/b' be equal, their integral parts and their proper fractional parts must be equal separately. For,

if this were not so, we should have, say $a/b = q + r/b$, $a'/b' = q' + r'/b'$, and $q + r/b = q' + r'/b'$; whence $q - q' = r'/b' - r/b = (r'b - rb')/bb'$. Now $r'b < b'b$ and $rb' < bb'$, hence $r'b - rb'$ is numerically $< bb'$. In other words, the integer $q - q'$ is equal to a proper fraction, which is impossible.

§ 13.] We can now prove that *an integer can be resolved into factors which are powers of primes in one way only.*

For, since the factors in question are powers of primes, they are prime to each other. Let, if possible, there be two such resolutions, namely, $a'b'c' \ldots$ and $a''b''c'' \ldots$ of the same integer N. Since $a'b'c' \ldots = a''b''c'' \ldots$, therefore $a'b'c' \ldots$ is exactly divisible by a''. Now, since a'' is a power of a prime, it will be prime to all the factors a', b', c', . . . save one, say a', which is a power of the same prime. Moreover, such a factor as a' (that is, a power of the prime of which a'' is a power) must occur, for, if it did not, then all the factors of $a'b'c' \ldots$ would be prime to a'', and a'' could not be a factor of N. It follows, then, that a' must be divisible by a''.

Again, since $a''b''c'' \ldots = a'b'c' \ldots$, therefore $a''b''c'' \ldots$ is divisible by a', and it follows as before that a'' is divisible by a'.

But, if two integers be such that each is divisible by the other, they must be equal (§ 6); hence $a'' = a'$.

Proceeding in this way we can show that each factor in the one resolution occurs in the other.

§ 14.] *Every remainder in the ordinary process for finding the G.C.M. of two positive integers a and b can be expressed in the form $\pm(\mathrm{A}a - \mathrm{B}b)$, where A and B are positive integral numbers. The upper sign being used for the 1st, 3rd, 5th, &c., and the lower for the 2nd, 4th, &c., remainders.*

For, by the equations in § 9, we have successively—

$$\begin{aligned} c &= +\{a - pb\} && (1); \\ d &= b - qc = b - q(a - pb), \\ &= -\{qa - (1 + pq)b\} && (2); \\ e &= c - rd, \\ &= \{a - pb\} + r\{qa - (1 + pq)b\}, \\ &= +\{(1 + qr)a - (p + r + pqr)b\} && (3); \end{aligned}$$

and so on. It is evident in fact that, if the theorem holds for any two successive remainders, it must hold for the next. Now equations (1), (2), and (3) prove it for the first three remainders; hence it holds for the fourth; hence for the fifth; and so on.

In the chapter on Continued Fractions, a convenient process will be given for calculating the successive values of A and B for each remainder. In the meantime it is sufficient to have established the existence of these numbers, and to have seen a straightforward way of finding them.

Cor. 1. *Since g, the G.C.M. of a and b, is the last remainder, we can always express g in the form*—

$$g = \pm (\mathrm{A}a - \mathrm{B}b) \qquad (4),$$

where A *and* B *are positive integers.*

Cor. 2. If a be prime to b, $g = 1$; hence, *If a and b be two integers prime to each other, two positive integers,* A *and* B, *can always be found such that*—

$$\mathrm{A}a - \mathrm{B}b = \pm 1 \qquad (5).$$

N.B.—It is clear that A *must be prime to* B. For, since a/g and b/g are integers, l and m say, we have, from (4),

$$1 = \pm (\mathrm{A}l - \mathrm{B}m);$$

hence, if A and B had any common factor it would divide 1 (by § 8 above).

Cor. 3. From Cor. 1 and § 8 we see that *every common factor of a and b must be a factor in their G.C.M.*

A result which may be proved otherwise, and will probably be considered obvious.

Cor. 4. Hence, *To find the G.C.M. of more than two integers a, b, c, d, . . ., we must first find g the G.C.M. of a and b, then g' the G.C.M. of g and c, then g'' the G.C.M. of g' and d, and so on, the last G.C.M. found being the G.C.M. of all the given integers.*

For every common factor of a, b, c must be a factor in a and b, that is, must be a factor in g; hence, to find the greatest common factor in a, b, c, we must find the greatest common factor in g and c; and so on.

From Cor. 2 we can also obtain an elegant proof of the conclusions in the latter part of § 10.

Example 1. To express the G.C.M. of 565 and 60 in the form A565 – B60. We have $565=9\times60+25$, $60=2\times25+10$, $25=2\times10+5$, $10=2\times5$.

Hence the G.C.M. is 5, and we have successively

$$\begin{aligned}25&=565-9\times60\,;\\10&=60-2\{565-9\times60\}\\&=-\{2\times565-19\times60\}\,;\\5&=25-2\times10\\&=565-9\times60+2\{2\times565-19\times60\}\\&=5\times565-47\times60.\end{aligned}$$

Example 2. Show that two integers A and B can be found so that

$$5\mathrm{A}-7\mathrm{B}=1.$$

We have $7=1\times5+2$, $5=2\times2+1$; whence $2=7-5$, $1=5-2(7-5)$ $=3\times5-2\times7$.

Hence $\mathrm{A}=3$, $\mathrm{B}=2$ are integers satisfying the requirements of the question.

Example 3. If $a, b, c, d, \ldots$ be a series of integers whose G.C.M. is g, show that integers (positive or negative) A, B, C, D, . . . can be found such that

$$g=\mathrm{A}a+\mathrm{B}b+\mathrm{C}c+\mathrm{D}d+\ldots$$

(Gauss's *Disquisitiones Arithmeticæ*, Th. 40).

Find A, B, C, D, when $a=36$, $b=24$, $c=18$, $d=30$.

This result may be easily arrived at by repeated application of corollaries 1 and 4 of this article.

Example 4. The proper fraction p/ab, where a is prime to b, can be decomposed, and that in one way only, into the form

$$\frac{a'}{a}+\frac{b'}{b}-k,$$

where a' and b' are both positive, $a'<a$, $b'<b$, and k is the integral part of $a'/a+b'/b$; that is to say, 0 or 1, according to circumstances.

Illustrate with 6/35.

Since a is prime to b, by Cor. 2 above,

$$\mathrm{A}a-\mathrm{B}b=\pm1\,;$$

multiplying this equation by $\pm p/ab$, we have

$$\pm\frac{p\mathrm{A}}{b}\mp\frac{p\mathrm{B}}{a}=\frac{p}{ab} \qquad (1).$$

If the upper sign has to be taken, resolve $p\mathrm{A}$ and $p\mathrm{B}$ as follows (§ 11):—

$$\begin{aligned}p\mathrm{A}&=lb+b'\ (b'\text{ positive }<b),\\p\mathrm{B}&=ma-a'\ (a'\text{ positive }<a).\end{aligned}$$

Then (1) becomes

$$\frac{p}{ab}=l-m+\frac{a'}{a}+\frac{b'}{b} \qquad (2).$$

Now, since p/ab is a proper fraction, the integral part on the right-hand side of (2) must vanish; hence, since the integral part of $a'/a+b'/b$ cannot exceed 1 we must have $l-m=0$, or $l-m=-1$.

If the lower sign has to be taken in (1), we have merely to take the resolutions

$$pA = lb - b' \ (b' \text{ positive} < b),$$
$$pB = ma + a' \ (a' \text{ positive} < a),$$

and then proceed as before. We leave the proof that the resolution is unique to the ingenuity of the reader.

Illustration. $35 = 5 \times 7$.

Now $3 \times 5 - 2 \times 7 = 1$ (see Example 2 above);

whence

$$\frac{6}{35} = \frac{6}{35}(3 \times 5 - 2 \times 7),$$
$$= \frac{18}{7} - \frac{12}{5},$$
$$= \frac{2 \times 7 + 4}{7} - \frac{3 \times 5 - 3}{5},$$
$$= 2 + \frac{4}{7} - 3 + \frac{3}{5},$$
$$= \frac{3}{5} + \frac{4}{7} - 1.$$

N.B.—If negative numerators are allowed, it is obvious that p/ab can always be decomposed (sometimes in more ways than one) into an algebraic sum of two fractions a'/a and b'/b, where a' and b' are numerically less than a and b respectively. For example, we have $6/35 = 3/5 - 3/7 = 4/7 - 2/5$.

Example 5. If the n integers $a, b, c, d, \ldots$ be prime to each other, the proper fraction $p/abcd \ldots$ may be resolved in one way only into the form

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} + \frac{\delta}{d} + \ldots - k,$$

where $\alpha, \beta, \gamma, \delta, \ldots$ are all positive, $\alpha < a$, $\beta < b$, $\gamma < c$, $\delta < d$, ... and k has, according to circumstances, one or other of the integral values

$$0, 1, 2, \ldots, n-1.$$

(Gauss's *Disquisitiones Arithmeticæ*, Th. 310).

This may be established by means of Example 3.

Example 6. Work out the resolution of Example 5 for the fraction 10729/17017.

§ 15.] We conclude this chapter with a proposition which is as old as Euclid (ix. 20),* namely—

The number of prime integers is infinite.

For let $\alpha, \beta, \gamma, \ldots, \kappa$ be any series of prime integers whatsoever, then we can show that an infinity of primes can be derived from these.

In fact the integer $\alpha\beta\gamma \ldots \kappa + 1$ is obviously not exactly

* Most of the foregoing propositions regarding integral numbers were known to the old Greek geometers.

divisible by any one of the primes α, β, γ, . . ., κ. It must therefore either be itself a prime different from any one of the series α, β, γ, . . ., κ, or it must be a power of a prime or a composite integer divisible by some prime not occurring among α, β, γ, . . ., κ. We thus derive from α, β, γ, . . ., κ at least one more prime, say λ. Then from α, β, γ, . . ., κ, λ we can in like manner derive at least one more prime, μ; and so on *ad infinitum.**

Exercises IV.

(1.) If the two fractions A/B, a/b be equal, and the latter be at its lowest terms, prove that $A=\mu a$, $B=\mu b$, where μ is an integer.

(2.) Prove that the sum or difference of two odd numbers is always even; the sum or difference of an odd and an even number always odd; the product of any number of odd numbers always odd; the quotient of one odd number by another always odd, if it be integral.

(3.) If a be prime to b, then—

1st. $(a+b)^m$ and $(a-b)^m$ have at most the G.C.M. 2^m;
2nd. a^m+b^m and a^m-b^m have at most the G.C.M. 2;
3rd. $a+b$ and a^2+b^2-ab have at most the G.C.M. 3.

(4.) The difference of the squares of any two odd numbers is exactly divisible by 8.

(5.) The sum of the squares of three consecutive odd numbers increased by 1 is a multiple of 12.

(6.) If each of two fractions be at its lowest terms, neither their sum nor their difference can be an integer unless the denominators be equal.

(7.) Resolve 45738 and 297675 into their prime factors.

(8.) Find the G.C.M. of 54643 and 91319, using negative remainders whenever it is of advantage to do so.

(9.) Prove that the L.C.M. of two integers is the quotient of their product by the G.C.M.

(10.) If g_1, g_2, g_3 be the G.C.M.'s, l_1, l_2, l_3 the L.C.M.'s, of b and c, c and a, a and b respectively, G the G.C.M., and L the L.C.M., of the three a, b, c, show that

$$\text{1st. } L=\frac{abcG}{g_1g_2g_3};$$

$$\text{2nd. } \frac{L}{G}=\sqrt{\left(\frac{l_1l_2l_3}{g_1g_2g_3}\right)}.$$

(11.) When x is divided by y, the quotient is u and the remainder v; show that, when x and uy are divided by v, the remainders are the same, and the quotients differ by unity.

---

* On this subject see Sylvester, *Nature*, vol. xxxviii. (1888), p. 261.

CHAPTER IV.

Distribution of Products—Multiplication of Rational Integral Functions—Resulting General Principles.

GENERALISED LAW OF DISTRIBUTION.

§ 1.] We proceed now to develop some of the more important consequences of the law of distribution. This law has already been stated in the most general manner for the case of two factors, each of which is the sum of a series of terms: namely, we multiply every term of the one factor by every term of the other, and set down all the partial products thus obtained each with the sign before it which results from a certain law of signs.

Let us now consider the case of three factors, say

$$(a+b+c+\ .\ .\ .)\,(a'+b'+c'+\ .\ .\ .)\,(a''+b''+c''+\ .\ .\ .).$$

First of all, we may replace the first two factors by the process just described, namely, we may write

$$(aa'+ab'+ac'+\ .\ .\ .+ba'+bb'+bc'+\ .\ .\ .)\,(a''+b''+c''+\ .\ .\ .).$$

Then we may repeat the process, and write

$$\begin{array}{l} \ aa'a''+aa'b''+aa'c''+\ .\ .\ . \\ +\ ab'a''+ab'b''+ab'c''+\ .\ .\ . \\ +\ ac'a''+ac'b''+ac'c''+\ .\ .\ . \\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\ +\ ba'a''+ba'b''+ba'c''+\ .\ .\ .\ \&c., \end{array}$$

where the original product is finally replaced by a sum of partial products, each of three letters. We have simplified the matter by writing + before every term in the original factors, but the proper application of the law of signs at each step will present no difficulty to the student.

The important thing to remark is that we might evidently have arrived at the final result by the following process, which is really an extension of the original rule for two factors:—

Form all possible partial products by taking a term from each factor (never more than one from each); determine the sign by the law of signs (that is, if there be an odd number of negative terms in the partial product, take the sign $-$; if an even number of such or none, take the sign $+$). Set down all the partial products thus obtained.

Cor. *The number of terms resulting from the distribution of a product of brackets which contain l, m, n, . . . terms respectively is* $l \times m \times n \times$ . . . For, taking the first two brackets alone, since each term of the first goes with each term of the second, the whole number of terms arising from the distribution of these is $l \times m$. Next, multiplying by the third bracket, each of the $l \times m$ terms already obtained must be taken with each of the n terms of the third. We thus get $(l \times m) \times n$, that is, $l \times m \times n$ terms. By proceeding in this way we establish the general result.

It should be noted, however, that all the terms are supposed to be unlike, and that no condensation or reduction, owing to like terms occurring more than once, or to terms destroying each other, is supposed to be made. Cases occur in § 2 below in which the number of terms is reduced in this way.

If the student have the least difficulty in following the above, he will quickly get over it by working out for himself the results stated below, first by successive distribution, and then by applying the law just given.

$$(a+b)(c+d)(e+f)$$
$$=ace+acf+ade+adf+bce+bcf+bde+bdf$$
$$(2 \times 2 \times 2 = 8 \text{ terms}),$$
$$(a-b)(c-d)(e-f)$$
$$=ace-acf-ade+adf-bce+bcf+bde-bdf;$$
$$(a-b)(c-d)(e+f+g)$$
$$=ace+acf+acg-ade-adf-adg-bce-bcf-bcg+bde+bdf+bdg$$
$$(2 \times 2 \times 3 = 12 \text{ terms}).$$

§ 2.] It was proved above that in the most general case of distribution the number of resulting terms is the product of the numbers of terms in the different factors of the product. An examination of the particular cases where reductions may be

afterwards effected will lead us to some important practical results, and will also bring to notice certain important principles.

Consider the product $(a+b)(a+b)$. By the general rule the distribution will give $2 \times 2 = 4$ terms. We observe, however, that only two letters, a and b, occur in the product, and that only three really distinct products of two factors, namely, $a \times a$, $a \times b$, $b \times b$, that is, a^2, ab, b^2, can be formed with these; hence among the four terms one at least must occur more than once. In fact, the term $a \times b$ (or $b \times a$) occurs twice, and the result of the distribution is, after collection,

$$(a+b)(a+b) = a^2 + 2ab + b^2.$$

This may of course be written

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (1).$$

Similarly

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad (2).$$

In the case

$$(a+b)(a-b) = a^2 - b^2 \qquad (3),$$

the term ab occurs twice, once with the $+$ and again with the $-$ sign, so that these two terms destroy each other when the final result is reduced.

Before proceeding to another example, let us write down all the possible products of three factors that can be made with two letters, a and b. These are a^3, a^2b, ab^2, b^3, four in all.

Hence in the distribution of $(a+b)^3$, that is, of $(a+b)(a+b)(a+b)$, which by the general rule would give $2 \times 2 \times 2 = 8$ terms, only four really distinct terms can occur. Let us see what terms recur, and how often they do so. a^3 and b^3 evidently occur each only once, because to get three a's, or three b's, one must be taken from each bracket, and this can be done in one way only. a^2b may be got by taking b from the first bracket and a from each of the others, or by taking the b from the second, or from the third, in all three ways; and the same holds for ab^2. Thus the result is

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (4).$$

In a similar way the student may establish for himself that

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \qquad (5),$$

$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4 \qquad (6).$$

and, remembering that the possible binary products of three letters, a, b, c, are a^2, b^2, c^2, bc, ca, ab, six in number, that—

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab \qquad (7),$$
$$(a+b-c)^2 = a^2 + b^2 + c^2 - 2bc - 2ca + 2ab \qquad (8),$$
&c.

The ternary products of three letters, a, b, c, are a^3, a^2b, a^2c, ab^2, ac^2, abc, b^3, b^2c, bc^2, c^3. The enumeration is made more certain and systematic by first taking those in which a occurs thrice, then those in which it occurs twice, then those in which it occurs once, and, lastly, those in which it does not occur at all.*

Bearing this in mind, the student, by following the method we are illustrating, will easily show that

$$\begin{aligned}(a+b+c)^3 &= (a+b+c)(a+b+c)(a+b+c),\\ &= a^3 + b^3 + c^3 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2\\ &\qquad + 3a^2b + 3ab^2 + 6abc \qquad (9),\end{aligned}$$

from which again he may derive, by substituting (see chap. i., § 24) $-c$ for c on both sides, the expansion of $(a+b-c)^3$, and so on. He should not neglect to verify these results by successive distributions, thus:—

$$\begin{aligned}(a+b+c)^3 &= (a+b+c)^2(a+b+c)\\ &= (a^2+b^2+c^2+2bc+2ca+2ab)(a+b+c)\\ &= a^3 + ab^2 + ac^2 + 2abc + 2ca^2 + 2a^2b\\ &\quad + a^2b + b^3 + bc^2 + 2b^2c + 2abc + 2ab^2\\ &\quad + ca^2 + b^2c + c^3 + 2bc^2 + 2c^2a + 2abc\\ &= \text{\&c.}\end{aligned}$$

It is by such means that he must convince himself of the coherency of algebraical processes, and gain for himself taste and skill in the choice of his methods.

* There is another way of classifying the products of a given degree which is even more important and which the student should notice, namely, according to *type*. All the terms that can be derived from one another by interchanges among the variables are said to be of the same type. For example, consider the ternary products of a, b, c. From a^3 we derive, by interchange of b and a, b^3; from this again, by interchange of b and c, c^3: no more can be got in this way, so that a^3, b^3, c^3 form one ternary type; b^2c, bc^2, c^2a, ca^2, a^2b, ab^2, form another ternary type; and abc a third. Thus the ternary products of three variables fall into three types.

Let us consider one more case, namely, $(b+c)(c+a)(a+b)$. Here even all the ten permissible ternary products of a, b, c cannot occur, for a^3, b^3, c^3 are excluded by the nature of the case, since a occurs in only two of the brackets, and the same is true of b and c. In fact, by the process of enumeration and counting of recurrences, we get

$$(b+c)(c+a)(a+b) = bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 2abc \quad (10).$$

In the product $(b-c)(c-a)(a-b)$ the term abc occurs twice with opposite signs, and there is a further reduction, namely,

$$(b-c)(c-a)(a-b) = bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b \quad (11).$$

Σ *Notation.*—Instead of writing out at length the sum of all the terms of the same type, say $bc+ca+ab$, the abbreviation Σbc is often used; that is to say, we write only one of the terms in question, and prefix the Greek letter Σ, which stands for "sum," or, more fully, "sum of all terms of the same type as." The exact meaning of Σ depends on the number of variables that are in question. For example, if there be only two variables, a and b, then Σab means simply ab; if there be four variables, a, b, c, d, then Σab means $ab+ac+ad+bc+bd+cd$. Again, if there be two variables, a, b, Σa^2b means a^2b+ab^2; if there be three, a, b, c, Σa^2b means $a^2b+ab^2+a^2c+ac^2+b^2c+bc^2$. Usually the context shows how many variables are understood; but, if this is not so, it may be indicated either by writing the variables under the Σ, thus $\sum\limits_{abcd} ab$, or otherwise.

This notation is much used in the higher mathematics, and will be found very useful in saving labour even in elementary work. For example, the results (4), (9), and (10) above may be written—

$$(a+b)^3 = \Sigma a^3 + 3\Sigma a^2b;$$
$$(a+b+c)^3 = \Sigma a^3 + 3\Sigma a^2b + 6abc;$$
$$(b+c)(c+a)(a+b) = \Sigma a^2b + 2abc.$$

By means of the ideas explained in the present article the reader should find no difficulty in establishing the following, which are generalisations of (1) and (9):—

$$(a+b+c+d+\ \ldots\ )^2 = \Sigma a^2 + 2\Sigma ab \quad (12),$$
$$(a+b+c+d+\ \ldots\ )^3 = \Sigma a^3 + 3\Sigma a^2b + 6\Sigma abc \quad (13),$$

the number of variables being any whatever.

Π *Notation.*—There is another abbreviative notation, closely allied to the one we have just been explaining, which is sometimes useful, and which often appears in Continental works. If we have a product of terms or functions of a given set of variables, which are all different, but of the same type (that is, derivable from each other by interchanges, see p. 52), this is contracted by writing only one of the terms or functions, and prefixing the Greek letter Π, which stands for "product of all of the same type as." Thus, in the case of three variables, a, b, c,

$$\Pi a^2b \text{ means } a^2b \times ab^2 \times a^2c \times ac^2 \times b^2c \times bc^2\,;$$
$$\Pi(b+c) \text{ means } (b+c)\,(c+a)\,(a+b)\,;$$
$$\Pi\left(\frac{b+c}{b^2+c^2}\right) \text{ means } \left(\frac{b+c}{b^2+c^2}\right)\left(\frac{c+a}{c^2+a^2}\right)\left(\frac{a+b}{a^2+b^2}\right);$$

and so on.

We might, for example, write (10) above—

$$\Pi(b+c) = \Sigma b^2c + 2abc.$$

§ 3.] Hitherto we have considered merely factors made up of letters preceded by the signs + and −. The case where they are affected by numerical coefficients is of course at once provided for by the principle of association. Or, what comes to the same thing, cases in which numerical coefficients occur can be derived by substitution from such as we have already considered. For example—

$$\begin{aligned}(3a+2b)^3 &= \{(3a)+(2b)\}^3\\ &= (3a)^3 + 3(3a)^2(2b) + 3(3a)(2b)^2 + (2b)^3,\end{aligned}$$

whence, by rules already established for monomials,

$$= 27a^3 + 54a^2b + 36ab^2 + 8b^3.$$

$$\begin{aligned}(a-2b+5c)^2 &= \{(a)+(-2b)+(5c)\}^2\\ &= (a)^2 + (-2b)^2 + (5c)^2 + 2(-2b)(5c) + 2(5c)(a) + 2(a)(-2b)\\ &= a^2 + 4b^2 + 25c^2 - 20bc + 10ca - 4ab.\end{aligned}$$

The student will observe that in the final result the general form by means of which this result was obtained has been lost, so far at least as the numerical coefficients are concerned.

§ 4.] It is very important to notice that the principle of substitution may also be used to deduce results for trinomials from results already obtained for binomials. Thus from $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, replacing b throughout by $b+c$, we have

$$\begin{aligned}\{a+(b+c)\}^3 &= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3\\ &= a^3 + 3a^2b + 3a^2c\\ &\quad + 3a(b^2+2bc+c^2)\\ &\quad + b^3 + 3b^2c + 3bc^2 + c^3\,;\end{aligned}$$

whence
$$\begin{aligned}(a+b+c)^3 &= a^3 + b^3 + c^3 + 3b^2c + 3bc^2 + 3c^2a + 3ca^2\\ &\quad + 3a^2b + 3ab^2 + 6abc.\end{aligned}$$

By association of parts of the factors, and by *partial* distribution in the earlier parts of a reduction, labour may often be saved and elegance attained.

For example—

$$
\begin{aligned}
&(a+b+c-d)(a-b+c+d)\\
&\qquad=\{(a+c)+(b-d)\}\{(a+c)-(b-d)\};\\
&\qquad=(a+c)^2-(b-d)^2,\\
&\qquad\qquad\qquad\qquad\text{by formula (3) above};\\
&\qquad=(a^2+2ac+c^2)-(b^2-2bd+d^2);\\
&\qquad=a^2-b^2+c^2-d^2+2ac+2bd.
\end{aligned}
$$

Again,

$$
\begin{aligned}
&(a+b+c)(b+c-a)(c+a-b)(a+b-c)\\
&\qquad=\{(b+c)+a\}\{(b+c)-a\}\{a-(b-c)\}\{a+(b-c)\};\\
&\qquad=\{(b+c)^2-a^2\}\{a^2-(b-c)^2\},\\
&\qquad\qquad\qquad\qquad\text{by a double application of formula (3)};\\
&\qquad=\{b^2+2bc+c^2-a^2\}\{a^2-b^2+2bc-c^2\};\\
&\qquad=\{2bc+(b^2+c^2-a^2)\}\{2bc-(b^2+c^2-a^2)\};\\
&\qquad=(2bc)^2-(b^2+c^2-a^2)^2,\\
&\qquad\qquad\qquad\qquad\text{by formula (3)};\\
&\qquad=4b^2c^2-(b^4+c^4+a^4+2b^2c^2-2c^2a^2-2a^2b^2);\\
&\qquad=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4,
\end{aligned}
$$

a result which the student will meet with again.

§ 5.] There is an important general theorem which follows so readily from the results established in §§ 1 and 2 that we may give it here. *If all the terms in all the factors of a product be simple letters unaccompanied by numerical coefficients and all affected with the positive sign, then the sum of the coefficients in the distributed value of the product will be* $l\times m\times n\times\ldots$, *where* $l, m, n, \ldots$ *are the numbers of the terms in the respective factors.*

This follows at once from the consideration that no terms can be lost since all are positive, and that the numerical coefficient of any term in the distribution is simply the number of times that that term occurs.

Thus in formulæ (4), (6), and (10) in § 2 above we have

$$
\begin{aligned}
1+3+3+1 &= 2\times2\times2,\\
1+4+6+4+1 &= 2\times2\times2\times2,\\
1+1+1+1+1+1+2 &= 2\times2\times2,\\
&\text{\&c.}
\end{aligned}
$$

In formulæ (8) and (11) of § 2, and in the formulæ of § 3, the theorem does not hold on account of the appearance of negative signs and numerical coefficients.

The following more general theorem, which includes the one just stated as a particular case, will, however, always apply:—

The algebraic sum of the coefficients in the expansion of any

product may be obtained from the product itself by replacing each of the variables by 1 throughout all the factors.

Thus, in the case of

$$(a+b-c)^2 = a^2+b^2+c^2-2bc-2ca+2ab,$$

we have $$(1+1-1)^2 = 1 = 1+1+1-2-2+2.$$

The general proof of the theorem consists merely in this—that any algebraical identity is established for all values of its variables: so that we may give each of the variables the value 1. When this is done, the expanded side reduces simply to the algebraic sum of its coefficients.

Exercises V.

(1.) How many terms are there in the distributed product $(a_1+a_2)(b_1+b_2+b_3)(c_1+c_2+c_3+c_4)(d_1+d_2+d_3+d_4+d_5)$?

Distribute, condense, and arrange the following:—

(2.) $(x+y)(x-y)(x^2-y^2)(x^2+y^2)^2$.

(3.) $(x^2+y^2)(x^2-y^2)(x^4+y^4)$.

(4.) $(x+y)^3(x-y)^3$.

(5.) $(x+2y)^4(x-2y)^4$.

(6.) $(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$.

(7.) $(x^2+x+1)^3$.

(8.) $(3a+2b-1)^3$.

(9.) $\left(x^2+x+1+\frac{1}{x}+\frac{1}{x^2}\right)^2$.

(10.) $(a+b+c)^4$, and $(a-b-c)^4$.

(11.) Write down all the quaternary products of the three letters x, y, z; point out how many different types they fall into, and how many products there are of each type.

(12.) Do the same thing for the ternary products of the four letters a, b, c, d.

(13.) Find the sum of the coefficients in the expansion of $(2a+3b+4c)^3$.

Distribute and condense the following, arranging terms of the same type together:—

(14.) $\left(\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}\right)\left(\frac{x}{b+c}+\frac{y}{c+a}+\frac{z}{a+b}\right)$.

(15.) $(x+y+z)^2-x(y+z-x)-y(z+x-y)-z(x+y-z)$.

(16.) $(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c)$.

(17.) $(b+c)(y+z)+(c+a)(z+x)+(a+b)(x+y)-(a+b+c)(x+y+z)$.

(18.) $\Sigma a(b+c-a)^2\Pi(b+c-a)$.*

---

* Wherever in this set of exercises the abbreviative symbols Σ and Π are used, it is understood that three letters only are involved. The student who finds difficulty with the latter part of this set of exercises, should postpone them until he has read the rest of this chapter.

Show that

(19.) $(x+y)^4=2(x^2+y^2)(x+y)^2-(x^2-y^2)^2$.

(20.)

$$a^4(x-b)^4-4a^3b(x-a)(x-b)^3+6a^2b^2(x-a)^2(x-b)^2-4ab^3(x-a)^3(x-b)+b^4(x-a)^4$$
$$=(a^4-4a^3b+6a^2b^2-4ab^3+b^4)x^4.$$

(21.) $(x^2-ay^2)(x'^2-ay'^2)=(xx'\pm ayy')^2-a(xy'\pm yx')^2$;

$$(x^2-ay^2)^3=(x^3+3axy^2)^2-a(3x^2y+ay^3)^2;$$

$$(x^2-By^2-Cz^2+BCu^2)(x'^2-By'^2-Cz'^2+BCu'^2)$$
$$=\{xx'+Byy'\pm C(zz'+Buu')\}^2-B\{xy'+x'y\pm C(uz'+u'z)\}^2$$
$$-C\{xz'-Byu'\pm(zx'-Buy')\}^2+BC\{yz'-xu'\pm(ux'-zy')\}^2.$$

Lagrange.

The theorems (21.) are of great importance in the theory of numbers; they show that the products and powers of numbers having a certain form are numbers of the same form. They are generalisations of the formulæ numbered V. in the table at the end of this chapter.

Distribute, condense, and arrange—

(22.) $\Sigma a\Sigma bc-\Pi(b+c)$.

(23.) $\Sigma a(\Sigma a^2+\Sigma bc)+\Sigma a\Sigma a^2-\Sigma(b+c)^3$.

(24.) $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$.

(25.) Distribute

$$\{(a+b)x^2-abxy+(a-b)y^2\}\{(a-b)x^2+abxy+(a+b)y^2\};$$

and arrange the result in the form

$$Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4.$$

Show that

(26.) $\{x^3-y^3+3xy(2x+y)\}^3+\{y^3-x^3+3xy(2y+x)\}^3$

$$=27xy(x+y)(x^2+xy+y^2)^3.$$

(27.) $\Sigma\{2(x^2+xy+y^2)(x^2+xz+z^2)-(y^2+yz+z^2)^2\}=3\{\Sigma yz\}^2$.

(28.) $\frac{1}{2}\{\Sigma a^2(b+c)^2+2abc\Sigma a\}=\{\Sigma bc\}^2$.

(29.) $\Sigma(a-b)(a-c)=\{\Sigma a^2-\Sigma bc\}$.

(30.) $\dfrac{(3abc-2b^3-a^2d)^2+4(ac-b^2)^3}{a^2}=\dfrac{(3dcb-2c^3-d^2a)^2+4(db-c^2)^3}{d^2}$.

GENERAL THEORY OF INTEGRAL FUNCTIONS.

§ 6.] As we have now made a beginning of the investigation of the properties of rational integral algebraical functions, it will be well to define precisely what is meant by this term.

We have already (chap. ii., § 5) defined a rational integral algebraical term as the product of a number of positive integral powers of various letters, x, y, z, . . ., called the variables, multiplied by a coefficient, which may be a positive or negative number, or a mere letter or function of a letter or letters, but must not contain or depend upon the variables.

A rational integral algebraical function is the algebraical sum of a series of rational integral algebraical terms. Thus, if $x, y, z, \ldots$ be the variables, $l, m, n, \ldots, l', m', n', \ldots, l'', m'', n'', \ldots$ positive integral numbers, and $C, C', C'', \ldots$ coefficients as above defined, then the type of such a function as we have defined is

$$Cx^l y^m z^n \ldots + C'x^{l'} y^{m'} z^{n'} \ldots + C''x^{l''} y^{m''} z^{n''} \ldots + \&c.$$

For shortness, we shall, when no ambiguity is to be feared, speak of it merely as an "integral function."

To fix the notion, we give a few special examples. Thus

(α) $3x^3+3xy+2y^2$ is an integral function of x and y;

(β) $ax^2+bxy+cy^2$, a, b, c being independent of x and y, is an integral function of x and y;

(γ) $3x^3-2x^2+3x+1$ is an integral function of x alone;

(δ) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1$ is an integral function, if x, y, z be regarded as the variables; but is not an *integral* function if the variables be taken to be x, y, z, a, b, c, or a, b, c alone.

Each term has a "degree," according to the definition of chap. ii., § 6, which is in fact the sum of the indices of the variables. The degrees of the various terms will not in general be alike; but *the degree of an integral function is defined to be the degree of the term of highest degree that occurs in it.*

For example, the degree of (α) above in x and y is the 3rd, of (β) the 2nd in x and y and the 1st in a, b, c, of (γ) in x the 3rd, of (δ) in x, y, z the 1st.

§ 7.] From what has already been shown in this chapter it appears that, in the result of the distribution of a product of any number of integral functions, each term arises as the product of a number of integral terms, and is therefore itself integral. Moreover, by chap. ii., § 7, the degree of each such term is the sum of the degrees of the terms from which it arises. Hence the following general propositions:—

The product of any number of integral functions is an integral function.

*The highest * term in the distributed product is the product of the*

* By "highest term" is meant term of highest degree, by "lowest term" term of lowest degree. If there be a term which does not contain the variables at all, its degree is said to be zero, and it of course would be the lowest term in an integral function, for example, $+1$ in (γ) above.

highest terms of the several factors, and the lowest term is the product of their lowest terms.

The degree of the product of a number of integral functions is the sum of the degrees of the several factors.

Every identity already given in this chapter, and all those that follow, will afford the student the means of verifying these propositions in particular cases. It is therefore needless to do more than call his attention to their importance. They form, it may be said, the corner-stones of the theory of algebraic forms.

INTEGRAL FUNCTIONS OF ONE VARIABLE.

§ 8.] The simplest case of an integral function is that where there is only one variable x. As this case is of great importance, we shall consider it at some length. The general type is

$$p_n x^n + p_{n-1}x^{n-1} + \ldots + p_1 x + p_0,$$

where $p_0, p_1, \ldots, p_n$ are the various coefficients and n is a positive integral number, which, being the index of the highest term, is the degree of the function. The function has in general $n+1$ terms, but of course some of these may be wanting, or, which amounts to the same thing, one or more of the letters $p_0, p_1, \ldots, p_n$ may have zero value.

§ 9.] When products of integral functions of one variable have to be distributed, it is usually required at the same time to arrange the result according to powers of x, as in the typical form above indicated. We proceed to give various instances of this process, using in the first place the method described in the earlier part of this chapter. The student should exercise himself by obtaining the same results by successive distribution or otherwise.

In the case of two factors $(x+a)(x+b)$, we see at once that the highest term is x^2, and the lowest ab. A term in x will be obtained in two ways, namely, ax and bx; hence

$$(x+a)(x+b) = x^2 + (a+b)x + ab \qquad (1).$$

This virtually includes all possible cases; for example, putting $-a$ for a we get

$$\begin{aligned}(x+(-a))(x+b) &= x^2 + ((-a)+b)x + (-a)b,\\ &= x^2 + (-a+b)x - ab.\end{aligned}$$

Similarly
$$\begin{aligned}(x-a)(x-b)&=x^2+(-a-b)x+ab,\\&=x^2-(a+b)x+ab.\\(x-a)(x-a)&=x^2+(-a-a)x+a^2,\\&=x^2-2ax+a^2,\ \text{\&c.}\end{aligned}$$

Cases in which numbers occur in place of a and b, or in which x is affected with coefficients in the two factors, may be deduced by specialisation or other modification of formula (1), for example,

$$\begin{aligned}(x-2)(x+3)&=x^2+(-2+3)x+(-2)(+3),\\&=x^2+x-6.\\(px+q)(rx+s)&=p\left(x+\frac{q}{p}\right)r\left(x+\frac{s}{r}\right),\\&=pr\left(x+\frac{q}{p}\right)\left(x+\frac{s}{r}\right),\\&=pr\left\{x^2+\left(\frac{q}{p}+\frac{s}{r}\right)x+\frac{q}{p}\frac{s}{r}\right\},\\&=prx^2+pr\left(\frac{q}{p}+\frac{s}{r}\right)x+pr\frac{qs}{pr},\\&=prx^2+(rq+ps)x+qs,\end{aligned}$$

which might of course be obtained more quickly by directly distributing the product and collecting the powers of x.

In the case of three factors of the first degree, say $(x+a_1)(x+a_2)(x+a_3)$, the highest term is x^3; terms in x^2 are obtained by taking for the partial products x from *two* of the three brackets only, then an a must be taken from the remaining bracket; we thus get a_1x^2, a_2x^2, a_3x^2; that is, $(a_1+a_2+a_3)x^2$ is the term in x^2. To get the term in x, x must be taken from one bracket, and a's from the two remaining in every possible way; this gives $(a_1a_2+a_1a_3+a_2a_3)x$. The last or absolute term is of course $a_1a_2a_3$. Thus $(x+a_1)(x+a_2)(x+a_3)$

$$=x^3+(a_1+a_2+a_3)x^2+(a_1a_2+a_1a_3+a_2a_3)x+a_1a_2a_3 \qquad (2).$$

By substitution all other cases may be derived from (2), for example,

$$\begin{aligned}&(x-a_1)(x-a_2)(x-a_3)\\&\quad=x^3-(a_1+a_2+a_3)x^2+(a_1a_2+a_1a_3+a_2a_3)x-a_1a_2a_3 \qquad (3);\\&(x+1)(x+2)(x-3)=x^3-7x-6,\ \text{and so on.}\end{aligned}$$

After what has been said it is easy to find the form of the distribution of a product of n factors of the first degree. The result is

$$\begin{aligned}&(x+a_1)(x+a_2)\ldots(x+a_n)\\&\quad=x^n+P_1x^{n-1}+P_2x^{n-2}+\ldots+P_{n-1}x+P_n \qquad (4),\end{aligned}$$

where P_1 signifies the algebraic sum of all the a's, P_2 the algebraic sum of all the products that can be formed by taking two of them at a time, P_3 the sum of all the products three at a time, and so on, P_n being the product of them all.

§ 10.] The formula (4) of § 9 of course includes (1) and (2) already given, and there is no difficulty in adapting it to special cases where negative signs, &c., occur. The following is particularly important:—

$$(x-a_1)(x-a_2)\dots(x-a_n)$$
$$=x^n-P_1x^{n-1}+P_2x^{n-2}-\dots+(-1)^{n-1}P_{n-1}x+(-1)^nP_n \quad (1).$$

Here P_1 P_2, &c., have a slightly different meaning from that attached to them in § 9 (4): P_3, for example, is not the sum of all the products of $-a_1, -a_2, \dots, -a_n$, taken three at a time, but the sum of the products of $+a_1, +a_2, \dots, +a_n$, taken three at a time; and the coefficient of x^{n-3} is therefore $-P_3$, since the concurrence of three negative signs gives a negative sign. As a special case of (1) let us take

$$(x-a)(x-2a)(x-3a)(x-4a)=x^4-P_1x^3+P_2x^2-P_3x+P_4.$$

Here $P_1=a+2a+3a+4a=10a$,

$$P_2=1\times 2a^2+1\times 3a^2+1\times 4a^2+2\times 3a^2+2\times 4a^2+3\times 4a^2$$
$$=35a^2,$$
$$P_3=2\times 3\times 4a^3+1\times 3\times 4a^3+1\times 2\times 4a^3+1\times 2\times 3a^3$$
$$=50a^3,$$
$$P_4=1\times 2\times 3\times 4a^4=24a^4.$$

So that
$$(x-a)(x-2a)(x-3a)(x-4a)$$
$$=x^4-10ax^3+35a^2x^2-50a^3x+24a^4.$$

§ 11.] Another important case of § 9 (4) is obtained by making $a_1=a_2=a_3=\dots=a_n$, each $=a$ say. The left-hand side then becomes $(x+a)^n$. Let us see what the values of $P_1, P_2, \dots, P_n$ become. P_1 obviously becomes na, and P_n becomes a^n. Consider any other, say P_r; the number of terms in it is the number of different sets of r things that we can choose out of n things. This number is, of course, independent of the nature of the things chosen; and, although we have no means as yet of calculating it, we may give it a name. The symbol generally in use

for it is ${}_nC_r$, the first suffix denoting the number of things chosen from, the second the number of things to be chosen. Again, each term of P_r consists of the product of r letters, and, since in the present case each of these is a, each term will be a^r. All the terms being equal, and there being ${}_nC_r$ of them, we have in the present case $P_r = {}_nC_r a^r$. Hence

$$(x+a)^n = x^n + nax^{n-1} + {}_nC_2a^2x^{n-2} + {}_nC_3a^3x^{n-3} + \ldots + a^n;$$

or, if we choose, since ${}_nC_1 = n$, ${}_nC_n = 1$, we may write

$$(x+a)^n = x^n + {}_nC_1ax^{n-1} + {}_nC_2a^2x^{n-2} + \ldots + {}_nC_{n-1}a^{n-1}x + {}_nC_na^n \quad (1).$$

This is the "binomial theorem" for positive integral exponents, and the numbers ${}_nC_1$, ${}_nC_2$, ${}_nC_3$, . . . *are called the binomial coefficients of the nth order.* They play an important part in algebra; in fact, the student has already seen that, besides their function in the binomial expansion, they answer a series of questions in the theory of combinations. When we come to treat that subject more particularly we shall investigate a direct expression for ${}_nC_r$ in terms of n and r. Later in this chapter we shall give a process for calculating the coefficients of the different orders by successive additions.

By substituting successively $-a$, $+1$, and -1 for a in (1) we get

$$(x-a)^n = x^n - {}_nC_1ax^{n-1} + {}_nC_2a^2x^{n-2} - {}_nC_3a^3x^{n-3} + \ldots + (-1)^n{}_nC_na^n \quad (2);$$

$$(x+1)^n = x^n + {}_nC_1x^{n-1} + {}_nC_2x^{n-2} + \ldots + {}_nC_n \quad (3);$$

$$(x-1)^n = x^n - {}_nC_1x^{n-1} + {}_nC_2x^{n-2} - \ldots + (-1)^n{}_nC_n \quad (4);$$

and an infinity of other results can of course be obtained by substituting various values for x and a.

§ 12.] In expanding and arranging products of two integral functions of one variable, the process which is sometimes called the *long rule for multiplication* is often convenient. It consists simply in taking one of the functions arranged according to descending powers of the variable and multiplying it successively by each of the terms of the other, beginning with the highest and proceeding to the lowest, arranging the like terms under one another. Thus we arrange the distribution of

as follows :—

$$(x^3 + 2x^2 + 2x + 1)(x^2 - x + 1)$$

$$\begin{array}{rrrrrr}
x^3 & + 2x^2 & + 2x & + 1 & & \\
x^2 & - x & + 1 & & & \\
\hline
x^5 + 2x^4 & + 2x^3 & + x^2 & & & \\
 - x^4 & - 2x^3 & - 2x^2 & - x & & \\
 & + x^3 & + 2x^2 & + 2x & + 1 & \\
\hline
x^5 + x^4 & + x^3 & + x^2 & + x & + 1 ,
\end{array}$$

or again

$$(px^2 + qx + r)(rx^2 + qx + p)$$

$$\begin{array}{lllll}
px^2 & + qx & + r & & \\
rx^2 & + qx & + p & & \\
\hline
prx^4 & + qrx^3 & + r^2x^2 & & \\
 & + pqx^3 & + q^2x^2 & + qrx & \\
 & & + p^2x^2 & + pqx & + pr \\
\hline
\end{array}$$

$$prx^4 + (pq + qr)x^3 + (p^2 + q^2 + r^2)x^2 + (pq + qr)x + pr.$$

The advantage of this scheme consists merely in the fact that like powers of x are placed in the same vertical column, and that there is an orderly exhaustion of the partial products, so that none are likely to be missed. It possesses none of the fundamental importance which might be suggested by its prominent position in English elementary text-books.

§ 13.] *Method of Detached Coefficients.*—When all the powers are present a good deal of labour may be saved by merely writing the coefficients in the scheme of § 12, which are to be multiplied together in the ordinary way. The powers of x can be inserted at the end of the operation, for we know that the highest power in the product is the product of the highest powers in the two factors, and the rest follow in order. Thus we may arrange the two multiplications given above as follows :—

$$\begin{array}{rrrrrr}
1 & + 2 & + 2 & + 1 & & \\
1 & - 1 & + 1 & & & \\
\hline
1 & + 2 & + 2 & + 1 & & \\
 & - 1 & - 2 & - 2 & - 1 & \\
 & & + 1 & + 2 & + 2 & + 1 \\
\hline
1 & + 1 & + 1 & + 1 & + 1 & + 1 ;
\end{array}$$

whence

$$(x^3+2x^2+2x+1)(x^2-x+1)=x^5+x^4+x^3+x^2+x+1.$$

Again,

$$\begin{array}{llll}
p & +q & +r & \\
r & +q & +p & \\
\hline
pr & +qr & +r^2 & \\
 & +pq & +q^2 & +qr \\
 & & +p^2 & +pq+pr \\
\hline
\multicolumn{4}{l}{pr+(qr+pq)+(p^2+q^2+r^2)+(pq+qr)+pr\,;}
\end{array}$$

whence

$$(px^2+qx+r)(rx^2+qx+p)$$
$$=prx^4+(pq+qr)x^3+(p^2+q^2+r^2)x^2+(pq+qr)x+pr.$$

The student should observe that the use of brackets in the last line of the scheme in the second example is necessary to preserve the identity of the several coefficients.

It has been said that this method is applicable directly only when all the powers are present in both factors, but it can be made applicable to cases where any powers of x are wanting by introducing these powers multiplied by zero coefficients. For example—

$$(x^4-2x^2+1)(x^4+2x^2+1)$$
$$=(x^4+0x^3-2x^2+0x+1)(x^4+0x^3+2x^2+0x+1)\,;$$

$$\begin{array}{l}
1+0-2+0+1 \\
1+0+2+0+1 \\
\hline
1+0-2+0+1 \\
\quad +0+0+0+0+0^* \\
\qquad +2+0-4+0+2 \\
\qquad\quad +0+0+0+0+0^* \\
\qquad\qquad +1+0-2+0+1 \\
\hline
1+0+0+0-2+0+0+0+1 \\
x^8+0x^7+0x^6+0x^5-2x^4+0x^3+0x^2+0x+1\,;
\end{array}$$

whence

$$(x^4-2x^2+1)(x^4+2x^2+1)=x^8-2x^4+1.$$

The process might, of course, be abbreviated by omitting the lines marked *, which contain only zeros, care being taken to place the commencement of the following lines in the proper columns; and, in writing out the result, the terms with zero coefficients might be omitted at once. With all these simplifications, the process in the present case is still inferior in brevity to the following, which depends on the use of the identities $(A+B)(A-B)=A^2-B^2$, and $(A+B)^2=A^2+2AB+B^2$.

$$\begin{aligned}(x^4-2x^2+1)(x^4+2x^2+1) &= \{(x^4+1)-2x^2\}\{(x^4+1)+2x^2\}\\ &= (x^4+1)^2-(2x^2)^2\\ &= x^8+2x^4+1-4x^4\\ &= x^8-2x^4+1.\end{aligned}$$

The method of detached coefficients can be applied with advantage in the case of integral functions of two letters which are homogeneous (see below, § 17), as will be seen by the following example:—

$$(x^2-xy+y^2)(x^3-2x^2y+2xy^2-y^3),$$

$$\begin{array}{rrrrrr}
1 & -2 & +2 & -1 & & \\
1 & -1 & +1 & & & \\
\hline
1 & -2 & +2 & -1 & & \\
 & -1 & +2 & -2 & +1 & \\
 & & +1 & -2 & +2 & -1 \\
\hline
1 & -3 & +5 & -5 & +3 & -1,
\end{array}$$

$$=x^5-3x^4y+5x^3y^2-5x^2y^3+3xy^4-y^5.$$

If the student will work out the above distribution, arrange his work after the pattern of the long rule, and then compare, he will at once see that the above scheme represents all the essential detail required for calculating the coefficients.

The reason of the applicability of the process is simply that the powers of x diminish by unity from left to right, and the powers of y in like manner from right to left.

We shall give some further examples of the method of detached coefficients, by using it to establish several important results.

§ 14.] *Addition Rule for calculating the Binomial Coefficients.*

We have to expand $(x+1)^2$, $(x+1)^3$, . . ., $(x+1)^n$. Let us proceed by successive distribution, using detached coefficients.

$$
\begin{array}{ll}
1+1 & \text{(The coefficients of } x+1\text{)},\\
1+1 & \\
\hline
1+1 & \\
\quad +1+1 & \\
\hline
1+2+1 & \text{(The coefficients of } (x+1)^2\text{)},\\
1+1 & \\
\hline
1+2+1 & \\
\quad +1+2+1 & \\
\hline
1+3+3+1 & \text{(The coefficients of } (x+1)^3\text{)}.
\end{array}
$$

The rule which here becomes apparent is as follows:—

To obtain the binomial coefficients of any order from those of the previous order—1st, Write down the first coefficient of the previous order; 2nd, Add the second of the previous order to the first of the same; 3rd, Add the third of the previous order to the second of the same; and so on, taking zeros when the coefficients of the previous order run out. We thus get in succession the first, second, third, &c., coefficients of the new order. For example, those of the fourth order are

$$1 + (1+3) + (3+3) + (3+1) + (1+0),$$

that is,
$$1 + 4 \qquad + 6 \qquad + 4 \qquad + 1,$$

which agrees with the result obtained by a different method above, § 2 (6).

We have only to show that this process is general. Suppose we had obtained the expansion of $(x+1)^n$, namely, using the notation of § 11,

$$(x+1)^n = x^n + {}_nC_1x^{n-1} + {}_nC_2x^{n-2} + {}_nC_3x^{n-3} + \ldots + {}_nC_{n-1}x + {}_nC_n.$$

Hence

$$
\begin{aligned}
(x+1)^{n+1} &= (x+1)^n \times (x+1)\\
&= (x^n + {}_nC_1x^{n-1} + {}_nC_2x^{n-2} + \ldots + {}_nC_{n-1}x + {}_nC_n)(x+1);
\end{aligned}
$$

using detached coefficients, we have the scheme

$$
\begin{array}{llllll}
1 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \ldots + {}_nC_{n-1} + {}_nC_n \\
1 + 1 \\
\hline
1 & + {}_nC_1 & + {}_nC_2 \ldots\ldots\ldots\ldots & + {}_nC_n & \\
 & + 1 & + {}_nC_1 \ldots\ldots\ldots\ldots & + {}_nC_{n-1} & + {}_nC_n \\
\hline
\end{array}
$$

$$1 + (1 + {}_nC_1) + ({}_nC_1 + {}_nC_2) + \ldots . + ({}_nC_{n-1} + {}_nC_n) + ({}_nC_n + 0).$$

Hence $$(x + 1)^{n+1}$$

$$= x^{n+1} + (1 + {}_nC_1)x^n + ({}_nC_1 + {}_nC_2)x^{n-1} + ({}_nC_2 + {}_nC_3)x^{n-2} + \ldots,$$

in which the coefficients are formed from the coefficients of the nth order, precisely after the law stated above, namely,

$${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}.$$

This law is therefore general, and enables us whenever we know the binomial coefficients of any rank to calculate those of the next, from these again those of the next, and so on. A table of these numbers (often called Pascal's Triangle) carried to a considerable extent is given at the end of this chapter, among the results and formulæ collected for reference there.

§ 15.] We may calculate the powers of $x^3 + x^2 + x + 1$ by means of the following scheme, in which the lines of coefficients of the constantly-recurring multiplier, namely, $1 + 1 + 1 + 1$, are for brevity omitted.

Power.

$$
\begin{array}{lccccccccccccc}
\text{1st.} & 1 & +1 & +1 & +1 & & & & & & & & & \\
 & & +1 & +1 & +1 & +1 & & & & & & & & \\
 & & & +1 & +1 & +1 & +1 & & & & & & & \\
 & & & & +1 & +1 & +1 & +1 & & & & & & \\
\hline
\text{2nd.} & 1 & +2 & +3 & +4 & +3 & +2 & +1 & & & & & & \\
 & & +1 & +2 & +3 & +4 & +3 & +2 & +1 & & & & & \\
 & & & +1 & +2 & +3 & +4 & +3 & +2 & +1 & & & & \\
 & & & & +1 & +2 & +3 & +4 & +3 & +2 & +1 & & & \\
\hline
\text{3rd.} & 1 & +3 & +6 & +10 & +12 & +12 & +10 & +6 & +3 & +1 & & & \\
 & & +1 & +3 & +6 & +10 & +12 & +12 & +10 & +6 & +3 & +1 & & \\
 & & & +1 & +3 & +6 & +10 & +12 & +12 & +10 & +6 & +3 & +1 & \\
 & & & & +1 & +3 & +6 & +10 & +12 & +12 & +10 & +6 & +3 & +1 \\
\hline
\text{4th.} & 1 & +4 & +10 & +20 & +31 & +40 & +44 & +40 & +31 & +20 & +10 & +4 & +1
\end{array}
$$

and so on.

The rule clearly is—*To get from the coefficients of any order the rth of the succeeding, add to the rth of that order the three preceding coefficients, taking zeros when the coefficients required by the rule do not exist.*

The rule for calculating the coefficients of the powers of $x^n + x^{n-1} + x^{n-2} + \ldots + x + 1$ is obtained from the above by putting n in place of 3.

These results may be regarded as a generalisation of the process of tabulating the binomial coefficients. They are useful in the Theory of Probability.

§ 16.] As the student will easily verify, we have

$$(x-y)(x^2+xy+y^2) = x^3 - y^3 \qquad (1),$$
$$(x+y)(x^2-xy+y^2) = x^3 + y^3 \qquad (2).$$

The following is a generalisation of the first of these:—

If n be any integer,

$$(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots+xy^{n-2}+y^{n-1}),$$

$$\begin{array}{l}
1+1+1+\ldots+1+1\\
1-1\\
\hline
1+1+1+\ldots+1+1\\
\quad\;\, -1-1-\ldots-1-1-1\\
\hline
1+0+0+\ldots+0+0-1
\end{array}$$
$$= x^n - y^n \qquad (3).$$

Again, n being an odd number,

$$(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ldots-xy^{n-2}+y^{n-1}),$$

(− sign going with odd powers of y)

$$\begin{array}{l}
1-1+1-\ldots-1+1\\
1+1\\
\hline
1-1+1-\ldots-1+1\\
\quad\;\, +1-1+\ldots+1-1+1\\
\hline
1+0+0+\ldots+0+0+1
\end{array}$$
$$= x^n + y^n \qquad (4).$$

And, similarly, n being an even number,

$$(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ .\ .\ .\ +xy^{n-2}-y^{n-1})$$
$$=x^n-y^n \qquad (5).$$

The last two may be considered as generalisations of (2) and of $(x+y)(x-y)=x^2-y^2$ respectively.

Exercises VI.

(1.) The variables being x, y, z, point out the integral functions among the following, and state their degree :—

(α) $3x^2+2xy+3y^2$;

(β) $\dfrac{3}{x^2}+\dfrac{2}{xy}+\dfrac{3}{y^2}$;

(γ) $x^2yz+y^2zx+z^2xy+x^3+y^3+z^3$;

(δ) $\dfrac{x^2y^2z^2}{x^2yz}+\dfrac{x^2y^2z^2}{xy^2z}+\dfrac{x^2y^2z^2}{xyz^2}$.

Distribute the following, and arrange according to powers of x :—

(2.) $6\{x-\frac{1}{2}(x-1)\}\{x-\frac{2}{3}(x-1)\}+20\{x-\frac{3}{4}(x-1)\}\{x-\frac{4}{5}(x-1)\}$.

(3.) $\dfrac{x(x+1)(x+3)}{3}-\dfrac{x(x+1)(2x+1)}{6}$.

(4.) $\{(x-2)(x-3)+(x-3)(x-1)+(x-1)(x-2)\}$
$\times\{(x+2)(x+3)+(x+3)(x+1)+(x+1)(x+2)\}$.

(5.) $\{x+a\}\{x^2+(b+c)x+bc\}\{x^3-(a+b+c)x^2+(bc+ca+ab)x-abc\}$.

(6.) $\{(x+p)(x-q)(x+1)\}\{(x-p)(x+q)(x-1)\}$.

(7.) $(x^2-y^2)(x^2-2y^2)(x^2-3y^2)(x^2-4y^2)(x^2-5y^2)$.

(8.) $\{ax+(b-c)y\}\{bx+(c-a)y\}\{cx+(a-b)y\}$;* and show that the sum of the coefficients of x^2y and y^3 is zero.

(9.) Show that

$$(x+\tfrac{5}{2}a)^4-10a(x+\tfrac{5}{2}a)^3+35a^2(x+\tfrac{5}{2}a)^2-50a^3(x+\tfrac{5}{2}a)+24a^4$$
$$=(x^2-\tfrac{1}{4}a^2)(x^2-\tfrac{9}{4}a^2).$$

(10.) Show that

$$\left(x+\frac{q}{r}y\right)\left(x+\frac{r}{p}y\right)\left(x+\frac{p}{q}y\right)-\left(x+\frac{r}{q}y\right)\left(x+\frac{p}{r}y\right)\left(x+\frac{q}{p}y\right)$$
$$=\frac{xy(x-y)(q-r)(r-p)(p-q)}{pqr}.$$

Distribute and arrange according to powers of x, the following :—

(11.) $\{(b+c)x^2+(c+a)x+(a+b)\}\{(b-c)x^2+(c-a)x+(a-b)\}$.

(12.) $(x^2-x+1)(x^2+x+1)(x^2-2x+1)(x^2+2x+1)$.

---

* In working some of these exercises the student will find it convenient to refer to the table of identities given at the end of this chapter.

(13.) $\{5x^2 - 4x(x-y) + (x-y)^2\}(2x+3y)$.
(14.) $(2x^2 - 3xy + 2y^2)(2x^2 + 3xy + 2y^2)$.
(15.) $\{(x^2+x+1)(x^2-x+1)(x^2-1)\}^2$.
(16.) $(x^3 - x^2 + x - 1)^2(x^3 + x^2 + x + 1)^2$.
(17.) $(\frac{3}{4}x^3 - \frac{1}{2}x^2 + \frac{1}{8}x + \frac{1}{9})(\frac{3}{4}x^3 + \frac{1}{2}x^2 - \frac{1}{8}x + \frac{1}{9})$.
(18.) $(x^4 - ax^3y + abx^2y^2 + bxy^3 + y^4)(ax^2 - abxy + by^2)$.
(19.) $(x^2+ax+b^2)^3 + (x^2+ax-b^2)^3 + (x^2-ax+b^2)^3 + (x^2-ax-b^2)^3$.
(20.) $(x^4 - 2a^2x^2 + a^4)^6$.
(21.) $(x^4 - a^5)^3$.
(22.) $(3x + \frac{1}{3})^7$.
(23.) $(a + bx^2)^8$.
(24.) $\{(x^6+y^3)(x^6-y^3)\}^9$.
(25.) $(1 + x + x^2 + x^3 + x^4)^3$.

(26.) Calculate the coefficient of x^4 in the expansion of $(1+x+x^2)^8$.
(27.) Calculate the coefficient of x^8 in $(1 - 2x + 3x^2 + 4x^3 - x^4)^5$.
(28.) Show that

$$(a+b)^3(a^5+b^5) + 5ab(a+b)^2(a^4+b^4) + 15a^2b^2(a+b)(a^3+b^3) + 35a^3b^3(a^2+b^2) + 70a^4b^4 = (a+b)^8.$$

(29.) Show that

$$ {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n = 2^n - 1\ ;$$
$$1 + {}_nC_2 + {}_nC_4 + \dots = {}_nC_1 + {}_nC_3 + {}_nC_5 + \dots\ ;$$
$${}_nC_r = {}_{n-2}C_r + 2\,{}_{n-2}C_{r-1} + {}_{n-2}C_{r-2}.$$

(30.) There are five boxes each containing five counters marked with the numbers 0, 1, 2, 3, 4; a counter is drawn from each of the boxes and the numbers drawn are added together. In how many different ways can the drawing be made so that the sum of the numbers shall be 8?

(31.) Show that

$$(x-y)^2(x^{n-2} + x^{n-3}y + \dots + xy^{n-3} + y^{n-2}) = x^n - x^{n-1}y - xy^{n-1} + y^n.$$

Exercises VII.

Distribute the following, and arrange according to descending powers of x:—

(1.) $(3x+4)(4x+5)(5x+6)(6x+7)$.
(2.) $(px+q-r)(qx+r-p)(rx+p-q)$.
(3.) $(x-a)(x-2a)(x-3a)(x-4a)(x+a)(x+2a)(x+3a)(x+4a)$.
(4.) $(x^3 + 3x^2 + 3x + 1)(x^3 - 3x^2 + 3x - 1)$.
(5.) $(\frac{1}{3}x^3 + \frac{1}{5}x^2 + \frac{5}{3}x + \frac{2}{3})(\frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + 1)$.
(6.) $(x - \frac{1}{2})(x^2 - \frac{1}{2}x + \frac{1}{4})(x + \frac{1}{2})(x^2 + \frac{1}{2}x + \frac{1}{4})$.
(7.) $\left(\frac{l}{m}x^2 + \frac{m}{n}x + \frac{n}{l}\right)\left(\frac{m}{n}x^2 + \frac{n}{l}x + \frac{l}{m}\right)\left(\frac{n}{l}x^2 + \frac{l}{m}x + \frac{m}{n}\right)$.
(8.) $(2x-3)^{11}$.
(9.) $\{(x+y)(x^2-xy+y^2)\}^8$.
(10.) $(x^2-1)^4(x+1)^{10}$.

(11.) In the product $(x+a)(x+b)(x+c)$, x^2 disappears, and in the product $(x-a)(x+b)(x+c)$, x disappears; also the coefficient of x in the former is equal to the coefficient of x^2 in the latter. Show that a is either 0 or 1.

Prove the following identities:—

(12.) $(b-c)(x-a)^2+(c-a)(x-b)^2+(a-b)(x-c)^2+(b-c)(c-a)(a-b)=0.$

(13.) $\Sigma(2a-b-c)(h-b)(h-c)=\Sigma(b-c)^2(h-a).$

(14.) $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3,$

where $$2s=a+b+c.$$

(15.) $$(s-a)^4+(s-b)^4+(s-c)^4$$
$$=2(s-b)^2(s-c)^2+2(s-c)^2(s-a)^2+2(s-a)^2(s-b)^2,$$

where $$3s=a+b+c.$$

(16.) $(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2$, where $s=a+b+c$.

(17.) $s(s-a-d)(s-d-b)(s-c-d)=(s-a)(s-b)(s-c)(s-d)-abcd,$

where $$2s=a+b+c+d.$$

(18.) $16(s-a)(s-b)(s-c)(s-d)=4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2,$

where $$2s=a+b+c+d.$$

(19.) $\Sigma(b-c)^6=3\Pi(b-c)^2+2(\Sigma a^2-\Sigma bc)^3.$

(20.) If $U_n=(b-c)^n+(c-a)^n+(a-b)^n$, then
$$U_{n+3}-(a^2+b^2+c^2-bc-ca-ab)U_{n+1}-(b-c)(c-a)(a-b)U_n=0.$$

(21.) If $p_1=a+b+c$, $p_2=bc+ca+ab$, $p_3=abc$, $s_n=a^n+b^n+c^n$, show that
$$s_1=p_1,\ s_2=p_1s_1-2p_2,\ s_3=p_1s_2-p_2s_1+3p_3,$$
$$s_n=p_1s_{n-1}-p_2s_{n-2}+p_3s_{n-3}.$$

(22.) If $p_2=(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$,
$$p_3=(b-c)(c-a)(a-b),$$
$$s_n=(b-c)^n+(c-a)^n+(a-b)^n,$$

show that
$$s_2=-2p_2,\quad s_3=3p_3,\quad s_4=2p_2^2,\quad s_5=-5p_2p_3,$$
$$s_6=-2p_2^3+3p_3^2,\quad s_7=7p_2^2p_3.\quad 25s_7s_3=21s_5^2.$$

HOMOGENEITY.

§ 17.] *An integral function of any number of variables is said to be "Homogeneous" when the degree of every term in it is the same.* In such a function the degree of the function (§ 6) is of course the same as the degree of every term, and the number of terms which (in the most general case) it can have is the number of different products of the given degree that can be formed with the given number of variables. If there be only two variables, and the degree be n, we have seen that the number of possible terms is $n+1$.

For example, the most general homogeneous integral functions of x and y of the 1st, 2nd, and 3rd degrees are *

$$Ax + By \qquad (1),$$

$$Ax^2 + Bxy + Cy^2 \qquad (2),$$

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 \qquad (3),$$

&c.,

A, B, C, &c., representing the coefficients as usual.

For three variables the corresponding functions are

$$Ax + By + Cz \qquad (4),$$

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy \qquad (5),$$

$$Ax^3 + By^3 + Cz^3 + Pyz^2 + P'y^2z + Qzx^2 + Q'z^2x + Rxy^2 + R'x^2y + Sxyz \quad (6),$$

&c.,

As the case of three variables is of considerable importance, we shall investigate an expression for the number of terms when the degree is n.

We may classify them into—1st, those that do not contain x; 2nd, those that contain x; 3rd, those that contain x^2; . . .; $n+1$th, those that contain x^n.

The first set will simply be the terms of the nth degree made up with y and z, $n+1$ in number; the second set will be the terms of the $(n-1)$th degree made up with y and z, n in number, each with x thrown in; the third set the terms in y and z of $(n-2)$th degree, $n-1$ in number, each with x^2 thrown in; and so on. Hence, if N denote the whole number of terms,

$$N = (n+1) + n + (n-1) + \ldots + 2 \quad + 1.$$

Reversing the right-hand side, we may write

$$N = \quad 1 + 2 + \quad 3 + \ldots + n + (n+1).$$

Now, adding the two left-hand and the two right-hand sides of these equalities, we get

$$\begin{aligned} 2N &= (n+2) + (n+2) + (n+2) + \ldots + (n+2) + (n+2); \\ &= (n+1)(n+2), \end{aligned}$$

since there are $n+1$ terms each $= n+2$.

Whence $$N = \tfrac{1}{2}(n+1)(n+2).$$

For example, let $n = 3$; $N = \frac{1}{2}(3+1)(3+2) = 10$, which is in fact the number of terms in (6), above.

In the above investigation we have been led incidentally to sum an arithmetical series (see chap. xx.); if we attempted the same problem for 4, 5, . . ., m variables, we should have to deal with more and more complicated series. A complete solution for a function of the nth degree in m variables will be given in the second part of this work.

---

* Homogeneous integral functions are called binary, ternary, &c., according as the number of variables is 2, 3, &c.; and quadric, cubic, &c., according as the degree is 2, 3, &c. Thus (3) would be called a binary cubic; (5) a ternary quadric; and so on.

The following is a fundamental property of homogeneous functions :—*If each of the variables in a homogeneous function of the nth degree be multiplied by the same quantity ρ, the result is the same as if the function itself were multiplied by ρ^n.*

Let us consider, for simplicity, the case of three variables ; and let

$$F = Ax^p y^q z^r + A' x^{p'} y^{q'} z^{r'} + \ldots,$$

where $p + q + r = p' + q' + r' =$ &c., each $= n$.

If we multiply x, y, z each by ρ, we have

$$F' = A(\rho x)^p(\rho y)^q(\rho z)^r + A'(\rho x)^{p'}(\rho y)^{q'}(\rho z)^{r'} + \ldots ;$$
$$= A\rho^{p+q+r} x^p y^q z^r + A' \rho^{p'+q'+r'} x^{p'} y^{q'} z^{r'} + \ldots,$$

by the laws of indices. Hence, since $p + q + r = p' + q' + r' =$ &c. $= n$, we have

$$F' = \rho^n \{ Ax^p y^q z^r + A' x^{p'} y^{q'} z^{r'} + \ldots \},$$
$$= \rho^n F,$$

which establishes the proposition in the present case. The reasoning is clearly general.*

* This property might be made the definition of a homogeneous function. Thus we might define a homogeneous function to be such that, when each of its variables is multiplied by ρ, its value is multiplied by ρ^n ; and define n to be its degree. If we proceed thus, we naturally arrive at the idea of homogeneous functions which are not integral or even rational ; and we extend the notion of degree in a corresponding way. For example, $(x^3 - y^3)/(x + y)$ is a homogeneous function of the 2nd degree, for $((\rho x)^3 - (\rho y)^3)/((\rho x) + (\rho y)) = \rho^2(x^3 - y^3)/(x + y)$. Similarly $\sqrt{(x^3 + y^3)}$, $1/(x^2 + y^2)$ are homogeneous functions, whose degrees are $\frac{3}{2}$ and -2 respectively (see chap. x.) Although these extensions of the notions of homogeneity and degree have not the importance of the simpler cases discussed in the text, they are occasionally useful. The distinction of homogeneous functions as a separate class is made by Euler in his *Introductio in Analysin Infinitorum* (1748), (t. i. chap. v.), in the course of an elementary classification of the various kinds of analytical functions. He there speaks, not only of homogeneous integral functions, but also of homogeneous fractional functions, and of homogeneous functions of fractional or negative degrees.

Example.

Consider the homogeneous integral function $3x^2-2xy+y^2$, of the 2nd degree. We have

$$3(\rho x)^2-2(\rho x)(\rho y)+(\rho y)^2=3\rho^2x^2-2\rho^2xy+\rho^2y^2,$$
$$=\rho^2(3x^2-2xy+y^2),$$

in accordance with the theorem above stated.

The following property is characteristic of *homogeneous* integral functions of the *first degree*.

If for the variables x, y, z, . . . we substitute $\lambda x_1+\mu x_2$, $\lambda y_1+\mu y_2$, $\lambda z_1+\mu z_2$, . . . respectively, the result is the same as that obtained by adding the results of substituting x_1, y_1, z_1, . . . and x_2, y_2, z_2, . . . respectively for x, y, z, . . . in the function, after multiplying these results by λ and μ respectively.

Example.

Consider the function $Ax+By+Cz$.

We have

$$\begin{aligned}A(\lambda x_1+\mu x_2)&+B(\lambda y_1+\mu y_2)+C(\lambda z_1+\mu z_2)\\&=A\lambda x_1+B\lambda y_1+C\lambda z_1+A\mu x_2+B\mu y_2+C\mu z_2\\&=\lambda(Ax_1+By_1+Cz_1)+\mu(Ax_2+By_2+Cz_2).\end{aligned}$$

This property is of great importance in Analytical Geometry.

§ 18.] *Law of Homogeneity.*—Since every term in the product of two homogeneous functions of the mth and nth degrees respectively is the product of a term (of the mth degree) taken from one function and a term (of the nth degree) taken from the other, we have the following important law:—

The product of two homogeneous integral functions, of the mth and nth degrees respectively, is a homogeneous integral function of the $(m+n)$th degree.

The student should never fail to use this rule to test the distribution of a product of homogeneous functions. If he finds any term in his result of a higher or lower degree than that indicated by the rule, he has certainly made some mistake. He should also see whether all possible terms of the right degree are present, and satisfy himself that, if any are wanting, it is owing to some peculiarity in the particular case in hand that this is so, and not to an accidental omission.

The rule has many other uses, some of which will be illustrated immediately.

§ 19.] If the student has fully grasped the idea of a homogeneous integral function, the most general of its kind, he will have no difficulty in rising to a somewhat wider generality, namely, the most general integral function of the nth degree in m variables, unrestricted by the condition of homogeneity or otherwise.

Since *any* integral term whose degree does not exceed the nth may occur in such a function, if we group the terms into such as are of the 0th, 1st, 2nd, 3rd, ... nth degrees respectively, we see at once that we obtain the most general type of such a function by simply writing down the sum of all the *homogeneous* integral functions of the m variables of the 0th, 1st, 2nd, 3rd, . . ., nth degrees, each the most general of its kind.

For example, the most general integral function of x and y of the third degree is

$$A+Bx+Cy+Dx^2+Exy+Fy^2+Gx^3+Hx^2y+Ixy^2+Jy^3.$$

The student will have no difficulty, after what has been done in § 17 above, in seeing that the number of terms in the general integral function of the nth degree in two variables is

$$\tfrac{1}{2}(n+1)(n+2).$$

Symmetry.

§ 20.] There is a peculiarity in certain of the functions we have been dealing with in this chapter that calls for special notice here. This peculiarity is denoted by the word "Symmetry"; and doubtless it has already caught the student's eye. What we have to do here is to show how a mathematically accurate definition of symmetry may be given, and how it may be used in algebraical investigations.

1st Definition.—*An integral function* is said to be symmetrical with respect to any two of its variables when the interchange of these two throughout the function leaves its value unaltered.*

---

* As a matter of fact these definitions and much of what follows are applicable to functions of any kind, as the student will afterwards learn. According to Baltzer, Lacroix (1797) was the first to use the term *Symmetric Function*, the older name having been *Invariable Function.*

For example, $$2a+3b+3c$$
becomes, by the interchange of b and c,
$$2a+3c+3b,$$
which is equal to $2a+3b+3c$ by the commutative law. Hence $2a+3b+3c$ is symmetrical with respect to b and c. The same is not true with respect to a and b, or a and c; for the interchange of a and b, for example, would produce $2b+3a+3c$, that is, $3a+2b+3c$, which is not in general equal to* $2a+3b+3c$.

2nd Definition.—*An integral function is said to be symmetrical (that is, symmetrical with respect to all its variables) when the interchange of any pair whatever of its variables would leave its value unaltered.*

For example, $3x+3y+3z$ is a symmetrical function of x, y, z. So are $yz+zx+xy$ and $2(x^2+y^2+z^2)+3xyz$. Taking the last, for instance, if we interchange y and z, we get
$$2(x^2+z^2+y^2)+3xzy,$$
that is, $$2(x^2+y^2+z^2)+3xyz,$$
and so for any other of the three possible interchanges.

On the other hand, $x^2y+y^2z+z^2x$ is not a symmetrical function of x, y, z, for the three interchanges x with y, x with z, y with z give respectively
$$y^2x+x^2z+z^2y,$$
$$z^2y+y^2x+x^2z,$$
$$x^2z+z^2y+y^2x,$$
and, although these are all equal to each other, no one of them is equal to the original function. It will be observed from this instance that asymmetrical functions have a property—which symmetrical functions have not—of assuming different values when the variables are interchanged: thus $x^2y+y^2z+z^2x$ is susceptible of two different values under this treatment, and is therefore a two-valued function. The study of functions from this point of view has developed into a great branch of modern algebra, called the theory of substitutions, which is intimately related with many other branches of mathematics, and, in particular, forms the basis of the theory of the algebraical solution of equations. (See Jordan, *Traité des Substitutions*, and Serret, *Cours d'Algèbre Supérieure*.)

All that we require here is the definition and its most elementary consequences.

3rd Definition.—*A function is said to be collaterally symmetrical in two sets of variables* $\begin{Bmatrix} x_1, x_2, \ldots, x_n \\ a_1, a_2, \ldots, a_n \end{Bmatrix}$, *each of the same number,*

---

* It may not be amiss to remind the student that for the present "equal to" means "transformable by the fundamental laws of algebra into."

when the simultaneous interchanges of two of the first set and of the corresponding two of the second set leave its value unaltered.

For example, $a^2x+b^2y+c^2z$
and $(b+c)x+(c+a)y+(a+b)z$
are evidently symmetrical in this sense.

Other varieties of symmetry might be defined, but it is needless to perplex the student with further definitions. If he fully master the 1st and 2nd, he will have no difficulty with the 3rd or any other case. At first he should adhere somewhat strictly to the formal use of, say, the 2nd definition; but, after a very little practice, he will find that in most cases his eye will enable him to judge without conscious effort as to the symmetry or asymmetry of any function.*

§ 21.] From the above definitions, and from the meaning of the word "equal" in the calculation of algebraical identities, we have at once the following

Rule of Symmetry.—*The algebraic sum, product, or quotient of two symmetrical functions is a symmetrical function.*

Observe, however, that the product, for example, of two *asymmetrical* functions is not necessarily asymmetrical.

Thus, $a+b+c$ and $bc+ca+ab$ being both symmetrical, their product,

$$(a+b+c)(bc+ca+ab)=b^2c+bc^2+c^2a+ca^2+a^2b+ab^2+3abc,$$

is symmetrical.

Again, a^2bc and ab^2c^2 are both asymmetrical functions of a, b, c, yet their product,

$$(a^2bc)\times(ab^2c^2)=a^3b^3c^3,$$

is a symmetrical function.

§ 22.] It will be interesting to see what alterations the restriction of symmetry will make on some of the general forms of integral functions written above.

Since the question of symmetry has nothing to do with degree, it can only affect the coefficients. Looking then at the

* There is a class of functions of great importance closely allied to symmetrical functions, which the student should note at this stage, namely, those that change their sign merely when any pair of the variables are interchanged. Such functions are called "alternating." An example is $(y-z)(z-x)(x-y)$. Obviously the product or quotient of two alternating functions of the same set of variables is a symmetric function. The term *Alternating Function* is due to Cauchy (1812).

homogeneous integral functions of two variables on page 72, we see that, in order that the interchange of x and y may produce no change of value, we must have $A = B$ in § 17 (1); $A = C$ in (2); $A = D$ and $B = C$ in (3).

Hence the *symmetrical* homogeneous integral functions of x and y of 1st, 2nd, 3rd, &c., degrees are

$$Ax + Ay \qquad (1),$$
$$Ax^2 + Bxy + Ay^2 \qquad (2),$$
$$Ax^3 + Bx^2y + Bxy^2 + Ay^3 \qquad (3),$$
&c.

The corresponding functions of x, y, z are

$$Ax + Ay + Az \qquad (4),$$
$$Ax^2 + Ay^2 + Az^2 + Byz + Bzx + Bxy \qquad (5),$$
$$Ax^3 + Ay^3 + Az^3 + Pyz^2 + Py^2z + Pzx^2 + Pz^2x + Pxy^2 + Px^2y + Sxyz \qquad (6),$$
&c.

The most general symmetrical integral function of x, y of the 3rd degree will be the algebraic sum of three functions, such as (1), (2), and (3), together with a constant term, namely,

$$F + Ax + Ay + Bx^2 + Cxy + By^2 + Dx^3 + Ex^2y + Exy^2 + Dy^3.$$

And so on.

If the student find any difficulty in detecting what terms ought to have the same coefficient, let him remark that they are all derivable from each other by interchanges of the variables. Thus, to get all the terms that have the same coefficient as x^3 in (6), putting y for x, we get y^3; putting z for x, we get z^3; and we cannot by operating in the same way upon any of these produce any more terms of the same type. Hence x^3, y^3, z^3 form one group, having the same coefficient. Next take yz^2; the interchanges x and y, x and z, y and z produce xz^2, yx^2, yz^2; applying these interchanges to the new terms, we get only two more new terms—zx^2, xy^2; hence the six terms yz^2, y^2z, zx^2, z^2x, xy^2, x^2y form another group; xyz is evidently unique, being itself symmetrical.

§ 23.] The rule of symmetry is exceedingly useful in abbreviating algebraical work.

Let it be required, for example, to distribute the product $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$, each of whose factors is symmetrical in a, b, c. The distributed product will be symmetrical in a, b, c. Now we see at once that the term a^3 occurs with the coefficient unity, hence the same must be true of b^3 and c^3. Again the term b^2c has the coefficient 0, so also by the principles of symmetry must each of the five other terms, bc^2, c^2a, ca^2, ab^2, a^2b, belonging to the same type. Lastly, the term $-abc$ is obtained by taking a from the first bracket, hence it must occur by taking b, and by taking c, that is, the

abc-term must have the coefficient -3. We have therefore shown that $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc$; and the principles of symmetry have enabled us to abbreviate the work by about two-thirds.

PRINCIPLE OF INDETERMINATE COEFFICIENTS.

§ 24.] A still more striking use of the general principles of homogeneity and symmetry can be best illustrated in conjunction with the application of another principle, which is an immediate consequence of the theory of integral functions.

We have laid down that the coefficients of an integral function are independent of the variables, and therefore are not altered by giving any special values to the variables. *If, therefore, on either side of any algebraic identity involving integral functions we determine the coefficients, either by general considerations regarding the forms of the functions involved, or by considering particular cases of the identity, then these coefficients are determined once for all.* This has (not very happily, it must be confessed) been called the *principle of indeterminate coefficients.* As applied to integral functions it results from the most elementary principles, as we have seen; when infinite series are concerned, its use requires further examination (see the chapter on Series in the second part of this work).

The following are examples:—

$(x+y)^2=(x+y)(x+y)$, being the product of two homogeneous symmetrical functions of x and y of the 1st degree, will be a homogeneous symmetrical integral function of the 2nd degree; therefore
$$(x+y)^2=Ax^2+Bxy+Ay^2 \qquad (1).$$

We have to determine the coefficients A and B.

Since the identity holds for all values of x and y, it must hold when $x=1$ and $y=0$, therefore
$$(1+0)^2=A1^2+B1\times 0+A0^2,$$
$$1=A.$$

We now have $\quad (x+y)^2=x^2+Bxy+y^2$;

this must hold when $\quad x=1$ and $y=-1$,

therefore $\quad (1-1)^2=1+B.1.(-1)+1$,

that is, $\quad 0=2-B$,

whence $\quad B=2$.

Thus finally $\quad (x+y)^2=x^2+2xy+y^2$.

This method of working may seem at first sight somewhat startling, but a little reflection will convince the learner of its soundness. We know, by the principles of homogeneity and symmetry, that a general identity of the form (1) exists, and we determine the coefficients by the consideration that the identity must hold in any particular case. The student will naturally ask how he is to be guided in selecting the particular cases in question, and whether it is material what cases he selects. The answer to the latter part of this question is that, except as to the labour involved in the calculation, the choice of cases is immaterial, provided enough are taken to determine all the coefficients. This determination will in general depend upon the solution of a system of simultaneous equations of the 1st degree, whose number is the number of the coefficients to be determined. (See below, chap. xvi.) So far as possible, the particular cases should be chosen so as to give equations each of which contains only one of the coefficients, so that we can determine them one at a time as was done above.

The student who is already familiar with the solution of simultaneous equations of the 1st degree may work out the values of the coefficients by means of particular cases taken at random. Thus, for example, putting $x=2$, $y=3$, and $x=1$, $y=4$ successively in (1) above, we get the equations

$$25=13A+6B,$$
$$25=17A+4B,$$

which, when solved in the usual way, give $A=1$ and $B=2$, as before.

We give one more example of this important process :—

By the principles of homogeneity and symmetry we must have

$$(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$
$$=A(x^3+y^3+z^3)+B(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+Cyxz.$$

Putting $x=1$, $y=0$, $z=0$, we get $1=A$.

Using this value of A, and putting $x=1$, $y=1$, $z=0$, we get

$$2\times 1=2+B\times 2,$$

that is, $$2=2+B\times 2,$$

therefore $$2B=0,$$

and therefore $$B=0.$$

Using these values of A and B, and putting $x=1$, $y=1$, $z=1$, we get

$$3\times 0=3+C,$$

that is, $$0=3+C,$$

therefore $$C=-3;$$

and we get finally

$$(x+y+z)(x^2+y^2+z^2-yz-zx-xy)=x^3+y^3+z^3-3xyz \qquad (2),$$

as in § 23.

§ 25.] *Reference Table of Identities.*—Most of the results given below will be found useful by the student in his occasional calculations of algebraical identities. Some examples of their use

have already been given, and others will be found among the Exercises in this chapter. Such of the results as have not already been demonstrated above may be established by the student himself as an exercise.

$$\left.\begin{aligned}
&(x+a)(x+b) = x^2 + (a+b)x + ab\,;\\
&(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2\\
&\qquad\qquad + (bc+ca+ab)x + abc\,;\\
&\text{and generally}\\
&(x+a_1)(x+a_2)\ldots(x+a_n) = x^n + P_1x^{n-1} + P_2x^{n-2}\\
&\qquad\qquad + \ldots + P_{n-1}x + P_n \text{ (see § 9).}
\end{aligned}\right\}\text{(I.)}$$

$$(x \pm y)^2 = x^2 \pm 2xy + y^2\,;$$
$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3\,;$$
&c.;

the numerical coefficients being taken from the following table of binomial coefficients :— (II.)*

Power.	Coefficients.												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1
	&c.												

* This table first occurs in the *Arithmetica Integra* of Stifel (1544), in connection with the extraction of roots. It does not appear that he was aware of the application to the expansion of a binomial. The table was discussed and much used by Pascal, and now goes by the name of Pascal's Arithmetical Triangle. The factorial formulæ for the binomial coefficients (see the second part of this work) were discovered by Newton.

$$(x \pm y)^2 \mp 4xy = (x \mp y)^2. \qquad \text{(III.)}$$

$$\left.\begin{array}{l}
(x+y)(x-y) = x^2 - y^2\,; \\
(x \pm y)(x^2 \mp xy + y^2) = x^3 \pm y^3\,; \\
\text{and generally} \\
(x-y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}) = x^n - y^n\,; \\
(x+y)(x^{n-1} - x^{n-2}y + \ldots \mp xy^{n-2} \pm y^{n-1}) = x^n \pm y^n, \\
\text{upper or lower sign according as } n \text{ is odd or even.}
\end{array}\right\}\text{(IV.)}$$

$$\left.\begin{array}{l}
(x^2+y^2)(x'^2+y'^2) = (xx' \mp yy')^2 + (xy' \pm yx')^2\,; \\
(x^2-y^2)(x'^2-y'^2) = (xx' \pm yy')^2 - (xy' \pm yx')^2\,; \\
(x^2+y^2+z^2)(x'^2+y'^2+z'^2) = (xx'+yy'+zz')^2 + (yz'-y'z)^2 \\
\qquad + (zx'-z'x)^2 + (xy'-x'y)^2\,; \\
(x^2+y^2+z^2+u^2)(x'^2+y'^2+z'^2+u'^2) = (xx'+yy'+zz'+uu')^2 \\
\qquad + (xy'-yx'+zu'-uz')^2 \\
\qquad + (xz'-yu'-zx'+uy')^2 \\
\qquad + (xu'+yz'-zy'-ux')^2.
\end{array}\right\}\text{(V.)*}$$

$$(x^2+xy+y^2)(x^2-xy+y^2) = x^4 + x^2y^2 + y^4. \qquad \text{(VI.)}$$

$$\left.\begin{array}{l}
(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad \\
\qquad + 2bc + 2bd + 2cd\,; \\
\text{and generally} \\
(a_1 + a_2 + \ldots + a_n)^2 = \text{sum of squares of } a_1, a_2, \ldots, a_n \\
\qquad + \text{twice sum of all partial products two and two.}
\end{array}\right\}\text{(VII.)}$$

$$\left.\begin{array}{l}
(a+b+c)^3 = a^3+b^3+c^3+3b^2c+3bc^2+3c^2a+3ca^2+3a^2b \\
\qquad + 3ab^2 + 6abc \\
\quad = a^3+b^3+c^3+3bc(b+c)+3ca(c+a) \\
\qquad + 3ab(a+b) + 6abc.
\end{array}\right\}\text{(VIII.)}$$

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab) = a^3+b^3+c^3-3abc. \qquad \text{(IX.)}$$

$$\left.\begin{array}{rl}
(b-c)(c-a)(a-b) = & -a^2(b-c) - b^2(c-a) - c^2(a-b), \\
= & a(b^2-c^2) + b(c^2-a^2) + c(a^2-b^2), \\
= & -bc(b-c) - ca(c-a) - ab(a-b), \\
= & +bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b.
\end{array}\right\}\text{(X.)}$$

* These identities furnish, *inter alia*, proofs of a series of propositions in the theory of numbers, of which the following is typical :—If each of two integers be the sum of two squares, their product can be exhibited in two ways as the sum of two integral squares.

$$\left.\begin{aligned}(b+c)(c+a)(a+b) &= a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc,\\ &= bc(b+c)+ca(c+a)+ab(a+b)+2abc,\\ &= bc^2+b^2c+ca^2+c^2a+ab^2+a^2b+2abc.\end{aligned}\right\}\text{(XI.)}$$

$$\left.\begin{aligned}(a+b+c)(a^2+b^2+c^2) = bc(b+c)+ca(c+a)+ab(a+b)\\ +a^3+b^3+c^3.\end{aligned}\right\}\text{(XII.)}$$

$$\left.\begin{aligned}(a+b+c)(bc+ca+ab) = a^2(b+c)+b^2(c+a)+c^2(a+b)\\ +3abc.\end{aligned}\right\}\text{(XIII.)}$$

$$\left.\begin{aligned}(b+c-a)(c+a-b)(a+b-c) = a^2(b+c)+b^2(c+a)\\ +c^2(a+b)-a^3-b^3-c^3-2abc.\end{aligned}\right\}\text{(XIV.)}$$

$$\left.\begin{aligned}(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2b^2c^2+2c^2a^2\\ +2a^2b^2-a^4-b^4-c^4.\end{aligned}\right\}\text{(XV.)*}$$

$$\left.\begin{aligned}(b-c)+(c-a)+(a-b)=0\,;\\ a(b-c)+b(c-a)+c(a-b)=0\,;\\ (b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)=0.\end{aligned}\right\}\text{(XVI.)}$$

Exercises VIII.

(1.) Write down the most general rational integral symmetrical function of x, y, z, u of the 3rd degree.

(2.) Distribute the product $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)$. Show that it is symmetrical; count the number of types into which its terms fall; and state how many of the types corresponding to its degree are missing.

(3.) Construct a homogeneous integral function of x and y of the 1st degree which shall vanish when $x=y$, and become 1 when $x=5$ and $y=2$.

(4.) Construct an integral function of x and y of the 1st degree which shall vanish when $x=x'$, $y=y'$, and also when $x=x''$, $y=y''$.

(5.) Construct a homogeneous integral function of x and y of the 2nd degree which shall vanish when $x=x'$, $y=y'$, and also when $x=x''$, $y=y''$, and shall become 1 when $x=1$, $y=1$.

(6.) If $A(x-3)(x-5)+B(x-5)(x-7)+C(x-7)(x-3)=8x-120$ for all values of x, determine the coefficients A, B, C.

(7.) Show that $5x^2+19x+18$ can be put into the form

$$l(x-2)(x-3)+m(x-3)(x-1)+n(x-1)(x-2)\,;$$

and find l, m, n.

(8.) Assuming that $(x-1)(x-2)(x-3)$ can be put into the form

$$l(x-1)(x+2)(x+3)+m(x-2)(x+3)(x+1)+n(x-3)(x+1)(x+2),$$

determine the numbers l, m, n.

---

* Important in connection with Hero's formula for the area of a planel triangle.

(9.) Find a rational integral function of x of the 3rd degree which shall have the values P, Q, R, S when $x=a$, $x=b$, $x=c$, $x=d$ respectively.

(10.) Find the coefficients of y^2z and yz^2 in the expansion of

$$(ax+by+cz)(a^2x+b^2y+c^2z)(a^3x+b^3y+c^3z).$$

(11.) Expand and simplify $\Sigma(y^2+z^2-x^2)(y+z-x)$.

Prove the following identities:—

(12.) $(ad+bc)^2+(a+b+c-d)(a+b-c+d)(b+d)(b-d)=(b^2-d^2+ab+cd)^2$.

*(13.) $\Sigma(b^2+c^2-a^2+bc+ca+ab)^2(c^2-b^2)=4(b^2-c^2)(c^2-a^2)(a^2-b^2)$.

(14.) $\Sigma(ca-b^2)(ab-c^2)=(\Sigma bc)(\Sigma bc-\Sigma a^2)$.

(15.) $\Sigma(bc'-b'c)(bc''-b''c)=\Sigma a^2\Sigma a'a''-\Sigma aa'\Sigma aa''$.

(16.) $3\Pi(y+z)-6\Sigma yz=\Sigma x(\Sigma x-1)(\Sigma x-2)-\Sigma x(x-1)(x-2)$.

(17.) $\Sigma(b^2+c^2-a^2)/2bc=(4p_1p_2-p_1^3-6p_3)/2p_3$, where $p_1=-\Sigma a$, $p_2=\Sigma bc$, $p_3=-abc$.

(18.) $\Pi(y+z)^2+2x^2y^2z^2-\Sigma x^4(y+z)^2=2(\Sigma yz)^3$.

(19.) $\Sigma(x+y-z)\{(y-z)^2-(z-x)(x-y)\}=\Sigma x^3-3xyz$.

(20.) $\Pi(a\pm b\pm c\pm d)=\Sigma a^8-4\Sigma a^6b^2+6\Sigma a^4b^4+4\Sigma a^4b^2c^2-40a^2b^2c^2d^2$.

(21.) Show that

$$(x^3+y^3+z^3-3xyz)^2=X^3+Y^3+Z^3-3XYZ,\text{ where } X=x^2+2yz, \&c.;$$

also that

$$(\Sigma x^3-3xyz)(\Sigma x'^3-3x'y'z')=\Sigma(xx'+yz'+y'z)^3-3\Pi(xx'+yz'+y'z).$$

(These identities have an important meaning in the theory of numbers.)

(22.) Show that, if n be a positive integer, then

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\ .\ .\ .\ -\frac{1}{n}\ (n\text{ even})=2\left(\frac{1}{n+2}+\frac{1}{n+4}+\ .\ .\ .\ +\frac{1}{2n}\right);$$

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\ .\ .\ .\ +\frac{1}{n}\ (n\text{ odd})=2\left(\frac{1}{n+1}+\frac{1}{n+3}+\ .\ .\ .\ +\frac{1}{2n}\right).$$

(*Blissard*).

---

* In this example, and in others of a similar kind, Σ is not used in its strict sense, but refers only to cyclical interchanges of a, b, c; that is, to interchanges in which a, b, c pass into b, c, a respectively, or into c, a, b respectively. Thus, $\Sigma a^2(b-c)$ is, strictly speaking, $=0$; but, if Σ be used in the present sense, it is $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

CHAPTER V.

Division of Integral Functions—Transformation of Quotients.

§ 1.] The operations of this chapter are for the most part inverse to those of last. Thus, A and D being any integral functions of one variable x,* and Q a function such that $D \times Q = A$, then Q is called the *quotient* of A by D; A is called the *dividend* and D the *divisor*. We symbolise Q by the notation $A \div D$, A/D, or $\frac{A}{D}$, as explained in chap. i.

The operation of finding Q is called *division*, but we prefer that the student should class the operations of this chapter under the title of *transformation of quotients*.

A and D being both integral functions, Q will be a rational function of x, but will not necessarily be an *integral* function.

When the quotient can be transformed so as to become integral, A *is said to be* exactly divisible *by* D.

When the quotient cannot be so transformed, the quotient is said to be fractional *or* essentially fractional.

It is of course obvious that an essentially integral function cannot be equal, in the identical sense, to an essentially fractional function.

§ 2.] *When the quotient is integral, its degree is the excess of the degree of the dividend over the degree of the divisor.* For, denoting

---

* For reasons partly explained below, the student must be cautious in applying many of the propositions of this chapter to functions of more variables than one; or at least in such cases he must select one of the variables at a time, and think of it as the variable for the purposes of this chapter.

the degrees of the functions represented by the various letters by suffixes, we have

$$A_m = Q_p D_n ;$$

therefore, by chap. iv., § 7, $m = p + n$, that is, $p = m - n$.

§ 3.] *If the degree of the dividend be less than that of the divisor, the quotient is essentially fractional.* For, m being $< n$, suppose, if possible, that the quotient is integral, of degree p say, then

$$A_m = Q_p D_n,$$

therefore $m = p + n$; but p cannot be less than 0 by our hypothesis, and m is already less than n, hence the quotient cannot be integral, that is, it must be fractional.

§ 4.] *If* A, D, Q, R *be all integral functions, and if* A = QD + R, *then* R *will be exactly divisible by* D *or not according as* A *is exactly divisible by* D *or not.*

For, since

$$A = QD + R,$$

$$\frac{A}{D} = \frac{QD + R}{D} = Q + \frac{R}{D} ;$$

therefore

$$\frac{R}{D} = \frac{A}{D} - Q.$$

Now, if A be exactly divisible by D, A/D will be integral, and A/D − Q will be integral, that is, R/D will be integral, that is, R will be exactly divisible by D.

Again, if A be not exactly divisible by D, A/D will be fractional. Hence R/D must be fractional, for, if it were integral, Q + R/D would be integral, that is, A/D would be integral, which is contrary to hypothesis.

INTEGRAL QUOTIENT AND REMAINDER.

§ 5.] The following is the fundamental theorem in the transformation of quotients.

A_m and D_n being integral functions of the degrees m and n respectively, we can always transform the quotient A_m/D_n as follows:—

$$\frac{A_m}{D_n} = P_{m-n} + \frac{R}{D_n},$$

where P_{m-n} *is an integral function of degree* $m-n$, *and* R (*if it do not vanish*) *an integral function whose degree is at most* $n-1$.

This transformation is effected by a series of steps. We shall first work out a particular case, and then give the general proof.

Let
$$A_6 = 8x^6 + 8x^5 - 20x^4 + 40x^3 - 50x^2 + 30x - 10,$$
$$D_4 = 2x^4 + 3x^3 - 4x^2 + 6x - 8,$$

multiply the divisor D_4 by the quotient of the highest term of the dividend by the highest term of the divisor (that is, multiply D_4 by $8x^6/2x^4 = 4x^2$), and subtract the result from the dividend A_6. We have

$$\begin{array}{rl} A_6 = & 8x^6 + 8x^5 - 20x^4 + 40x^3 - 50x^2 + 30x - 10 \\ 4x^2D_4 = & 8x^6 + 12x^5 - 16x^4 + 24x^3 - 32x^2 \\ \hline A_6 - 4x^2D_4 = & \quad - 4x^5 - 4x^4 + 16x^3 - 18x^2 + 30x - 10 \\ & = A_5 \text{ say}; \end{array}$$

therefore
$$A_6 = 4x^2D_4 + A_5 \qquad (1).$$

Repeat the same process with the residue A_5 in place of A_6, and we have

$$\begin{array}{rl} A_5 = & -4x^5 - 4x^4 + 16x^3 - 18x^2 + 30x - 10 \\ -2xD_4 = & -4x^5 - 6x^4 + 8x^3 - 12x^2 + 16x \\ \hline A_5 + 2xD_4 = & \quad 2x^4 + 8x^3 - 6x^2 + 14x - 10 \\ & = A_4 \text{ say}; \end{array}$$

therefore
$$A_5 = -2xD_4 + A_4 \qquad (2).$$

And again with A_4,

$$\begin{array}{rl} A_4 = & 2x^4 + 8x^3 - 6x^2 + 14x - 10 \\ 1 \times D_4 = & 2x^4 + 3x^3 - 4x^2 + 6x - 8 \\ \hline A_4 - D_4 = & \quad 5x^3 - 2x^2 + 8x - 2 \\ & = A_3 \text{ say}; \end{array}$$

therefore
$$A_4 = D_4 + A_3 \qquad (3).$$

Here the process must stop, unless we agree to admit fractional multipliers of D_4; for the quotient of the highest term of A_3 by the highest term of D_4 is $5x^3/2x^4$, that is, $\frac{5}{2}/x$, which is a fractional function of x. Such a continuation of the process does not concern us now, but will be considered below.

Meantime, from (1) we have

$$A_6 = 4x^2D_4 + A_5 \qquad (1)$$

and, using (2) to replace A_5,

$$A_6 = 4x^2D_4 - 2xD_4 + A_4 \qquad (5);$$

and finally, using (3),

$$\begin{aligned} A_6 &= 4x^2D_4 - 2xD_4 + D_4 + A_3, \\ &= (4x^2 - 2x + 1)D_4 + A_3 \qquad (6). \end{aligned}$$

Hence
$$\begin{aligned} \frac{A_6}{D_4} &= \frac{(4x^2 - 2x + 1)D_4 + A_3}{D_4}, \\ &= 4x^2 - 2x + 1 + \frac{A_3}{D_4}; \end{aligned}$$

or, replacing the capital letters by the functions they represent,

$$\frac{8x^6+8x^5-20x^4+40x^3-50x^2+30x-10}{2x^4+3x^3-4x^2+6x-8}$$
$$=4x^2-2x+1+\frac{5x^3-2x^2+8x-2}{2x^4+3x^3-4x^2+6x-8} \qquad (7).$$

Since $6-4=2$, it will be seen that we have established the above theorem for this special case. It so happens that the degrees of the residues A_5, A_4, A_3 diminish at each operation by unity only; but the student will easily see that the diminution might happen to be more rapid; and, in particular, that the degree of the first residue whose degree falls under that of the divisor might happen to be less than the degree of the divisor by more than unity. But none of these possibilities will affect the proof in any way.

We shall return to the present case immediately, but in the first place we may give a *general form* to the proof of the important proposition which we are illustrating.

§ 6.] Let $\quad A_m = p_0x^m + p_1x^{m-1} + p_2x^{m-2} + \&c.$;
$$D_n = q_0x^n + q_1x^{n-1} + q_2x^{n-2} + \&c.$$

Multiplying D_n by the quotient p_0x^m/q_0x^n, that is, by $(p_0/q_0)x^{m-n}$, and subtracting the result from A_m, we get

$$A_m - \frac{p_0}{q_0}x^{m-n}D_n = \left(p_1 - \frac{p_0q_1}{q_0}\right)x^{m-1} + \left(p_2 - \frac{p_0q_2}{q_0}\right)x^{m-2} + \&c.,$$
$$= A_{m-1} \text{ say,}$$

whence, denoting p_0/q_0 by r for shortness, we get

$$A_m = rx^{m-n}D_n + A_{m-1} \qquad (1).$$

Treating A_{m-1} in the same way, we get

$$A_{m-1} = sx^{m-n-1}D_n + A_{m-2} \qquad (2).$$

And so on, so long as the degree of the residue is not less than n, the last such equation obtained being—

$$A_n = wD_n + R \qquad (3),$$

where R is of degree $n-1$ at the utmost. Using all these equations in succession we get

$$A_m = rx^{m-n}D_n + sx^{m-n-1}D_n + \ . \ . \ . \ + wD_n + R$$
$$= (rx^{m-n} + sx^{m-n-1} + \ . \ . \ . \ + w)D_n + R\,;$$

whence, dividing both sides by D_n, and distributing on the right,

$$\frac{A_m}{D_n} = rx^{m-n} + sx^{m-n-1} + \ldots + w + \frac{R}{D_n},$$

which, if we bear in mind the character of R, gives a general proof of the proposition in question.

§ 7.] We have shown that the transformation of § 5 can always be effected in a particular way, but this gives no assurance that the final result will always be the same. The proof that this really is so is furnished by the following proposition:—

The quotient A/D *of two integral functions can be put into the form* P + R/D, *where* P *and* R *are integral functions and the degree of* R *is less than that of* D, *in one way only.*

If possible let

$$\frac{A}{D} = P + \frac{R}{D},$$

and

$$\frac{A}{D} = P' + \frac{R'}{D},$$

where P, R and P′, R′ both satisfy the above requirements;

then

$$P + \frac{R}{D} = P' + \frac{R'}{D};$$

subtracting $P' + \frac{R}{D}$ from both sides, we have

$$P - P' = \frac{R'}{D} - \frac{R}{D};$$

whence

$$\frac{R' - R}{D} = P - P'.$$

Now, since the degrees R and R′ are both less than the degree of D, it follows that the degree of R′ − R is less than that of D. Therefore, by § 3, the left-hand side, (R′ − R)/D, is essentially fractional, and cannot be equal to the right, which is integral, unless R′ − R = 0, in which case we must also have P − P′ = 0, that is, R = R′, and P = P′.

§ 8.] The two propositions of §§ 5, 7 give a peculiar importance to the functions P and R, of which the following definition may now legitimately be given:—

If the quotient A/D *be transformed into* P + R/D, P *and* R *being*

integral and R of degree less than D, P is called the integral quotient, and R the remainder of A when divided by D.

§ 9.] We can now express the condition that one integral function A may be exactly divisible by another D. For, if R be the remainder, as above defined, we have, P being an integral function,—

$$\frac{A}{D} = P + \frac{R}{D},$$

whence, subtracting P from both sides,

$$\frac{A}{D} - P = \frac{R}{D}.$$

Now, if A be exactly divisible by D, A/D will be integral, and therefore A/D − P will be integral. Hence R/D must be integral; but, since the degree of R is less than that of D, this cannot be the case unless R vanish identically.

The necessary and sufficient condition for exact divisibility is therefore that the remainder shall vanish.

When the divisor is of the nth degree, the remainder will in general be of the $(n-1)$th degree, and will contain n coefficients, every one of which must vanish if the remainder vanish. *In general, therefore, when the divisor is of the nth degree, n conditions are necessary to secure exact divisibility.*

§ 10.] Having examined the exact meaning and use of the integral quotient and remainder, we proceed to explain a convenient method for calculating them. The process is simply a succinct arrangement of the calculation of §§ 5, 6. It will be sufficient to take the particular case of § 5.

The work may be arranged as follows:—

$$\begin{array}{rrrrrrr|l}
8x^6 + & 8x^5 - & 20x^4 + & 40x^3 - & 50x^2 + & 30x - & 10 & 2x^4 + 3x^3 - 4x^2 + 6x - 8 \\ \cline{8-8}
8x^6 + & 12x^5 - & 16x^4 + & 24x^3 - & 32x^2 & & & 4x^2 - 2x + 1 \\ \cline{1-7}
 & -\ 4x^5 - & 4x^4 + & 16x^3 - & 18x^2 + & 30x - & 10 & \\
 & -\ 4x^5 - & 6x^4 + & 8x^3 - & 12x^2 + & 16x & & \\ \cline{2-7}
 & & 2x^4 + & 8x^3 - & 6x^2 + & 14x - & 10 & \\
 & & 2x^4 + & 3x^3 - & 4x^2 + & 6x - & 8 & \\ \cline{3-7}
 & & & 5x^3 - & 2x^2 + & 8x - & 2 &
\end{array}$$

Or, observing that the term -10 is not wanted till the last operation, and therefore need not be taken down from the upper line until that stage is reached, and observing further that the method of detached coefficients is clearly applicable here just as in multiplication, we may arrange the whole thus :—

$$
\begin{array}{rrrrrrr|l}
8+ & 8- & 20+ & 40- & 50+ & 30- & 10 & 2+3-4+6-8 \\
\cline{8-8}
8+ & 12- & 16+ & 24- & 32 & & & 4-2+1 \\
\cline{1-7}
 & -4- & 4+ & 16- & 18+ & 30 & & \\
 & -4- & 6+ & 8- & 12+ & 16 & & \\
\cline{2-6}
 & & 2+ & 8- & 6+ & 14- & 10 & \\
 & & 2+ & 3- & 4+ & 6- & 8 & \\
\cline{3-7}
 & & & 5- & 2+ & 8- & 2 &
\end{array}
$$

Therefore, Integral quotient $= 4x^2 - 2x + 1$;

Remainder $= 5x^3 - 2x^2 + 8x - 2$.

The process may be verbally described as follows :—

Arrange both dividend and divisor according to descending powers of x, filling in missing powers with zero coefficients. Find the quotient of the highest term of the dividend by the highest term of the divisor ; the result is the highest term of the "integral quotient."

Multiply the divisor by the term thus obtained, and subtract the result from the dividend, taking down only one term to the right beyond those affected by the subtraction ; the result thus obtained will be less in degree than the dividend by one at least. Divide the highest term of this result by the highest of the divisor ; the result is the second term of the "integral quotient."

Multiply the divisor by the new term just obtained, and subtract, &c., as before.

The process continues until the result after the last subtraction is less in degree than the divisor ; this last result is the remainder as above defined.

§ 11.] The following are some examples of the use of the "long rule" for division.

Example 1.

$$\left(x^4 - \frac{5}{6}x^3 + \frac{19}{36}x^2 - \frac{5}{36}x + \frac{1}{36}\right) \div \left(x^2 - \frac{1}{2}x + \frac{1}{4}\right).$$

$$\begin{array}{l|l}
1 - \frac{5}{6} + \frac{19}{36} - \frac{5}{36} + \frac{1}{36} & 1 - \frac{1}{2} + \frac{1}{4} \\
\underline{1 - \frac{1}{2} + \frac{1}{4}\qquad\qquad} & \overline{1 - \frac{1}{3} + \frac{1}{9}} \\
\quad - \frac{1}{3} + \frac{5}{18} - \frac{5}{36} & \\
\quad \underline{- \frac{1}{3} + \frac{1}{6} - \frac{1}{12}} & \\
\qquad\quad \frac{1}{9} - \frac{1}{18} + \frac{1}{36} & \\
\qquad\quad \underline{\frac{1}{9} - \frac{1}{18} + \frac{1}{36}} & \\
\qquad\qquad 0 + 0 &
\end{array}$$

The remainder vanishes, therefore the division is exact, and the quotient is $x^2 - \frac{1}{3}x + \frac{1}{9}$.

Example 2.

$$(x^3 + px^2 + qx + r) \div (x - a).$$

$$\begin{array}{l|l}
1 + p \qquad + q \qquad\qquad + r & 1 - a \\
\underline{1 - a\qquad\qquad\qquad\qquad\qquad\qquad} & \overline{1 + (a + p) + (a^2 + ap + q)} \\
\quad (a + p) + q & \\
\quad \underline{(a + p) - (a^2 + ap)} & \\
\qquad\quad (a^2 + ap + q) + r & \\
\qquad\quad \underline{(a^2 + ap + q) - (a^3 + a^2p + aq)} & \\
\qquad\qquad\qquad (a^3 + a^2p + aq + r) &
\end{array}$$

Hence the integral quotient is

$$x^2 + (a + p)x + (a^2 + ap + q)\,;$$

and the remainder is

$$a^3 + a^2p + aq + r.$$

The student should observe the use of brackets throughout to preserve the identity of the coefficients.

Example 3.

$$(a^4 - 3a^3b + 6a^2b^2 - 3ab^3 + b^4) \div (a^2 - ab + b^2).$$

1st. Let us consider a as the variable. Since the expressions are homogeneous, we may omit the powers of b in the coefficients, and use the numbers merely.

$$\begin{array}{l|l}
1 - 3 + 6 - 3 + 1 & 1 - 1 + 1 \\
\underline{1 - 1 + 1\qquad\quad} & \overline{1 - 2 + 3} \\
\quad - 2 + 5 - 3 & a^2 - 2ab + 3b^2 \\
\quad \underline{- 2 + 2 - 2} & \\
\qquad\quad 3 - 1 + 1 & \\
\qquad\quad \underline{3 - 3 + 3} & \\
\qquad\qquad 2 - 2 & \\
\qquad\quad 2ab^3 - 2b^4 &
\end{array}$$

whence

$$\frac{a^4-3a^3b+6a^2b^2-3ab^3+b^4}{a^2-ab+b^2}=a^2-2ab+3b^2+\frac{2ab^3-2b^4}{a^2-ab+b^2}.$$

2nd. Let us consider b as the variable. We must then arrange according to descending powers of b, thus—

$$(b^4-3ab^3+6a^2b^2-3a^3b+a^4)\div(b^2-ab+a^2).$$

Detach the coefficients, and proceed as before. It happens in this particular case that the mere numerical part of the work is exactly the same as before; the only difference is in the insertion of the powers of a and b at the end. Thus the integral quotient is $b^2-2ba+3a^2$, and the remainder is $2ba^3-2a^4$, whence

$$\frac{a^4-3a^3b+6a^2b^2-3ab^3+b^4}{a^2-ab+b^2}=3a^2-2ab+b^2+\frac{2a^3b-2a^4}{a^2-ab+b^2}.$$

§ 12.] The process of long division may be still further abbreviated (after expertness and accuracy have been acquired) by combining the operations of multiplying the divisor and subtracting. Then only the successive residues need be written. Thus contracted, the numerical part of the operations of Example 3 in last paragraph would run thus:—

$$\begin{array}{rrrrrrrrr|l}
1 & -3 & +6 & -3 & +1 & & & & & 1-1+1 \\ \cline{10-10}
 & -2 & +5 & -3 & +1 & & & & & 1-2+3 \\
 & & 3 & -1 & +1 & & & & & \\
 & & & 2 & -2 & & & & &
\end{array}$$

BINOMIAL DIVISOR—REMAINDER THEOREM.

§ 13.] The case of a binomial divisor of the 1st degree is of special importance. Let the divisor be $x-a$, and the dividend $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ .\ .\ .\ +p_{n-1}x+p_n$.

Then, if we employ the method of detached coefficients, the calculation runs as follows:—

$$\begin{array}{l|l}
p_0+p_1 \qquad +p_2+\ .\ .\ . \qquad +p_{n-1}+p_n & 1-a \\ \cline{2-2}
\underline{p_0-p_0a} & p_0 \\
\quad (p_0a+p_1)+p_2 & +(p_0a+p_1) \\
\quad \underline{(p_0a+p_1)-(p_0a+p_1)a} & +(p_0a^2+p_1a+p_2) \\
\qquad\qquad (p_0a^2+p_1a+p_2)+p_3 & \quad\vdots \\
\qquad\qquad \underline{(p_0a^2+p_1a+p_2)-(p_0a^2+p_1a+p_2)a} & \quad\vdots \\
\qquad\qquad\qquad\qquad (p_0a^3+p_1a^2+p_2a+p_3) & \quad\vdots \\
\qquad\qquad\qquad\qquad\qquad\vdots &
\end{array}$$

The integral quotient is therefore

$$p_0x^{n-1} + (p_0a + p_1)x^{n-2} + (p_0a^2 + p_1a + p_2)x^{n-3} + \ldots$$

The law of formation of the coefficients is evidently as follows:—

The first is the first coefficient of the dividend;

The second is obtained by multiplying its predecessor by a and adding the second coefficient of the dividend;

The third by multiplying the second just obtained by a and adding the third coefficient of the dividend; and so on.

It is also obvious that the remainder, which in the present case is of zero degree in x (that is, does not contain x), is obtained from the last coefficient of the integral quotient by multiplying that coefficient by a and adding the last coefficient of the dividend.

The operations in any numerical instance may be conveniently arranged as follows:—*

Example 1.

$$(2x^4 - 3x^2 + 6x - 4) \div (x - 2).$$

$$\begin{array}{r} 2+0-3+\ \ 6-\ \ 4 \\ 0+4+8+10+32 \\ \hline 2+4+5+16+28 \end{array}$$

Integral quotient $= 2x^3 + 4x^2 + 5x + 16$;
Remainder $= 28$.

The figures in the first line are the coefficients of the dividend.

The first coefficient in the second line is 0.

The first coefficient in the third line results from the addition of the two above it.

The second figure in the second line is obtained by multiplying the first coefficient in the third line by 2.

The second figure in the third line by adding the two over it.

And so on.

Example 2.

If the divisor be $x+2$, we have only to observe that this is the same as

---

* The student should observe that this arrangement of the calculation of the remainder is virtually a handy method for calculating the value of an integral function of x for any particular value of x, for 28 is $2\times2^4 - 3\times2^2 + 6\times2 - 4$, that is to say, the value of $2x^4 - 3x^2 + 6x - 4$ when $x=2$ (see § 14). This method is often used, and always saves arithmetic when some of the coefficients are negative and others positive. It was employed by Newton; see Horsley's edition, vol. i. p. 270.

$x-(-2)$; and we see that the proper result will be obtained by operating throughout as before, using -2 for our multiplier instead of $+2$.

$$\begin{aligned}&(2x^4-3x^2+6x-4)\div(x+2)\\ &=(2x^4-3x^2+6x-4)\div(x-(-2)).\end{aligned}$$

$$\begin{array}{l} 2+0-3+\ \ 6-4 \\ 0-4+8-10+8 \\ \hline 2-4+5-\ \ 4+4. \end{array}$$

$$\begin{aligned}&\text{Integral quotient}=2x^3-4x^2+5x-4\,;\\ &\text{Remainder}\qquad\quad =4.\end{aligned}$$

Example 3.

The following example will show the student how to bring the case of any binomial divisor of the 1st degree under the case of $x-a$.

$$\begin{aligned}\frac{3x^4-2x^3+3x^2-2x+3}{3x+2}&=\frac{3x^4-2x^3+3x^2-2x+3}{3(x+\frac{2}{3})}\\ &=\tfrac{1}{3}\left\{\frac{3x^4-2x^3+3x^2-2x+3}{x-(-\frac{2}{3})}\right\}.\end{aligned}$$

Transforming now the quotient inside the bracket $\{\ \}$, we have

$$\begin{array}{l} 3-2+\ 3\ -\ 2\ +\ 3 \\ 0-2+\frac{8}{3}-\frac{34}{9}+\frac{104}{27} \\ \hline 3-4+\frac{17}{3}-\frac{52}{9}+\frac{185}{27}. \end{array}$$

$$\begin{aligned}&\text{Integral quotient}=3x^3-4x^2+\tfrac{17}{3}x-\tfrac{52}{9}.\\ &\text{Remainder}\qquad\quad =\tfrac{185}{27}.\end{aligned}$$

Whence

$$\begin{aligned}\frac{3x^4-2x^3+3x^2-2x+3}{3x+2}&=\tfrac{1}{3}\left\{3x^3-4x^2+\tfrac{17}{3}x-\tfrac{52}{9}+\frac{\frac{185}{27}}{x-(-\frac{2}{3})}\right\}\\ &=x^3-\tfrac{4}{3}x^2+\tfrac{17}{9}x-\tfrac{52}{27}+\frac{\frac{185}{27}}{3x+2}.\end{aligned}$$

Hence, for the division originally proposed, we have—

$$\begin{aligned}&\text{Integral quotient}=x^3-\tfrac{4}{3}x^2+\tfrac{17}{9}x-\tfrac{52}{27}\,;\\ &\text{Remainder}\qquad\quad =\tfrac{185}{27}.\end{aligned}$$

The process employed in Examples 2 and 3 above is clearly applicable in general, and the student should study it attentively as an instance of the use of a little transformation in bringing cases apparently distinct under a common treatment.

§ 14.] Reverting to the general result of last section, we see that the remainder, when written out in full, is

$$p_0a^n+p_1a^{n-1}+\ .\ \ .\ \ .+p_{n-1}a+p_n.$$

Comparing this with the dividend

$$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n,$$

we have the following "*remainder theorem*":—

When an integral function of x is divided by $x - a$, the remainder is obtained by substituting a for x in the function in question.

In other words, the remainder is the same function of a as the dividend is of x.

Partly on account of the great importance of this theorem, partly as an exercise in general algebraical reasoning, we give another proof of it.

Let us, for shortness, denote

$$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n \text{ by } f(x),$$

$f(a)$ will then, naturally, denote the result of substituting a for x in $f(x)$, that is,

$$p_0a^n + p_1a^{n-1} + \ldots + p_{n-1}a + p_n.$$

Let $\chi(x)$ denote the integral quotient, and R the remainder, when $f(x)$ is divided by $x - a$. Then $\chi(x)$ is an integral function of x of degree $n - 1$, and R is a constant (that is, is independent of x), and we have

$$\frac{f(x)}{x-a} = \chi(x) + \frac{R}{x-a},$$

whence, multiplying by $x - a$, we get the identity

$$f(x) = (x - a)\chi(x) + R.$$

Since this holds for all values of x, we get, putting $x = a$ throughout,

$$f(a) = (a - a)\chi(a) + R,$$

where R remains the same as before, since it does not depend on x, and therefore is not altered by giving any particular value to x.

Since $\chi(a)$ is finite if a be finite, $(a - a)\chi(a) = 0 \times \chi(a) = 0$; and we get finally

$$f(a) = R,$$

which, if we remember the meaning of $f(a)$, proves the "remainder theorem."

Cor. 1. Since $x + a = x - (-a)$, it follows that

The remainder, when an integral function $f(x)$ is divided by $x + a$, is $f(-a)$.

For example, the remainder, when $x^4 - 3x^3 + 2x^2 - 5x + 6$ is divided by $x + 10$, is $(-10)^4 - 3(-10)^3 + 2(-10)^2 - 5(-10) + 6 = 13256$.

Cor. 2. *The remainder, when an integral function of x, $f(x)$, is divided by $ax + b$, is $f(-b/a)$.*

This is simply the generalisation of Example 3, § 13, above.

By substitution we may considerably extend the application of the remainder theorem, as the following example will show:—

Consider $p_m(x^n)^m + p_{m-1}(x^n)^{m-1} + \ldots + p_1(x^n) + p_0$ and $x^n - a^n$. Writing for a moment ξ in place of x^n, and α in place of a^n, we have to deal with $p_m\xi^m + p_{m-1}\xi^{m-1} + \ldots + p_1\xi + p_0$ and $\xi - \alpha$. Now the remainder, when the former of these is divided by the latter, is $p_m\alpha^m + p_{m-1}\alpha^{m-1} + \ldots + p_1\alpha + p_0$. Hence the remainder, when $p_m(x^n)^m + p_{m-1}(x^n)^{m-1} + \ldots + p_1x^n + p_0$ is divided by $x^n - a^n$, is $p_m(a^n)^m + p_{m-1}(a^n)^{m-1} + \ldots + p_1a^n + p_0$.

APPLICATION OF REMAINDER THEOREM TO THE DECOMPOSITION OF AN INTEGRAL FUNCTION INTO LINEAR FACTORS.

§ 15.] *If $a_1, a_2, \ldots, a_r$ be r different values of x, for which the integral function of the nth degree $f(x)$ vanishes, where $r < n$, then $f(x) = (x - a_1)(x - a_2) \ldots (x - a_r)\phi_{n-r}(x)$, $\phi_{n-r}(x)$ being an integral function of x of the $(n - r)$th degree.*

For, since the remainder, $f(a_1)$, when $f(x)$ is divided by $x - a_1$, vanishes, therefore $f(x)$ is exactly divisible by $x - a_1$, and we have

$$f(x) = (x - a_1)\phi_{n-1}(x),$$

where $\phi_{n-1}(x)$ is an integral function of x of the $(n - 1)$th degree. Since this equation subsists for all values of x, we have

$$f(a_2) = (a_2 - a_1)\phi_{n-1}(a_2),$$

that is, by hypothesis, $\quad 0 = (a_2 - a_1)\phi_{n-1}(a_2)$.

Now, since a_1 and a_2 are different by hypothesis, $a_2 - a_1 \neq 0$; therefore $\phi_{n-1}(a_2) = 0$.

Hence, $\qquad \phi_{n-1}(x)$ is divisible by $(x - a_2)$,

that is, $\qquad \phi_{n-1}(x) = (x - a_2)\phi_{n-2}(x)$;

whence $\qquad f(x) = (x - a_1)(x - a_2)\phi_{n-2}(x)$.

From this again,

$$0 = f(a_3) = (a_3 - a_1)(a_3 - a_2)\phi_{n-2}(a_3),$$

which gives, since a_1, a_2, a_3 are all unequal, $\phi_{n-2}(a_3) = 0$; whence $\phi_{n-2}(x) = (x - a_3)\phi_{n-3}(x)$; so that

$$f(x) = (x - a_1)(x - a_2)(x - a_3)\phi_{n-3}(x).$$

Proceeding in this way step by step, we finally establish the theorem for any number of factors not exceeding n.

Cor. 1. *If an integral function be divisible by the factors* $x - a_1$, $x - a_2$, . . ., $x - a_r$, *all of the* 1*st degree, and all different, it is divisible by their product; and, conversely, if it is divisible by the product of any number of such factors, all of the* 1*st degree and all different, it is divisible by each of them separately.* The proof of this will form a good exercise in algebraical logic.

Cor. 2. The particular case of the above theorem where the number of factors is equal to the degree of the function is of special interest. We have then

$$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)\mathrm{P}.$$

Here P is of zero degree, that is, is a constant. To determine it we have only to compare the coefficients of x^n on the left and right hand sides, which must be equal by chap. iv., § 24. Now $f(x)$ stands for $p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$. Hence $\mathrm{P} = p_0$, and we have

$$f(x) = p_0(x - a_1)(x - a_2) \ldots (x - a_n).$$

In other words—*If n different values of x can be found for which the integral function $f(x)$ vanishes, then $f(x)$ can be resolved into n factors of the* 1*st degree, all different.*

The student must observe the "if" here. We have not shown that n such particular values of x can always be found, or how they can be found, but only that *if they can be found* the factorisation may be effected. The question as to the finding of a_1, a_2, . . ., &c., belongs to the Theory of Equations, into which we are not yet prepared to enter.

§ 16.] The student who has followed the above theory will naturally put to himself the question, "Can more than n values

of x be found for which an integral function of x of the nth degree vanishes, and, if so, what then?" The following theorem will answer this question, and complete the general theory of factorisation so far as we can now follow it.

If an integral function of x of the nth degree vanish for more than n different values of x, it must vanish identically, that is, each of its coefficients must vanish.

Let $a_1, a_2, \ldots, a_n$ be n of the values for which $f(x)$ vanishes, then by § 15 above, if p_0 be the coefficient of the highest power of x in $f(x)$, we have

$$f(x) = p_0(x - a_1)(x - a_2) \ldots (x - a_n) \qquad (1).$$

Now let β be another value (since there are more than n) for which $f(x)$ vanishes, then, since (1) is true for all values of x, we have

$$0 = f(\beta) = p_0(\beta - a_1)(\beta - a_2) \ldots (\beta - a_n) \qquad (2).$$

Since, by hypothesis, $a_1, a_2, \ldots, a_n$ and β are all different, none of the differences $\beta - a_1, \beta - a_2, \ldots, \beta - a_n$, can vanish, and therefore their product cannot vanish. Hence (2) gives $p_0 = 0$.

This being so, $f(x)$ reduces to $p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$. We have now, therefore, a function of the $(n - 1)$th degree which vanishes for more than n, therefore for more than $(n - 1)$, values of its variable. We can, by a repetition of the above reasoning, prove that the highest coefficient p_1 of this function vanishes. Proceeding in this way we can show, step by step, that all the coefficients of $f(x)$ vanish.

As an example of this case the student may take the following:—

The integral function

$$(\beta-\gamma)(x-\beta)(x-\gamma)+(\gamma-\alpha)(x-\gamma)(x-\alpha)+(\alpha-\beta)(x-\alpha)(x-\beta)$$
$$+(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$$

vanishes when $x=\alpha$, when $x=\beta$, and when $x=\gamma$; but it is only of the 2nd degree in x. We therefore infer that the function vanishes for all values of x, that is to say, that we have identically

$$(\beta-\gamma)(x-\beta)(x-\gamma)+(\gamma-\alpha)(x-\gamma)(x-\alpha)+(\alpha-\beta)(x-\alpha)(x-\beta)$$
$$+(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta) \qquad =0.$$

That this is so the reader may readily verify by expanding and arranging the left-hand side.

Cor. *If two integral functions of x, whose degrees are m and n respectively, m being $> n$, be equal in value for more than m different values of x,* a fortiori, *if they be equal for all values of x, that is to say, identically equal, then the coefficients of like powers of x in the two must be equal.*

We may, without loss of generality, suppose the two functions to be each of degree m, for, if they be not equal in degree, this simply means that the coefficients of $x^{n+1}, x^{n+2}, \ldots, x^m$ in one of them are zero. We have therefore, by hypothesis,

$$p_0x^m + p_1x^{m-1} + \ldots + p_m = q_0x^m + q_1x^{m-1} + \ldots + q_m,$$

and therefore

$$(p_0 - q_0)x^m + (p_1 - q_1)x^{m-1} + \ldots + (p_m - q_m) = 0,$$

for more than m values of x.

Hence we must have

$$p_0 - q_0 = 0, \quad p_1 - q_1 = 0, \quad \ldots, \quad p_m - q_m = 0;$$

that is,

$$p_0 = q_0, \quad p_1 = q_1, \quad \ldots, \quad p_m = q_m.$$

This is of course merely another form of the principle of *indeterminate coefficients.* The present view of it is, however, important and instructive, for we can now say that, *if any function of x can be transformed into an integral function of x, then this transformation is unique.*

§ 17.] The danger with the theory we have just been expounding is not so much that the student may refuse his assent to the demonstrations given, as that he may fail to apprehend fully the scope and generality of the conclusions. We proceed, therefore, before leaving the subject, to illustrate very fully the use of the remainder theorem in various particular cases.

To help the student, we shall distinguish, in the following examples, between identical and conditional equations by using the sign "$\equiv$" for the former and the sign "$=$" for the latter.

Example 1. To determine the value of the constant k in order that

$$x^3 + 6x^2 + 4x + k$$

may be exactly divisible by $x + 2$.

The remainder, after dividing by $x + 2$, that is, by $x - (-2)$, is $(-2)^3$

$+6(-2)^2+4(-2)+k$, that is, $8+k$. The condition for exact divisibility is therefore $8+k=0$, that is, $k=-8$.

Example 2. To determine whether

$$3x^3-2x^2-7x-2 \qquad (1)$$

is divisible by $(x+1)(x-2)$.

If we put $x=-1$ in the function (1) we get

$$-3-2+7-2=0,$$

hence it is divisible by $x+1$.

If we put $x=2$ we get

$$24-8-14-2=0,$$

hence it is divisible by $x-2$.

Hence by § 15 it is divisible by $(x+1)(x-2)$. The quotient in this case is easily obtained, for, since the degree of (1) is the 3rd, we must have

$$3x^3-2x^2-7x-2\equiv(x+1)(x-2)(ax+b) \qquad (2),$$

where a and b are numbers to be determined.

If we observe that the highest term $3x^3$ on the left must be equal to the product $x\times x\times ax$ of the three highest terms of the factors on the right, we see that $3x^3\equiv ax^3$, hence $a=3$. And, since the product of the three lowest terms of the factors on the right must be equal to -2, the lowest term on the left, we get $-2b=-2$, that is, $b=1$. Hence finally

$$3x^3-2x^2-7x-2\equiv(x+1)(x-2)(3x+1).$$

Example 3. If n be a positive integer,

when	is divided by	the rem. is	that is
x^n-a^n	$x-a$	a^n-a^n	0 always.
x^n-a^n	$x+a$	$(-a)^n-a^n$	0 if n be even, $-2a^n$ if n be odd.
x^n+a^n	$x-a$	a^n+a^n	$2a^n$ always.
x^n+a^n	$x+a$	$(-a)^n+a^n$	0 if n be odd, $2a^n$ if n be even.

Hence x^n-a^n is exactly divisible by $x-a$ for all integral values of n, and by $x+a$ if n be even. x^n+a^n is exactly divisible by $x+a$ if n be odd, but is never exactly divisible by $x-a$ (so long as $a\neq 0$). These results agree with those given above, chap. iv., § 16.

Example 4. To prove that

$$a^3(b-c)+b^3(c-a)+c^3(a-b)\equiv-(a+b+c)(b-c)(c-a)(a-b).$$

First of all regard $P\equiv a^3(b-c)+b^3(c-a)+c^3(a-b)$ as a function of a. P is an integral function of a of the 3rd degree; and, if we put $a=b$,

$$\begin{aligned}P&=b^3(b-c)+b^3(c-b)+c^3(b-b)\\&=0;\end{aligned}$$

and similarly, if we put $a=c$, $P=0$; therefore P is exactly divisible both by $a-b$ and by $a-c$.

Again, regard P as a function of b alone. It is an integral function of b, and vanishes when $b=c$, hence it is exactly divisible by $b-c$. We have, therefore,

$$P\equiv Q(a-b)(a-c)(b-c).$$

Since P, regarded as a function of a, b, and c, is of the 4th degree, it follows that Q must be an integral function of a, b, c of the 1st degree. Hence, l, m, n being mere numbers which we have still to determine, we have

$$\begin{aligned} a^3(b-c)+b^3(c-a)+c^3(a-b) &\equiv (la+mb+nc)(b-c)(a-c)(a-b) \\ &\equiv -(la+mb+nc)(b-c)(c-a)(a-b). \end{aligned}$$

To determine l we have merely to compare the coefficients of a^3b on both sides. It thus results by inspection that $l=1$; and similarly $m=1$, $n=1$; the last two inferences being also obvious by the law of symmetry. We have therefore finally

$$a^3(b-c)+b^3(c-a)+c^3(a-b)\equiv -(a+b+c)(b-c)(c-a)(a-b).$$

Example 5. To show that

$$\begin{aligned} P &\equiv 2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4 \\ &\equiv (a+b+c)(-a+b+c)(a-b+c)(a+b-c). \end{aligned}$$

First, regarding P as an integral function of a, and dividing it by $a+b+c$, that is, by $a-(-b-c)$, we have for the remainder

$$\begin{aligned} &2b^2c^2+2c^2(-b-c)^2+2b^2(-b-c)^2-(-b-c)^4-b^4-c^4 \\ &\quad\equiv 2b^2c^2+2b^2c^2+4bc^3+2c^4+2b^4+4b^3c+2b^2c^2 \\ &\qquad -b^4-4b^3c-6b^2c^2-4bc^3-c^4-b^4-c^4 \\ &\quad\equiv 0. \end{aligned}$$

Hence P is exactly divisible by $a+b+c$.

Observing that the change of a into $-a$, or of b into $-b$, or of c into $-c$ does not alter P, all the powers of these letters therein occurring being even, we see that P must also be divisible by $-a+b+c$, $a-b+c$, and $a+b-c$. We have thus found four factors of the 1st degree in a, b, c, and there can be no more, since P itself is of the 4th degree in these letters. This being established, it is easy to prove, as in Example 4, that the constant multiplier is $+1$: and thus the result is established.

EXPANSION OF RATIONAL FRACTIONS IN SERIES BY MEANS OF CONTINUED DIVISION.

§ 18.] If we refer back to §§ 5 and 6, and consider the analysis there given, we shall see that every step in the process of long division gives us an algebraical identity of the form

$$\frac{A_m}{D_n}=Q'+\frac{R'}{D_n},$$

where Q' is the part of the quotient already found, and R' the residue at the point where we suppose the operation arrested.

For example, if we stop at the end of the second operation,

$$\frac{8x^6+8x^5-20x^4+40x^3-50x^2+30x-10}{2x^4+3x^3-4x^2+6x-8}$$
$$=4x^2-2x+\frac{2x^4+8x^3-6x^2+14x-10}{2x^4+3x^3-4x^2+6x-8}.$$

Again, instead of confining ourselves to integral terms in x, and therefore arresting the process when the remainder, *strictly so called*, is reached, we may *continue* the operation to any extent. In this case, if we stop after any step we still get an identity of the form

$$\frac{A_m}{D_n}=Q'+\frac{R'/x^q}{D_n} \qquad (1),$$

where

$$Q'=Ax^p+Bx^{p-1}+\ldots+Kx+L+M/x+\ldots+T/x^q \qquad (2),$$

and

$$R'=Ux^{n-1}+Vx^{n-2}+\ldots+Z \qquad (3).$$

This process may be called *Descending Continued Division.*

For example, consider

$$\frac{x^3+2x^2+3x+4}{x^2+x+1},$$

and let us conduct the division in the contracted manner of § 12, but insert the powers of x for greater clearness.

$$\begin{array}{l|l}
x^3+2x^2+3x+4 & x^2+x+1 \\
\quad +\ x^2+2x+4 & x+1+\frac{1}{x}+\frac{2}{x^2}-\frac{3}{x^3}+\frac{1}{x^4}+\frac{2}{x^5} \\
\qquad\qquad x+3 & \\
\qquad\qquad\quad +2-\frac{1}{x} & \\
\qquad\qquad\qquad -\frac{3}{x}-\frac{2}{x^2} & \\
\qquad\qquad\qquad\quad +\frac{1}{x^2}+\frac{3}{x^3} & \\
\qquad\qquad\qquad\qquad +\frac{2}{x^3}-\frac{1}{x^4} & \\
\qquad\qquad\qquad\qquad\quad -\frac{3}{x^4}-\frac{2}{x^5}, &
\end{array}$$

whence $$\frac{x^3+2x^2+3x+4}{x^2+x+1}=x+1+\frac{1}{x}+\frac{2}{x^2}-\frac{3}{x^3}+\frac{1}{x^4}+\frac{2}{x^5}+\frac{(-3x-2)/x^5}{x^2+x+1} \qquad (4),$$

an identity which the student should verify by multiplying both sides by x^2+x+1.

§ 19.] When we prolong the operation of division indefinitely in this way we may of course arrange either dividend or divisor, or both, according to ascending powers of x. Taking the latter arrangement we get a new kind of result, which may be illustrated with the fraction used above.

We now have

$$\begin{array}{l|l} 4+3x+2x^2+x^3 & 1+x+x^2 \\ \hline \quad -x-2x^2+x^3 & 4-x-x^2+3x^3-2x^4-x^5 \\ \qquad -x^2+2x^3 & \\ \qquad\quad +3x^3+x^4 & \\ \qquad\qquad -2x^4-3x^5 & \\ \qquad\qquad\quad -x^5+2x^6 & \\ \qquad\qquad\qquad +3x^6+x^7, & \end{array}$$

whence
$$\frac{x^3+2x^2+3x+4}{x^2+x+1}=4-x-x^2+3x^3-2x^4-x^5+\frac{(3+x)x^6}{1+x+x^2} \qquad (5).$$

And, in general, proceeding in a similar way with any two integral functions, A_m and D_n, we get

$$\frac{A_m}{D_n}=Q''+\frac{R''x^{q+1}}{D_n} \qquad (6),$$

where
$$Q''=A+Bx+\ldots+Kx^q \qquad (7),$$
$$R''=L+Mx+\ldots+Tx^{n-1} \qquad (8).$$

This process may be called *Ascending Continued Division.*

§ 20.] *The results of §§ 18, 19 show us that we can, by the ordinary process of continued long division, expand any rational fraction as a "series" either of descending or ascending powers of x, containing as many terms as we please, plus a "residue," which is itself a rational fraction.*

And there is no difficulty in showing that, when the integer q is given, each of the expansions (2) and (6) of §§ 18 and 19 is unique.

The proof depends on the theory of degree, and may be left as an exercise for the reader.

These series (the Q' or Q'' parts above) belong, as we shall see hereafter, to the general class of "Recurring Series."

The following are simple examples of the processes we have been describing:—

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\frac{x^{n+1}}{1-x} \qquad (9).$$

$$=-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\ldots-\frac{1}{x^n}+\frac{1/x^n}{1-x} \qquad (10).$$

$$\frac{1}{1+x}=1-x+x^2-\ldots+(-1)^n x^n-\frac{(-1)^n x^{n+1}}{1+x} \qquad (11).$$

$$=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\ldots-\frac{(-1)^n}{x^n}+\frac{(-1)^n/x^n}{1+x} \qquad (12).$$

$$\frac{1}{1+x+\ldots+x^n}=1-x+x^{n+1}-x^{n+2}+x^{2n+2}-x^{2n+3}+\ldots$$

$$+x^{nr+r}-x^{nr+r+1}+\frac{x^{(r+1)(n+1)}}{1+x+\ldots+x^n} \qquad (13).$$

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots+(n+1)x^n+\frac{(n+2)x^{n+1}-(n+1)x^{n+2}}{(1-x)^2} \qquad (14).$$

EXPRESSION OF ONE INTEGRAL FUNCTION IN POWERS OF ANOTHER.

§ 21.] We shall have occasion in a later chapter to use two particular cases of the following theorem.

If P *and* Q *be integral functions of the mth and nth degrees respectively* ($m>n$), *then* P *may always be put into the form*

$$P=R_0+R_1Q+R_2Q^2+\ldots+R_pQ^p \qquad (1),$$

where $R_0, R_1, \ldots, R_p$ *are integral functions, the degree of each of which is* $n-1$ *at most, and* p *is a positive integer, which cannot exceed* m/n.

Proof.—Divide P by Q, and let the quotient be Q_0 and the remainder R_0.

If the degree of Q_0 be greater than that of Q, divide Q_0 by Q, and let the quotient be Q_1 and the remainder R_1.

Next divide Q_1 by Q, and let the quotient be Q_2 and the remainder R_2, and so on, until a quotient Q_{p-1} is reached whose degree is less than the degree of Q. Q_{p-1}, for convenience, we call also R_p. We thus have

$$\left.\begin{aligned}
P &= Q_0Q+R_0\\
Q_0 &= Q_1Q+R_1\\
Q_1 &= Q_2Q+R_2\\
&\ldots\ldots\\
Q_{p-2} &= Q_{p-1}Q+R_{p-1}\\
Q_{p-1} &= R_p.
\end{aligned}\right\} \qquad (2).$$

Now, using in the first of these the value of Q_0 given by the second, we obtain

$$\begin{aligned} P &= (Q_1 Q + R_1)Q + R_0, \\ &= R_0 + R_1 Q + Q_1 Q^2. \end{aligned}$$

Using the value of Q_1 given by the third, we obtain

$$P = R_0 + R_1 Q + R_2 Q^2 + Q_2 Q^3 \qquad (3).$$

And so on.

We thus obtain finally the required result; for, $R_0, R_1, \ldots, R_p$ being remainders after divisions by Q (whose degree is n), none of these can be of higher degree than $n-1$; moreover, since the degrees of $Q_0, Q_1, Q_2, \ldots, Q_{p-1}$ are $m-n$, $m-2n$, $m-3n$, . . ., $m-np$, p cannot exceed m/n.

The two most important particular cases are those in which $Q = x - \alpha$ and $Q = x^2 + \beta x + \gamma$. We then have

$$P = a_0 + a_1(x-\alpha) + \ldots + a_n(x-\alpha)^n,$$

where $a_0, a_1, \ldots, a_n$ are constants;

$$P = (a_0 + b_0 x) + (a_1 + b_1 x)(x^2 + \beta x + \gamma) + \ldots + (a_p + b_p x)(x^2 + \beta x + \gamma)^p,$$

where $a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_p$ are constants, and $p \ngtr m/2$.

Example 1.

Let

$$\begin{aligned} P &= 5x^3 - 11x^2 + 10x - 2, \\ Q &= x - 1. \end{aligned}$$

The calculation of the successive remainders proceeds as follows (see § 13):—

$$\begin{array}{llll} 5 & -11 & +10 & -2 \\ 0 & +\ 5 & -\ 6 & +4 \\ \hline 5 & -\ 6 & +\ 4 & \vert +2 \\ 0 & +\ 5 & -\ 1 & \\ \hline 5 & -\ 1 & \vert +\ 3 & \\ 0 & +\ 5 & & \\ \hline 5 & \vert +\ 4 & & \\ 0 & & & \\ \hline \vert +5\ ; & & & \end{array}$$

and we find

$$5x^3 - 11x^2 + 10x - 2 = 2 + 3(x-1) + 4(x-1)^2 + 5(x-1)^3.$$

Example 2.

$$\begin{aligned} P &= x^8 + 3x^7 + 4x^6 + 4x^2 + 3x + 1, \\ Q &= x^2 - x + 1. \end{aligned}$$

The student will find

$$R_0 = 11x, \quad R_1 = -22x + 7, \quad R_2 = 19x - 22, \quad R_3 = 7x + 15, \quad R_4 = 1\ ;$$

so that

$$P = 11x + (-22x + 7)(x^2 - x + 1) + (19x - 22)(x^2 - x + 1)^2 + (7x + 15)(x^2 - x + 1)^3 + (x^2 - x + 1)^4.$$

§ 22.] *If $a_1, a_2, \ldots, a_n$ be n constants, any two or more of which may be equal, then any integral function of x of the nth degree may be put into the form*

$$A_0 + A_1(x - a_1) + A_2(x - a_1)(x - a_2) + A_3(x - a_1)(x - a_2)(x - a_3) + \ldots + A_n(x - a_1)(x - a_2)\ldots(x - a_n) \qquad (1),$$

where $A_0, A_1, A_2, \ldots, A_n$ are constants, any one of which except A_n may be zero.

Let P_n be the given integral function, then, dividing P_n by $x - a_1$, we have

$$P_n = P_{n-1}(x - a_1) + A_0 \qquad (2),$$

where A_0 is the constant remainder, which may of course in any particular case be zero.

Next, dividing P_{n-1} by $(x - a_2)$, we have

$$P_{n-1} = P_{n-2}(x - a_2) + A_1 \qquad (3);$$

and so on.

Finally, $$P_1 = A_n(x - a_n) + A_{n-1} \qquad (n + 1).$$

Using these equations, we get successively

$$P_n = A_0 + A_1(x - a_1) + (x - a_1)(x - a_2)P_{n-2},$$
$$= A_0 + A_1(x - a_1) + A_2(x - a_1)(x - a_2) + (x - a_1)(x - a_2)(x - a_3)P_{n-3},$$

. . . .

$$P_n = A_0 + A_1(x - a_1) + A_2(x - a_1)(x - a_2) + A_3(x - a_1)(x - a_2)(x - a_3) + \ldots + A_n(x - a_1)(x - a_2)\ldots(x - a_n).$$

This kind of expression for an integral function is often useful in practice.

Knowing *a priori* that the expansion is possible, we can, if we choose, determine the coefficients by giving particular values to x. But the most rapid process in general is simply to carry out the divisions indicated in the proof, exactly as in Example 1 of last paragraph.

Thus, to express $x^3 - 1$ in the form $A_0 + A_1(x - 1) + A_2(x - 1)(x - 2) + A_3(x - 1)(x - 2)(x - 3)$, we calculate as follows :—

$$\begin{array}{r|l}
 & 1 \;+0 \;+0 \;-1 \\
1 & \underline{0 \;+1 \;+1 \;+1} \\
 & 1 \;+1 \;+1 \,|\, 0 \\
2 & \underline{0 \;+2 \;+6} \\
 & 1 \;+3 \,|\, +7 \\
3 & \underline{0 \;+3} \\
 & 1 \,|\, +6
\end{array}$$

Hence $x^3 - 1 = 0 + 7(x-1) + 6(x-1)(x-2) + (x-1)(x-2)(x-3)$.

Exercises IX.

Transform the following quotients, finding both integral quotient and remainder where the quotient is fractional.

(1.) $(x^5 - 5x^3 + 5x^2 - 1)/(x^2 + 3x + 1)$.

(2.) $(6x^6 + 2x^5 - 19x^4 + 31x^3 - 37x^2 + 27x - 7)/(2x^2 - 3x + 1)$.

(3.) $(4x^5 - 2x^4 + 3x^3 - x + 1)/(x^2 - 2x + 1)$.

(4.) $(x^2 - 8x + 15)(x^2 + 8x + 15)/(x^2 - 25)$.

(5.) $\{(x-1)(x-2)(x-3)(x-4)(x-5) - 760(x-6) + 120(x-7)\} \div (x-6)(x-7)$.

(6.) $(x^6 + 4x^5 - 3x^4 - 16x^3 + 2x^2 + x + 3)/(x^3 + 4x^2 + 2x + 1)$.

(7.) $(27x^6 + 10x^3 + 1)/(3x^2 - 2x + 1)$.

(8.) $(x^3 - 9x^2 + 23x - 15)(x - 7)/(x^2 - 8x + 7)$.

(9.) $(x^5 + \frac{13}{12}x^3 + \frac{7}{12}x^2 + \frac{5}{24}x + \frac{1}{4})/(x^2 - \frac{1}{2}x + 1)$.

(10.) $(x^4 + \frac{1}{2}x^3 + \frac{1}{5}x^2 + \frac{1}{7}x + \frac{1}{8})/(x^2 + 2x + 1)$.

(11.) $(x^7 + \frac{1}{128})/(2x + 1)$.

(12.) $(x^2 - x + 1)(x^3 - 1)/(x^4 + x^2 + 1)$.

(13.) $(x^{15}y^8 - x^8y^{15})/(x - y)$.

(14.) $(9a^4 + 2a^2b^2 + b^4)/(3a^2 + 2ab + b^2)$.

(15.) $(a^7 + b^7)/(a + b)$.

(16.) $(x^4 + y^4 - 7x^2y^2)/(x^2 + 3xy + y^2)$.

(17.) $(x^5 - 2x^4y + 4x^3y^2 - 8x^2y^3 + 16xy^4 - 32y^5)/(x^3 - 8y^3)$.

(18.) $(x^4 + 5x^3y + 7x^2y^2 + 15xy^3 + 12y^4)/(x + 4y)$.

(19.) $(1 + x + x^2 + x^3 + x^4 + x^6 + x^7 + x^8 + x^9 + x^{15})/(1 - x^5 + x^6)$.

(20.) $(x^6 - 5x^3 + 8)/(x^2 + x + 2)$.

(21.) $\{abx^3 + (ac - bd)x^2 - (af + cd)x + df\}/(ax - d)$.

(22.) $\{a^2b^2 + a^2c^2 + b^2c^2 + 2a^2bc - 2ab^2c - 2abc^2\} \div \{a^2 - (a-b)(a-c)\}$.

(23.) $(1 + b + c - bc - b^2c - bc^2)/(1 - bc)$.

(24.) $\{(ax+by)^3 + (ax-by)^3 - (ay-bx)^3 + (ay+bx)^3\}/\{(a+b)^2x^2 - 3ab(x^2-y^2)\}$.

(25.) $\{(a^2 + b^2)^3 + b^3a^3\}/\{(a+b)^2 - ba\}$.

(26.) $\{(x^2 + xy + y^2)^4 - (x^2 - xy + y^2)^4\}/\{x^4 + 3x^2y^2 + y^4\}$.

(27.) $\{(x+y)^7 - x^7 - y^7\}/(x^2 + xy + y^2)^2$.

(28.) $\{(x+1)^6 - x^6 - 1\}/\{x^2 + x + 1\}$.

(29.) $\{ab(x^2 + y^2) + xy(a^2 + b^2)\}/\{ab(x^2 - y^2) - xy(a^2 - b^2)\}$.

(30.) $(a^5 + 2a^3b^2 + 2a^2b^3 - 3b^5)/(a^2 - 2ab + b^2)$.

(31.) $(x^4-3x^2-2x+4)/(x+2)$.

(32.) $(x^4-4x^3-34x^2+76x+105)/(x-7)$.

(33.) Find the remainder when x^3-6x^2+8x-9 is divided by $2x+3$.

(34.) Find the remainder when px^3+qx^2+qx+p is divided by $x-1$; and find the condition that the function in question be exactly divisible by x^2-1.

(35.) Find the condition that $Ax^{2m}+Bx^my^n+Cy^{2n}$ be exactly divisible by Px^m+Qy^n.

(36.) Find the conditions that x^3+ax^2+bx+c be exactly divisible by x^2+px+q.

(37.) If $x-a$ be a factor of $x^2+2ax-3b^2$, then $a=\pm b$.

(38.) Determine λ, μ, ν, in order that $x^4+3x^3+\lambda x^2+\mu x+\nu$ be exactly divisible by $(x^2-1)(x+2)$.

(39.) If $x^4+4x^3+6px^2+4qx+r$ be exactly divisible by x^3+3x^2+9x+3, find p, q, r.

(40.) Show that $px^3+(p^2+q)x^2+(2pq+r)x+q^2+s$ and $px^3+(p^2-q)x^2+rx-q^2+s$ either both are, or both are not, exactly divisible by x^2+px+q.

(41.) Find the condition that $(x^m+x^{m-1}+\ldots+1)/(x^n+x^{n-1}+\ldots+1)$ be integral.

(42.) Expand $1/(3x+1)$ in a series of ascending, and also in a series of descending, powers of x; and find in each case the residue after $n+1$ terms.

(43.) Express $1/(a^2-ax+x^2)$ in the form $A+Bx+Cx^2+Dx^3+R$, where A, B, C, D are constants and R a certain rational function of x.

(44.) Divide $1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots$ by $1-x$.

(45.) Show that, if $y<1$, then approximately $1/(1+y)=1-y$, $1/(1-y)=1+y$, the error in each case being $100y^2$ per cent.

Find similar approximations for $1/(1+y)^n$ and $1/(1-y)^n$, where n is a positive integer.

(46.) If $a>1$, show that $a^n>1+n(a-1)$, n being a positive integer. Hence, show that when n is increased without limit a^n becomes infinitely great or infinitely small according as $a>$ or <1.

(47.) Show that when an integral function $f(x)$ is divided by $(x-a_1)(x-a_2)$ the remainder is $\{f(a_2)(x-a_1)-f(a_1)(x-a_2)\}/(a_2-a_1)$. Generalise this theorem.

(48.) Show that $f(x)-f(a)$ is exactly divisible by $x-a$; and that, if $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n$, then the quotient is $\chi(x)=p_0x^{n-1}+(p_0a+p_1)x^{n-2}+(p_0a^2+p_1a+p_2)x^{n-3}+\ldots+(p_0a^{n-1}+p_1a^{n-2}+\ldots+p_{n-1})$.

Hence show that when $f(x)$ is divided by $(x-a)^2$ the remainder is $\chi(a)(x-a)+f(a)$,

where
$$f(a)=p_0a^n+p_1a^{n-1}+\ldots+p_n,$$
$$\chi(a)=np_0a^{n-1}+(n-1)p_1a^{n-2}+\ldots+p_{n-1}.$$

(49.) If $x^n+p_1x^{n-1}+\ldots+p_n$ and $x^{n-1}+q_1x^{n-2}+\ldots+q_{n-1}$ have the same linear factors with the exception of $x-a$, which is a factor in the first only, find the relations connecting the coefficients of the two functions.

(50.) If, when $y+c$ is substituted for x in $x^n+a_1x^{n-1}+\ldots+a_n$, the

result is $y^n+b_1y^{n-1}+\ldots+b_n$, show that $b_n, b_{n-1}, \ldots, b_1$ are the remainders when the original function is divided by $x-c$, and the successive quotients by $x-c$. Hence obtain the result of substituting $y+3$ for x in $x^5-15x^4+20x^3-17x^2-x+3$.

(51.) Express $(x^2+3x+1)^4$ in the form $A+B(x+2)+C(x+2)^2+\&c.$, and also in the form $Ax+B+(Cx+D)(x^2+x+1)+(Ex+F)(x^2+x+1)^2+\&c.$

(52.) Express $x^4+x^3+x^2+x+1$ in the form $A_0+A_1(x+1)+A_2(x+1)(x+3)+A_3(x+1)(x+3)(x+5)+A_4(x+1)(x+3)(x+5)(x+7)$.

(53.) If, when P and P′ are divided by D, the remainders are R and R′, show that, when PP′ and RR′ are divided by D, the remainders are identical.

(54.) When P is divided by D the remainder is R; and when the integral quotient obtained in this division is divided by D′ the remainder is S and the integral quotient Q. R′, S′, Q′ are the corresponding functions obtained by first dividing by D′ and then by D. Show that $Q=Q'$, and that each is the integral quotient when P is divided by DD′; also that $SD+R=S'D'+R'$, and that each of these is the remainder when P is divided by DD′.

CHAPTER VI.

Greatest Common Measure and Least Common Multiple.

§ 1.] Having seen how to test whether one given integral function is exactly divisible by another, and seen how in certain cases to find the divisors of a given integral function, we are naturally led to consider the problem—Given two integral functions, to find whether they have any common divisor or not.

We are thus led to lay down the following definitions:—

Any integral function of x which divides exactly two or more given integral functions of x is called a common measure of these functions.

The integral function of highest degree in x which divides exactly each of two or more given integral functions of x is called the greatest common measure (G.C.M.) of these functions.

§ 2.] A more general definition might be given by supposing that there are any number of variables, x, y, z, u, &c.; in that case the functions must all be integral in x, y, z, u, &c., and the degree must be reckoned by taking all these variables into account. This definition is, however, of comparatively little importance, as it has been applied in practice only to the case of monomial functions, and even there it is not indispensable. As it has been mentioned, however, we may as well exemplify its use before dismissing it altogether.

Let the monomials be $432a^4b^2x^2y^4z$, $270a^2b^3x^2y^6z^2$, $90a^3bx^3y^3z^2$, the variables being x, y, z, then the G.C.M. is x^2y^3z, or Cx^2y^3z, where C is a constant coefficient (that is, does not depend on the variables x, y, z).

The general rule, of which the above is a particular case, is as follows:—

The G.C.M. of any number of monomials is the product of the variables, each raised to the lowest power in which it occurs in any one of the given functions.*

This product may of course be multiplied by any constant coefficient.

G.C.M. OBTAINED BY INSPECTION.

§ 3.] Returning to the practically important case of integral functions of one variable x, let us consider the case of a number of integral functions, P, P′, P″, &c., each of which has been resolved into a product of positive integral powers of certain factors of the 1st degree, say $x-\alpha$, $x-\beta$, $x-\gamma$, &c.; so that

$$\begin{aligned}
P &= p(x-\alpha)^a(x-\beta)^b(x-\gamma)^c\ldots,\\
P' &= p'(x-\alpha)^{a'}(x-\beta)^{b'}(x-\gamma)^{c'}\ldots,\\
P'' &= p''(x-\alpha)^{a''}(x-\beta)^{b''}(x-\gamma)^{c''}\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

By § 15 of chap. v., we know that every measure of P can contain only powers of those factors of the 1st degree that occur in P, and can contain none of those factors in a higher power than that in which it occurs in P, and the same is true for P′, P″, &c. Hence every common measure of P, P′, P″, &c., can contain only such factors as are common to P, P′, P″, &c. *Hence the greatest common measure of* P, P′, P″, *&c., contains simply all the factors that are common to* P, P′, P″, *&c., each raised to the lowest power in which it occurs in any one of these functions.*

Since mere numbers or constant letters have nothing to do with questions relating to the integrality or degree of algebraical functions, the G.C.M. given by the above rule may of course be multiplied by any numerical or constant coefficient.

Example 1.

$$\begin{aligned}
P &= 2x^2-6x+\ \ 4=2(x-1)(x-2),\\
P' &= 6x^2-6x-12=6(x+1)(x-2).
\end{aligned}$$

Hence the G.C.M. of P and P′ is $x-2$.

Example 2.

$$\begin{aligned}
P\ &= x^5-5x^4+7x^3+x^2-8x+4=(x-1)^2(x+1)(x-2)^2,\\
P'\ &= x^6-7x^5+17x^4-13x^3-10x^2+20x-8=(x-1)^2(x+1)(x-2)^3,\\
P'' &= x^5-3x^4-x^3+7x^2-4=(x-1)(x+1)^2(x-2)^2.
\end{aligned}$$

The G.C.M. is $(x-1)(x+1)(x-2)^2$, that is, $x^4-4x^3+3x^2+4x-4$.

* If any variable does not occur at all in one or more of the given functions, it must of course be omitted altogether in the G.C.M.

§ 4.] It will be well at this stage to caution the student against being misled by the analogy between the algebraical and the arithmetical G.C.M. He should notice that no mention is made of arithmetical magnitude in the definition of the algebraical G.C.M. The word "greatest" used in that definition refers merely to degree. It is not even true that the arithmetical G.C.M. of the two numerical values of two given functions of x, obtained by giving x any particular value, is the arithmetical value of the algebraical G.C.M. when that particular value of x is substituted therein; nor is it possible to frame any definition of the algebraical G.C.M. so that this shall be true.*

The student will best satisfy himself of the truth of this remark by studying the following example:—

The algebraical G.C.M. of x^2-3x+2 and x^2-x-2 is $x-2$. Now put $x=31$. The numerical values of the two functions are 870 and 928 respectively; the numerical value of $x-2$ is 29; but the arithmetical G.C.M. of 870 and 928 is not 29 but 58.

LONG RULE FOR G.C.M.

§ 5.] In chap. v. we have seen that in certain cases integral functions can be resolved into factors; but no general method for accomplishing this resolution exists apart from the theory of equations. Accordingly the method given in § 3 for finding the G.C.M. of two integral functions is not one of perfectly general application.

The problem admits, however, of an elementary solution by a method which is fundamental in many branches of algebra. This solution rests on the following proposition:—

If $A=BQ+R$, A, B, Q, R *being all integral functions of* x, *then the G.C.M. of* A *and* B *is the same as the G.C.M. of* B *and* R.

To prove this we have to show—1st, that every common

* To avoid this confusion some writers on algebra have used instead of the words "greatest common measure" the term "*highest* common *factor*." We have adhered to the time-honoured nomenclature because the innovation in this case would only be a partial reform. The very word *factor* itself is used in totally different senses in algebra and in arithmetic; and the same is true of the words *fractional* and *integral*, with regard to which confusion is no less common. As no one seriously proposes to alter the whole of the terminology of the four species in algebra, it seems scarcely worth the while to disturb an old friend like the G.C.M.

divisor of B and R divides A and B, and, 2nd, that every common divisor of A and B divides B and R.

Now, since $A = BQ + R$, it follows, by § 4 of chap. v., that every common divisor of B and R divides A, hence every common divisor of B and R divides A and B.

Again, $R = A - BQ$, hence every common divisor of A and B divides R, hence every common divisor of A and B divides B and R.

Let now A *and* B *be two integral functions whose G.C.M. is required; and let* B *be the one whose degree is not greater than that of the other. Divide* A *by* B, *and let the quotient be* Q_1, *and the remainder* R_1.

Divide B *by* R_1, *and let the quotient be* Q_2, *and the remainder* R_2.

Divide R_1 *by* R_2, *and let the quotient be* Q_3, *and the remainder* R_3, *and so on.*

Since the degree of each remainder is less by unity at least than the degree of the corresponding divisor, R_1, R_2, R_3, *&c., go on diminishing in degree, and the process must come to an end in one or other of two ways.*

I. *Either the division at a certain stage becomes exact, and the remainder vanishes;*

II. *Or a stage is reached at which the remainder is reduced to a constant.*

Now we have, by the process of derivation above described,

$$\left.\begin{aligned} A &= BQ_1 + R_1 \\ B &= R_1Q_2 + R_2 \\ R_1 &= R_2Q_3 + R_3 \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ R_{n-2} &= R_{n-1}Q_n + R_n \end{aligned}\right\} (1).$$

Hence by the fundamental proposition the pairs of functions

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} \left.\begin{matrix} B \\ R_1 \end{matrix}\right\} \left.\begin{matrix} R_1 \\ R_2 \end{matrix}\right\} \left.\begin{matrix} R_2 \\ R_3 \end{matrix}\right\} \dots \left.\begin{matrix} R_{n-2} \\ R_{n-1} \end{matrix}\right\} \left.\begin{matrix} R_{n-1} \\ R_n \end{matrix}\right\} \text{all have the same G.C.M.}$$

In Case I. $R_n = 0$ and $R_{n-2} = Q_nR_{n-1}$. Hence the G.C.M. of R_{n-2} and R_{n-1}, that is, of Q_nR_{n-1} and R_{n-1}, is R_{n-1}, for this divides both, and no function of higher degree than itself can divide R_{n-1}. Hence R_{n-1} is the G.C.M. of A and B.

In Case II. R_n = constant. In this case A and B have no G.C.M., for their G.C.M. is the G.C.M. of R_{n-1} and R_n, that is, their G.C.M. divides the constant R_n. But no integral function

(other than a constant) can divide a constant exactly. Hence A and B have no G.C.M. (other than a constant).

If, therefore, the process ends with a zero remainder, the last divisor is the G.C.M.; if it ends with a constant, there is no G.C.M.

§ 6.] It is important to remark that it follows from the nature of the above process for finding the G.C.M., which consists essentially in substituting for the original pair of functions pair after pair of others which have the same G.C.M., that *we may, at any stage of the process, multiply either the divisor or the remainder by an integral function, provided we are sure that this function and the remainder or divisor, as the case may be, have no common factor. We may similarly remove from either the divisor or the remainder a factor which is not common to both. We may remove a factor which is common to both, provided we introduce it into the G.C.M. as ultimately found. It follows of course,* a fortiori, *that a numerical factor may be introduced into or removed from divisor or remainder at any stage of the process.* This last remark is of great use in enabling us to avoid fractions and otherwise simplify the arithmetic of the process. In order to obtain the full advantage of it, the student should notice that, in what has been said, "remainder" may mean, not only the remainder properly so called at the end of each separate division, but also, if we please, the "*remainder in the middle of any such division,*" or "residue," as we called it in § 18, chap. v.

Some of these remarks are illustrated in the following examples:—

Example 1.

To find the G.C.M. of $x^5-2x^4-2x^3+8x^2-7x+2$ and x^4-4x+3.

$$
\begin{array}{rrrrrrrl}
 & x^5 & -2x^4 & -2x^3 & +8x^2 & -7x & +2 & \big| \; x^4-4x+3 \\
 & x^5 & & & -4x^2 & +3x & & \big| \; x+1 \\ \hline
-2) & & -2x^4 & -2x^3 & +12x^2 & -10x & +2 & \\ \hline
 & & x^4 & +x^3 & -6x^2 & +5x & -1 & \\
 & & x^4 & & & -4x & +3 & \\ \hline
 & & & x^3 & -6x^2 & +9x & -4 &
\end{array}
$$

$$
\begin{array}{rrrrrrl}
 & x^4 & & & -4x & +3 & \big| \; x^3-6x^2+9x-4 \\
 & x^4 & -6x^3 & +9x^2 & -4x & & \big| \; x+2 \\ \hline
3) & & 6x^3 & -9x^2 & & +3 & \\ \hline
 & & 2x^3 & -3x^2 & & +1 & \\
 & & 2x^3 & -12x^2 & +18x & -8 & \\ \hline
9) & & & 9x^2 & -18x & +9 & \\ \hline
 & & & x^2 & -2x & +1 &
\end{array}
$$

$$\begin{array}{r|l}
x^3-6x^2+9x-4 & x^2-2x+1 \\
x^3-2x^2+x & x+1 \\
\hline
-4)\ \ -4x^2+8x-4 & \\
x^2-2x+1 & \\
x^2-2x+1 & \\
\hline
0 &
\end{array}$$

Hence the G.C.M. is x^2-2x+1.

It must be observed that what we have written in the place of quotients are not really quotients in the ordinary sense, owing to the rejection of the numerical factors here and there. In point of fact the quotients are of no importance in the process, and need not be written down; neglecting them, carrying out the subtractions mentally, and using detached coefficients, we may write the whole calculation in the following compact form :—

$$\begin{array}{rr|rl}
 & 1-2-2+8-7+2 & 1+0+0-4+3 & \\
\div -2 & -2-2+12-10+2 & 6-9+0+3 & \div 3 \\
 & 1+1-6+5-1 & 2-3+0+1 & \\
 & 1-6+9-4 & 9-18+9 & \div 9 \\
\div -4 & -4+8-4 & 1-2+1 & \\
 & 1-2+1 & & \\
 & 0 & &
\end{array}$$

G.C.M., x^2-2x+1.

Example 2.

Required the G.C.M. of $4x^4+26x^3+41x^2-2x-24$ and $3x^4+20x^3+32x^2-8x-32$.

Bearing in mind the general principle on which the rule for finding the G.C.M. is founded, we may proceed as follows, in order to avoid large numbers as much as possible :—

$$\begin{array}{rr|rl}
 & 4+26+41-2-24 & 3+20+32-8-32 & \\
\times 2 & 1+6+9+6+8 & 2+5-26-56 & \\
 & 2+12+18+12+16 & -53-318-424 & \div -53 \\
 & 7+44+68+16 & 1+6+8 & \\
 & 1+29+146+184 & & \\
\div 23 & 23+138+184 & & \\
 & 1+6+8 & & \\
 & 0 & &
\end{array}$$

The G.C.M. is x^2+6x+8.

Here the second line on the left is obtained from the first by subtracting the first on the right. By the general principle referred to, the function $x^4+6x^3+9x^2+6x+8$ thus obtained and $3x^4+20x^3+32x^2-8x-32$ have the same G.C.M. as the original pair. Similarly the fifth line on the left is the result of subtracting from the line above three times the second line on the right.

If the student be careful to pay more attention to the principle underlying the rule than to the mere mechanical application of it, he will have little difficulty in devising other modifications of it to suit particular cases.

METHOD OF ALTERNATE DESTRUCTION OF HIGHEST AND LOWEST TERMS.

§ 7.] *If l, m, p, q be constant quantities (such that $lq - mp$ is not zero), and if*

$$P = lA + mB \qquad (1),$$
$$Q = pA + qB \qquad (2),$$

where A *and* B, *and therefore* P *and* Q, *are integral functions, then the G.C.M. of* P *and* Q *is the G.C.M. of* A *and* B.

For it is clear from the equations as they stand that every divisor of A and B divides both P and Q. Again, we have

$$qP - mQ = q(lA + mB) - m(pA + qB) = (lq - mp)A \qquad (3),$$
$$-pP + lQ = -p(lA + mB) + l(pA + qB) = (lq - mp)B \qquad (4);$$

hence (provided $lq - mp$ does not vanish), since l, p, m, q, and therefore $lq - mp$, are all constant, it follows that every divisor of P and Q divides A and B. Thus the proposition is proved.

In practice l and m and p and q are so chosen that the highest term shall disappear in $lA + mB$, and the lowest in $pA + qB$. The process will be easily understood from the following example:—

Example 1.

$$\text{Let } A = 4x^4 + 26x^3 + 41x^2 - 2x - 24,$$
$$B = 3x^4 + 20x^3 + 32x^2 - 8x - 32;$$

then
$$-3A + 4B = 2x^3 + 5x^2 - 26x - 56,$$
$$4A - 3B = 7x^4 + 44x^3 + 68x^2 + 16x.$$

Rejecting now the factor x, which clearly forms no part of the G.C.M., we have to find the G.C.M. of

$$A' = 7x^3 + 44x^2 + 68x + 16,$$
$$B' = 2x^3 + 5x^2 - 26x - 56.$$

Repeating the above process—

$$2A' - 7B' = 53x^2 + 318x + 424,$$
$$7A' + 2B' = 53x^3 + 318x^2 + 424x.$$

the G.C.M. of which is $53x^2+318x+424$. Hence this, or, what is equivalent so far as the present quest is concerned, x^2+6x+8, is the G.C.M. of the two given functions.

When the functions differ in degree, we may first destroy the lowest term in the function of higher degree, divide the result by x, and replace the function of higher degree by the new function thus obtained. We can proceed in this way until we arrive at two functions of the same degree, which can in general be dealt with by destroying alternately the highest and lowest terms.

Detached coefficients may be used as in the following example:

Example 2.

To find the G.C.M. of $x^4-3x^3+2x^2+x-1$ and x^3-x^2-2x+2, we have the following calculation:—

A	$1-3+2+1-1$
B	$1-1-2+2$
$A' = (2A+B)/x$	$2-5+3+0$
$B' = B$	$1-1-2+2$
$A'' = A'/x$	$2-5+3$
$B'' = (-A'+2B')$	$3-7+4$
$A''' = (-4A''+3B'')/x$	$1-1$
$B''' = -3A''+2B''$	$1-1$

Hence the G.C.M. is $x-1$.

The failing case of the original process, where $lq-mp=0$, may be treated in a similar manner, the exact details of which we leave to be worked out as an exercise by the learner.

§ 8.] The following example shows how, by a semi-tentative process, the desired result may often be obtained very quickly:—

Example.

$$A=2x^4-3x^3-3x^2+4,$$
$$B=2x^4-x^3-9x^2+4x+4.$$

Every common divisor of A and B divides A − B, that is, $-2x^3+6x^2-4x$, that is, rejecting the numerical factor -2, $x(x^2-3x+2)$, that is, $x(x-1)(x-2)$. We have therefore merely to select those factors of $x(x-1)(x-2)$ which divide both A

and B. x clearly is not a common divisor, but we see at once, by the remainder theorem (§ 13, chap. v.), that both $x-1$ and $x-2$ are common divisors. Hence the G.C.M. is $(x-1)(x-2)$, or x^2-3x+2.

§ 9.] The student should observe that the process for finding the G.C.M. has the valuable peculiarity not only of furnishing the G.C.M., but also of indicating when there is none.

Example.

$$A = x^2 - 3x + 1,$$
$$B = x^2 - 4x + 6.$$

Arranging the calculation in the abridged form, we have

$$\begin{array}{r|r} 1-3+1 & 1-4+6 \\ 2+1 & -1+5 \\ \hline 11 & \end{array}$$

The last remainder being 11, it follows that there is no G.C.M.

G.C.M. OF ANY NUMBER OF INTEGRAL FUNCTIONS.

§ 10.] It follows at once, by the method of proof given in § 5, that *every common divisor of two integral functions* A *and* B *is a divisor of their G.C.M.*

This principle enables us at once to find the G.C.M. of any number of integral functions by successive application of the process for two. Consider, for example, four functions, A, B, C, D. Let G_1 be the G.C.M. of A and B, then G_1 is divisible by every common divisor of A and B. Find now the G.C.M. of G_1 and C, G_2 say. Then G_2 is the divisor of highest degree that will divide A, B, and C. Finally, find the G.C.M. of G_2 and D, G_3 say. Then G_3 is the G.C.M. of A, B, C, and D.

GENERAL PROPOSITIONS REGARDING ALGEBRAICAL PRIMENESS.

§ 11.] We now proceed to establish a number of propositions for integral functions analogous to those given for integral numbers in chap. iii., again warning the student that he must not confound the algebraical with the arithmetical results;

although he should allow the analogy to lead him in seeking for the analogous propositions, and in devising methods for proving them.

Definition.—*Two integral functions are said to be prime to each other when they have no common divisor.*

Proposition.—A *and* B *being any two integral functions, there exist always two integral functions,* L *and* M, *prime to each other, such that, if* A *and* B *have a G.C.M.,* G, *then*

$$LA + MB = G\,;$$

and, if A *and* B *be prime to each other,*

$$LA + MB = 1.$$

To prove this, we show that any one of the remainders in the process for finding the G.C.M. of A and B may be put into the form PA + QB, where P and Q are integral functions of x.

We have, from the equalities of § 5,

$$R_1 = A - Q_1B \qquad (1),$$
$$R_2 = B - Q_2R_1 \qquad (2),$$
$$R_3 = R_1 - Q_3R_2 \qquad (3),$$
$$\cdots\cdots\cdots$$
$$R_n = R_{n-2} - Q_nR_{n-1} \qquad (4).$$

Equation (1) at once establishes the result for R_1 (only here $P = 1$, $Q = -Q_1$).

From (2), using the value of R_1 given by (1),

$$R_2 = B - Q_2(A - Q_1B) = (-Q_2)A + (+1 + Q_1Q_2)B,$$

which establishes the result for R_2.

From (3), using the results already obtained, we get

$$\begin{aligned} R_3 &= A - Q_1B - Q_3\{(-Q_2)A + (+1 + Q_1Q_2)B\} \\ &= (1 + Q_2Q_3)A + (-Q_1 - Q_3 - Q_1Q_2Q_3)B, \end{aligned}$$

which establishes the result for R_3, since Q_1, Q_2, Q_3 are all integral functions. Similarly we establish the result for R_4, R_5, &c.

Now, if A and B have a G.C.M., this is the last remainder which does not vanish, and therefore we must have

$$G = LA + MB \qquad (I.),$$

where L and M are integral functions; and these must be prime to each other, for, since G divides both A and B, A/G ($= a$ say) and B/G ($= b$ say) are integral functions; we have therefore, dividing both sides of (I.) by G,

$$1 = La + Mb;$$

so that any common divisor of L and M would divide unity.

If A and B have no G.C.M., the last remainder, R_n, is a constant; and we have, say, $R_n = L'A + M'B$, where L′ and M′ are integral functions. Dividing both sides by the constant R_n, and putting $L = L'/R_n$, $M = M'/R_n$, so that L and M are still integral functions, we have

$$1 = LA + MB \qquad \text{(II.)}.$$

Here again it is obvious that L and M have no common divisor, for such divisor, if it existed, would divide unity.

The proposition just proved is of considerable importance in algebraical analysis. We proceed to deduce from it several conclusions, the independent proof of which, by methods more analogous to those of chap. iii., § 10, we leave as an exercise to the learner. Unless the contrary is stated, all the letters used denote integral functions of x.

§ 12.] *If* A *be prime to* B, *then any common divisor of* AH *and* B *must divide* H.

For, since A is prime to B, we have

$$LA + MB = 1,$$

whence

$$LAH + MBH = H,$$

which shows that any common divisor of AH and B divides H.

If A and B have a G.C.M. a somewhat different proposition may be established by the help of equation (I.) of § 11. The discovery and proof of this may be left to the reader.

Cor. 1. *If* B *divide* AH *and be prime to* A, *it must divide* H.

Cor. 2. *If* A′ *be prime to each of the functions* A, B, C, *&c.*, *it is prime to their product* ABC . . .

Cor. 3. *If each of the functions* A, B, C, . . . *be prime to each of the functions* A′, B′, C′, . . ., *then the product* ABC . . . *is prime to the product* A′B′C′. . . .

Cor. 4. *If* A *be prime to* A′, *then* A^a *is prime to* $A'^{a'}$, *a and a′ being any positive integers.*

Cor. 5. *If a given set of integral functions be each resolved into a product of powers of the integral factors* A, B, C, . . ., *which are prime to each other, then the G.C.M. of the set is found by writing down the product of all the factors that are common to all the given functions, each raised to the lowest power in which it occurs in any of these functions.*

This is a generalisation of § 3 above.

After what has been done it seems unnecessary to add detailed proofs of these corollaries.

LEAST COMMON MULTIPLE.

§ 13.] Closely allied to the problem of finding the G.C.M. of a set of integral functions is the problem of finding *the integral function of least degree which is divisible by each of them.* This function is called their *least common multiple* (*L.C.M.*).

§ 14.] As in the case of the G.C.M., the degree may, if we please, be reckoned in terms of more variables than one; thus the L.C.M. of the monomials $3x^3yz^2$, $6x^2y^3z^4$, $8xyzu$, the variables being x, y, z, u, is $x^3y^3z^4u$, or any constant multiple thereof.

The general rule clearly is to *write down all the variables, each raised to the highest power in which it occurs in any of the monomials.*

§ 15.] Confining ourselves to the case of integral functions of a single variable x, let us investigate what are the essential factors of every common multiple of two given integral functions A and B. Let G be the G.C.M. of A and B (if they be prime to each other we may put G = 1); then

$$A = aG, \quad B = bG,$$

where a and b are two integral functions which are prime to each

other. Let M be any common multiple of A and B. Since M is divisible by A, we must have

$$M = PA,$$

where P is an integral function of x.

Therefore $$M = PaG.$$

Again, since M is divisible by B, that is, by bG, therefore M/bG, that is, PaG/bG, that is, Pa/b must be an integral function. Now b is prime to a; hence, by § 12, b must divide P, that is, $P = Qb$, where Q is integral. Hence finally

$$M = QabG.$$

This is the general form of all common multiples of A and B.

Now a, b, G are given, and the part which is arbitrary is the integral function Q. Hence we get the *least* common multiple by making the degree of Q as small as possible, that is, by making Q any constant, unity say. The L.C.M. of A and B is therefore abG, or $(aG)(bG)/G$, that is, AB/G. In other words, *the L.C.M. of two integral functions is their product divided by their G.C.M.*

§ 16.] The above reasoning also shows that *every common multiple of two integral functions is a multiple of their least common multiple.*

The converse proposition, that every multiple of the L.C.M. is a common multiple of the two functions, is of course obvious.

These principles enable us to find the L.C.M. of a set of any number of integral functions A, B, C, D, &c. For, if we find the L.C.M., L_1 say, of A and B; then the L.C.M., L_2 say, of L_1 and C; then the L.C.M., L_3 say, of L_2 and D, and so on, until all the functions are exhausted, it follows that the last L.C.M. thus obtained is the L.C.M. of the set.

§ 17.] The process of finding the L.C.M. has neither the theoretical nor the practical importance of that for finding the G.C.M. In the few cases where the student has to solve the problem he will probably be able to use the following more direct process, the foundation of which will be obvious after what has been already said.

If a set of integral functions can all be exhibited as powers of a

set of integral factors A, B, C, &c., which are either all of the 1st degree and all different, or else are all prime to each other, then the L.C.M. of the set is the product of all these factors, each being raised to the highest power in which it occurs in any of the given functions.

For example, let the functions be

$$(x-1)^2(x^2+2)^3(x^2+x+1),$$
$$(x-2)^2(x-3)\,(x^2-x+1)^2,$$
$$(x-1)^5(x-2)^3(x-3)^4(x^2+x+1)^3,$$

then, by the above rule, the L.C.M. is

$$(x-1)^5(x-2)^3(x-3)^4(x^2+2)^3(x^2+x+1)^3(x^2-x+1)^2.$$

EXERCISES X.

Find the G.C.M. of the following, or else show that they have no C.M.

(1.) $(x^2-1)^2, \quad x^6-1.$

(2.) $x^6-1, \quad x^4-2x^3+3x^2-2x+1.$

(3.) $x^4-x^2+1, \quad x^4+x^2+1.$

(4.) $x^9+1, \quad x^{11}+1.$

(5.) $x^3-x^2-8x+12, \quad x^3+4x^2-3x-18.$

(6.) $x^4-7x^3-22x^2+139x+105, \quad x^4-8x^3-11x^2+116x+70.$

(7.) $x^4-286x^2+225, \quad x^4+14x^3-480x^2-690x-225.$

(8.) $x^6-x^4-8x^2+12, \quad x^6+4x^4-3x^2-18.$

(9.) $x^5-2x^4-2x^3+4x^2+x-2, \quad x^5+2x^4-2x^3-8x^2-7x-2.$

(10.) $x^8+6x^6-8x^4+1, \quad x^{12}+7x^{10}-3x^8-3x^2-2.$

(11.) $12x^3+13x^2+6x+1, \quad 16x^3+16x^2+7x+1.$

(12.) $5x^3+38x^2-195x-600, \quad 4x^3-15x^2-38x+65.$

(13.) $16x^4-56x^3-88x^2+278x+105, \quad 16x^4-64x^3-44x^2+232x+70.$

(14.) $7x^4+6x^3-8x^2-6x+1, \quad 11x^4+15x^3-2x^2-5x+1.$

(15.) $x^4+64a^4, \quad (x+2a)^4-16a^4.$

(16.) $9x^4+4x^2+1, \quad 3\sqrt{2}x^3+x^2+1.$

(17.) $x^3+3px^2-(1+3p), \quad px^3-3(1+3p)x+(3+8p).$

(18.) $x^3-3(a-b)x^2+(4a^2-3ab)x-2a^2(2a-3b),$
$\quad x^4-(3a+b)x^3+(5a^2+2ab)x^2-a^2(5a+3b)x+2a^3(a+b).$

(19.) $nx^{n+1}-(n+1)x^n+1, \quad x^n-nx+(n-1).$

(20.) Show that x^3+px^2+qx+1, x^3+qx^2+px+1 cannot have a common measure, unless either $p=q$ or $p+q+2=0$.

(21.) Show that, if ax^2+bx+c, cx^2+bx+a have a common measure of the 1st degree, then $a\pm b+c=0$.

(22.) Find the value of a for which $\{x^3-ax^2+19x-a-4\}/\{x^3-(a+1)x^2+23x-a-7\}$ admits of being expressed as the quotient of two integral functions of lower degree.

(23.) If ax^3+3bx^2+d, $bx^3+3dx+e$ have a common measure, then $(ae-4bd)^3=27(ad^2+b^2e)^2$.

(24.) $Ax^2+Bxy+Cy^2$, $Bx^2-2(A-C)xy-By^2$ cannot have a common measure unless the first be a square.

(25.) ax^3+bx^2+cx+d, dx^3+cx^2+bx+a will have a common measure of the 2nd degree if

$$\frac{abc-a^2b-b^2d+acd}{ac-bd}=\frac{ac^2-bcd-a^3+ad^2}{ab-cd}=\frac{d(ac-bd)}{a^2-d^2};$$

show that these conditions are equivalent to only one, namely, $ac-bd=a^2-d^2$.

(26.) Find two integral functions P and Q, such that

$$P(x^2-3x+2)+Q(x^2+x+1)=1.$$

(27.) Find two integral functions P and Q, such that

$$P(2x^3-7x^2+7x-2)+Q(2x^3+x^2+x-1)=2x-1.$$

Find the L.C.M. of the following:—

(28.) a^5-ab^4, a^9+a^8b, $a^6+b^6+a^2b^2(a^2+b^2)$.

(29.) $x^3-x^2-14x+24$, x^3-2x^2-5x+6, x^2-4x+3.

(30.) $3x^3+x^2-8x+4$, $3x^3+7x^2-4$, x^3+2x^2-x-2, $3x^3+2x^2-3x-2$.

(31.) $x^3-12x+16$, $x^4-4x^3-x^2+20x-20$, $x^4+3x^3-11x^2-3x+10$.

(32.) $x^6+2ax^5+a^2x^4+5a^5x+a^6$, $x^3+a^2x-ax^2-a^3$.

(33.) If x^2+ax+b, $x^2+a'x+b'$ have a common measure of the 1st degree, then their L.C.M. is

$$x^3+\frac{ab-a'b'}{b-b'}x^2+\left\{aa'-\left(\frac{b-b'}{a-a'}\right)^2\right\}x+bb'\frac{a-a'}{b-b'}.$$

(34.) Show that the L.C.M. of two integral functions A and B can always be expressed in the form PA+QB, where P and Q are integral functions.

CHAPTER VII.

On the Resolution of Integral Functions into Factors.

§ 1.] Having seen how to determine whether any given integral function is a factor in another or not, and how to determine the factor of highest degree which is common to two integral functions, it is natural that we should put to ourselves the question, How can any given integral function be resolved into integral factors?

TENTATIVE METHODS.

§ 2.] Confining ourselves at present to the case where factors of the 1st degree, whose coefficients are rational integral functions of the coefficients of the given function, are suspected or known to exist, we may arrive at these factors in various ways.

For example, every known identity resulting from the distribution of a product of such factors, when read backwards, gives a factorisation.

Thus $(x+y)(x-y)=x^2-y^2$ tells us that x^2-y^2 may be resolved into the product of two factors, $x+y$ and $x-y$. In a similar way we learn that $x+y+z$ is a factor in $x^3+y^3+z^3-3xyz$. The student should again refer to the tables of identities given on pp. 81-83, and study it from this point of view.

When factors of the 1st degree with rational integral coefficients are known to exist, it is usually not difficult to find them by a tentative process, because the number of possible factors is limited by the nature of the case.

Example 1.

Consider $x^2-12x+32$, and let us assume that it is resolvable into $(x-a)(x-b)$.

Then we have

$$x^2-12x+32=x^2-(a+b)x+ab,$$

and we have to find a and b so that

$$ab=+32, \quad a+b=+12.$$

We remark, first, that a and b must have the same sign, since their product is positive; and that that sign must be $+$, since their sum is positive. Further, the different ways of resolving 32 into a product of integers are 1×32, 2×16, 4×8; and of these we must choose the one which gives $a+b=+12$, namely, the last, that is, $a=4$, $b=8$.

So that

$$x^2-12x+32=(x-4)(x-8).$$

Example 2.

$$x^3-2x^2-23x+60=(x-a)(x-b)(x-c) \text{ say.}$$

Here $$-abc=+60.$$

Now the divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60; and we have therefore to try $x\pm 1$, $x\pm 2$, $x\pm 3$, &c. The theorem of remainders (chap. v., § 14) at once shows that $x+1$, $x-1$, $x+2$, $x-2$, are all inadmissible. On the other hand, for $x-3$ we have (see chap. v., § 13)

$$\begin{array}{r} 1-2-23+60 \\ 0+3+\ \ 3-60 \\ \hline 1+1-20+\ \ 0 \end{array}$$

that is, $x-3$ *is* a factor; and the other factor is x^2+x-20, which we resolve by inspection, or as in Example 1, into $(x-4)(x+5)$.

Hence $$x^3-2x^2-23x+60=(x-3)(x-4)(x+5).$$

Example 3.

$$6x^2-19x+15=(ax+b)(cx+d).$$

Here $ac=+6$, $bd=+15$; and we have more cases to consider. We might have any one of the 32 factors, $x\pm 1$, $x\pm 3$, $x\pm 5$, $x\pm 15$, $2x\pm 1$, $2x\pm 3$, $2x\pm 5$, $2x\pm 15$, &c. A glance at the middle coefficient, -19, at once excludes a large number of these, and we find, after a few trials,

$$6x^2-19x+15=(2x-3)(3x-5).$$

§ 3.] In cases like those of last section, we can often detect a factor by suitably grouping the terms of the given function. For it follows from the general theory of integral functions already established that, if P can be arranged as the sum of a series of groups in each of which Q is a factor, then Q is a factor in P; and, if P can be arranged as the sum of a series of groups in each of which Q is a factor, plus a group in which Q is not a factor, then Q is not a factor in P.

Example 1.

$$\begin{aligned}&x^3-2x^2-23x+60\\&=x^2(x-2)-23(x-2)+14,\end{aligned}$$

that is, $x-2$ is not a factor.

$$\begin{aligned}&x^3-2x^2-23x+60\\&=x^2(x-3)+x^2-23x+60\\&=x^2(x-3)+x(x-3)-20x+60\\&=x^2(x-3)+x(x-3)-20(x-3),\end{aligned}$$

that is, $x-3$ is a factor.

Example 2.

$$\begin{aligned}&px^2+(1+pq)xy+qy^2\\&=px^2+xy+pqxy+qy^2\\&=x(px+y)+qy(px+y),\end{aligned}$$

that is, $px+y$ is a factor, the other being $x+qy$.

Example 3.

$$\begin{aligned}&x^3+(m+n+1)x^2a+(m+n+mn)xa^2+mna^3\\&=x^3+x^2a+(m+n)(x^2a+xa^2)+mn(xa^2+a^3)\\&=x^2(x+a)+(m+n)xa(x+a)+mna^2(x+a)\\&=\{x^2+(m+n)xa+mna^2\}(x+a)\\&=\{x(x+ma)+na(x+ma)\}(x+a)\\&=(x+ma)(x+na)(x+a).\end{aligned}$$

GENERAL SOLUTION FOR A QUADRATIC FUNCTION.

§ 4.] For tentative processes, such as we have been illustrating, no general rule can be given; and skill in this matter is one of those algebraical accomplishments which the student must cultivate by practice. There is, however, one case of great importance, namely, that of the integral function of the 2nd degree in one variable, for which a systematic solution can be given.

We remark, first of all, that every function of the form x^2+px+q can be made a complete square, so far as x is concerned, by the addition of a constant. Let the constant in question be a, so that we have

$$x^2+px+q+a=(x+\beta)^2=x^2+2\beta x+\beta^2,$$

β being by hypothesis another constant. Then we must have

$$p=2\beta,\quad q+a=\beta^2.$$

The first of these equations gives $\beta=p/2$, the second $a=\beta^2-q=(p/2)^2-q$. Thus our problem is solved by adding to x^2+px+q the constant $(p/2)^2-q$.

The same result is obtained for the more general form, ax^2+bx+c, as follows:—

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right).$$

Now, from the case just treated, we see that $x^2+(b/a)x+c/a$ is made a complete square in x by the addition of $(b/2a)^2-c/a$, that is, $(b^2-4ac)/4a^2$. Hence ax^2+bx+c will be made a complete square in x by the addition of $a(b^2-4ac)/4a^2$, that is, $(b^2-4ac)/4a$. We have, in fact,

$$ax^2+bx+c+\frac{b^2-4ac}{4a}=a\left(x+\frac{b}{2a}\right)^2.$$

§ 5.] The process of last article at once suggests that ax^2+bx+c can always be put into the form $a\{(x+l)^2-m^2\}$, where l and m are constant.

In point of fact we have

$$\begin{aligned}ax^2+bx+c&=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}\\&=a\left\{x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right\}\\&=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right\}.\end{aligned}$$

In other words, our problem is solved if we make $l=b/2a$ and find m, so that $m^2=(b^2-4ac)/4a^2$.

This being done, the identity $X^2-A^2=(X-A)(X+A)$ at once gives us the factorisation of ax^2+bx+c; for we have

$$\begin{aligned}ax^2+bx+c&=a\{(x+l)^2-m^2\}\\&=a\{(x+l)+m\}\{(x+l)-m\}.\end{aligned}$$

Example 1.

Consider $6x^2-19x+15$; we have

$$\begin{aligned}6x^2-19x+15&=6\left\{x^2-\frac{19}{6}x+\frac{15}{6}\right\}\\&=6\left\{x^2-2\frac{19}{12}x+\left(\frac{19}{12}\right)^2-\frac{19^2}{12^2}+\frac{15}{6}\right\}\\&=6\left\{\left(x-\frac{19}{12}\right)^2-\left(\frac{1}{12}\right)^2\right\}.\end{aligned}$$

Here $l=-\frac{19}{12}$, and $m^2=\left(\frac{1}{12}\right)^2$; so that our problem is solved if we take $m=\frac{1}{12}$.

We get, therefore,

$$\begin{aligned}6x^2 - 19x + 15 &= 6\left\{\left(x - \frac{19}{12}\right) + \frac{1}{12}\right\}\left\{\left(x - \frac{19}{12}\right) - \frac{1}{12}\right\}\\ &= 6\left(x - \frac{3}{2}\right)\left(x - \frac{5}{3}\right)\\ &= 6\left(\frac{2x-3}{2}\right)\left(\frac{3x-5}{3}\right)\\ &= (2x-3)(3x-5);\end{aligned}$$

the same result as we obtained above (in § 2, Example 3), by a tentative process.

Example 2.

Consider $x^6 - 5x^3 + 6$. We may regard this as $(x^3)^2 - 5(x^3) + 6$, that is to say, as an integral function of x^3 of the 2nd degree. We thus see that

$$\begin{aligned}x^6 - 5x^3 + 6 &= (x^3)^2 - 5(x^3) + 6,\\ &= (x^3 - 3)(x^3 - 2).\end{aligned}$$

INTRODUCTION INTO ALGEBRA OF SURD AND IMAGINARY NUMBERS.

§ 6.] The necessities of algebraic generality have already led us to introduce essentially negative quantity. So far, algebraic quantity consists of all conceivable multiples positive or negative of 1. To give this scale of quantity order and coherence, we introduce an extended definition of the words *greater than* and *less than*, as follows :—*a is said to be greater or less than b, according as $a - b$ is positive or negative.*

Example.

$(+3) - (+2) = +1$ therefore $+3 > +2$; $(-3) - (-5) = +2$ therefore $-3 > -5$;
$(+3) - (-5) = +8$ therefore $+3 > -5$; $(-7) - (-3) = -4$ therefore $-7 < -3$.

Hence it appears that, according to the above definition, any negative quantity, however great numerically, is less than any positive quantity, however small numerically ; and that, in the case of negative quantities, descending order of numerical magnitude is ascending order of algebraical magnitude.

We may therefore represent the whole ascending series of algebraical quantity, so far as we have yet had occasion to consider it, as follows :—

$$-\infty \;\ldots\; -1 \;\ldots\; -\tfrac{1}{2} \;\ldots\; 0 \;\ldots\; +\tfrac{1}{2} \;\ldots\; +1 \;\ldots\; +\infty.^*$$

* The symbol ∞ is here used as an abbreviation for a real quantity as great as we please.

The most important part of the operations in the last paragraph is the finding of the quantity m, whose square shall be equal to a given algebraical quantity. We say *algebraical*, for we must contemplate the possibility of $(b^2-4ac)/4a^2$, say k for shortness, assuming any value between $-\infty$ and $+\infty$. *When m is such that $m^2=k$, then m is called the square root of k, and we write $m=\sqrt{k}$.* We are thus brought face to face with the problem of finding the square root of any algebraical quantity; and it behoves us to look at this question somewhat closely, as it leads us to a new extension of the field of algebraical operations, similar to that which took place when we generalised addition and subtraction and thus introduced negative quantity.

1st. Let us suppose that k is a positive number, and either a square integer $=+\kappa^2$, say, or the square of a rational number $=+(\kappa/\lambda)^2$, say, where κ and λ are both integers, or, which is the same thing [since $(\kappa/\lambda)^2=\kappa^2/\lambda^2$], the quotient of two square integers. Then our problem is solved if we take

$$m=+\kappa, \text{ or } m=-\kappa$$

in the one case, or

$$m=+\kappa/\lambda, \text{ or } m=-\kappa/\lambda$$

in the other;

for
$$m^2=(\pm\kappa)^2=\kappa^2=k,$$

$$m^2=\left(\pm\frac{\kappa}{\lambda}\right)^2=\left(\frac{\kappa}{\lambda}\right)^2=k,$$

which is the sole condition required.

It is interesting to notice that we thus obtain two solutions of our problem; and it will be afterwards shown that there are no more. Either of these will do, so far as the problem of factorisation in § 5 is concerned, for all that is there required is *any one value of the square root.*

More to the present purpose is it to remark that this is the only case in which m can be rational; for if m be rational, that is, $=\pm\kappa/\lambda$ where κ and λ are integers, then $m^2=(\kappa/\lambda)^2$, that is, $k=(\kappa/\lambda)^2$, that is, k must be the square of a rational number.

2nd. Let k be positive, but *not* the square of a rational

number; then everything is as before, except that no exact arithmetical expression can be found for m. We can, by the arithmetical process for finding the square root, find a rational value of m, say v, such that $m^2 = (\pm v)^2$ shall differ from k by less than any assigned quantity, however small; but no such rational expression can be absolutely exact. In this case m is called a *surd number*. When k is positive, and not a square number, as in the present case, it is usual to use $\sqrt{k}$ to denote the mere (signless) arithmetical value of the square root, which has an actual existence, although it is not capable of exact arithmetical expression; and to denote the two algebraical values of m by $\pm\sqrt{k}$. Thus, if $k = +2$, we write $m = \pm\sqrt{2}$. In any *practical* application we use some rational approximation of sufficient accuracy; for example, if $k = +2$, and it is necessary to be exact to the 1/10,000th, we may use $m = \pm 1{\cdot}4142$.

A special chapter will be devoted to the discussion of surd numbers; all that it is necessary in the meantime to say further concerning them is, that they, or the symbols representing them, are of course to be subject to all the laws of ordinary algebra.†

3rd. Let k be negative $= -k'$, say, where k' is a mere arithmetical number. A new difficulty here arises; for, since the square of every algebraical quantity between $-\infty$ and $+\infty$ (except 0, which, of course, is not in question unless $k' = 0$) is positive, there exists no quantity m in the range of algebraical quantity, as at present constituted, which is such that $m^2 = -k'$. *If we are as hitherto to maintain the generality of all algebraical operations, the only resource is to widen the field of algebraical quantity still farther. This is done by introducing an ideal, so-called imaginary, unit commonly denoted by the letter i,* whose definition is, that it is such that*

$$i^2 = -1.$$

It is, of course, at once obvious that i has no arithmetical existence whatsoever, and does not admit of any arithmetical expression, approximate or other. We form multiples and submultiples of this unit, positive or negative, by combining it with

* Occasionally also by ι.

† See vol. ii. chap. xxv. § 28-41.

quantities of the ordinary algebraical, now for distinction called *real*, series, namely,

$$-\infty \ldots -1 \ldots -\tfrac{1}{2} \ldots 0 \ldots +\tfrac{1}{2} \ldots +1 \ldots +\infty.$$

We thus obtain a new series of *purely imaginary* quantity:—

$$-\infty i \ldots -i \ldots -\tfrac{1}{2}i \ldots 0i \ldots +\tfrac{1}{2}i \ldots +i \ldots +\infty i.$$

These new imaginary quantities must of course, like every other quantity in the science, be subject to all the ordinary laws of algebra when combined either with real quantities or with one another. All that the student requires to know, so far at least as operations with them are concerned, beyond the laws already laid down, is the defining property of the new unit i, namely, $i^2 = -1$.

When purely real and purely imaginary numbers are combined by way of algebraical addition, forms arise like $p + qi$, where p and q are real numbers positive or negative. Such forms are called *complex numbers*; and it will appear later that every algebraical function of a complex number can itself be reduced to a complex number. In other words, it comes out in the end that the field of ordinary algebraical quantity is rendered complete by this last extension.

The further consequences of the introduction of complex numbers will be developed in a subsequent chapter. In the meantime we have to show that these ideal numbers suffice for our present purpose. That this is so is at once evident; for, if we denote by $\sqrt{k'}$ the square root of the arithmetical number k', so that $\sqrt{k'}$ may be either rational or surd as heretofore, but certainly real, then $m = \pm i\sqrt{k'}$ gives two solutions of the problem in hand, since we have

$$\begin{aligned} m^2 &= (\pm i\sqrt{k'})^2 \\ &= (\pm i\sqrt{k'}) \times (\pm i\sqrt{k'}), \end{aligned}$$

upper signs going together or lower together,

$$\begin{aligned} &= (i^2) \times (\sqrt{k'})^2 \\ &= (-1) \times (k') \\ &= -k'. \end{aligned}$$

§ 7.] We have now to examine the bearing of the discussions of last paragraph on the problem of the factorisation of $ax^2 + bx + c$.

It will prevent some confusion in the mind of the student if we confine ourselves in the first place to the supposition that a, b, c denote positive or negative *rational* numbers. Then $l = b/2a$ is in all cases a real rational number, and we have the following cases :—

1st. If $b^2 - 4ac$ is the positive square of a rational number, then m has a real rational value, and

$$ax^2 + bx + c = a(x + l + m)(x + l - m)$$

is the product of two linear factors whose coefficients are real rational numbers. Example 1, § 5, will serve as an illustration of this case.

2nd. If $b^2 - 4ac$ is positive, but not the square of a rational number, then m is real, but not rational; and the coefficients in the factors are irrational.

Example 1.

$$\begin{aligned} x^2 + 2x - 1 &= x^2 + 2x + 1 - 2, \\ &= (x+1)^2 - (\sqrt{2})^2, \\ &= (x + 1 + \sqrt{2})(x + 1 - \sqrt{2}). \end{aligned}$$

3rd. If $b^2 - 4ac$ is negative, then m is imaginary, and the coefficients in the factors are complex numbers.

Example 2.

$$\begin{aligned} x^2 + 2x + 5 &= x^2 + 2x + 1 + 4, \\ &= (x+1)^2 - (2i)^2, \\ &= (x + 1 + 2i)(x + 1 - 2i). \end{aligned}$$

Example 3.

$$\begin{aligned} x^2 + 2x + 3 &= x^2 + 2x + 1 + 2, \\ &= (x+1)^2 - (i\sqrt{2})^2, \\ &= (x + 1 + i\sqrt{2})(x + 1 - i\sqrt{2}). \end{aligned}$$

4th. There is another case, which forms the transition between the cases where the coefficients in the factors are real and the case where they are imaginary.

If $b^2 - 4ac = 0$, then $m = 0$,

and we have $$ax^2 + bx + c = a(x + l)^2 ;$$

in other words, $ax^2 + bx + c$ is a complete square, so far as x is concerned. The two factors are now $x + l$ and $x + l$, that is, both real, but identical.

We have, therefore, incidentally the important result *that $ax^2 + bx + c$ is a complete square in x if $b^2 - 4ac = 0$.*

Example 4.

$$3x^2 - 3x + \tfrac{3}{4} = 3(x^2 - 2.\tfrac{1}{2}x + \tfrac{1}{4}), = 3(x - \tfrac{1}{2})^2.$$

* $b^2 - 4ac$ is called the Discriminant of the quadratic function $ax^2 + bx + c$.

§ 8.] There is another point of view which, although usually of less importance than that of last section, is sometimes taken.

Paying no attention to the values of a, b, c, but regarding them merely as functions of certain other letters which they may happen to contain, we may inquire under what circumstances the coefficients of the *factors* will be *algebraically rational* functions of those letters.

In order that this may be the case it is clearly necessary and sufficient that $b^2 - 4ac$ be a complete square in the letters in question, $= P^2$ say.
Then

$$ax^2 + bx + c = a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{P}{2a}\right)^2\right\},$$
$$= a\left(x + \frac{b}{2a} + \frac{P}{2a}\right)\left(x + \frac{b}{2a} - \frac{P}{2a}\right),$$

which is rational, since P is so.

If $b^2 - 4ac = -P^2$, where P is rational in the present sense, then

$$ax^2 + bx + c = a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{P}{2a}i\right)^2\right\},$$
$$= a\left(x + \frac{b}{2a} + \frac{P}{2a}i\right)\left(x + \frac{b}{2a} - \frac{P}{2a}i\right)$$

where the coefficients are rational, but not real.

Example 5.

$$px^2 + (p+q)x + q$$
$$= p\left\{x^2 + \frac{p+q}{p}x + \frac{q}{p}\right\},$$
$$= p\left\{x^2 + 2\left(\frac{p+q}{2p}\right)x + \left(\frac{p+q}{2p}\right)^2 - \left(\frac{p+q}{2p}\right)^2 + \frac{q}{p}\right\},$$
$$= p\left\{\left(x + \frac{p+q}{2p}\right)^2 - \left(\frac{p-q}{2p}\right)^2\right\},$$
$$= p\left(x + \frac{p+q}{2p} + \frac{p-q}{2p}\right)\left(x + \frac{p+q}{2p} - \frac{p-q}{2p}\right),$$
$$= p\left(x+1\right)\left(x + \frac{q}{p}\right),$$
$$= (x+1)(px+q);$$

a result which would, of course, be more easily obtained by the tentative processes of §§ 2, 3.

§ 9.] It should be observed that the factorisation for ax^2+bx+c leads at once to the factorisation of the homogeneous function $ax^2+bxy+cy^2$ of the 2nd degree in two variables; for

$$\begin{aligned}
&ax^2+bxy+cy^2\\
&=ay^2\left\{\left(\frac{x}{y}\right)^2+\frac{b}{a}\left(\frac{x}{y}\right)+\frac{c}{a}\right\},\\
&=ay^2\left\{\frac{x}{y}+\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}\right\}\left\{\frac{x}{y}+\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}\right\};\\
&=a\left\{x+\left(\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}}\right)y\right\}\left\{x+\left(\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}\right)y\right\}.
\end{aligned}$$

By operating in a similar way any homogeneous function of two variables may be factorised, provided a certain non-homogeneous function of one variable, having the same coefficients, can be factorised.

Example 1. From

$$x^2+2x+3=(x+1+i\sqrt{2})(x+1-i\sqrt{2}),$$

we deduce

$$x^2+2xy+3y^2=\{x+(1+i\sqrt{2})y\}\{x+(1-i\sqrt{2})y\}.$$

Example 2. From

$$x^3-2x^2-23x+60=(x-3)(x-4)(x+5),$$

we deduce

$$x^3-2x^2y-23xy^2+60y^3=(x-3y)(x-4y)(x+5y).$$

§ 10.] By using the principle of substitution a great many apparently complicated cases may be brought under the case of the quadratic function, or under other equally simple forms. The following are some examples:—

Example 1.

$$\begin{aligned}
x^4+x^2y^2+y^4&=(x^2+y^2)^2-(xy)^2,\\
&=(x^2+y^2+xy)(x^2+y^2-xy),\\
&=\left\{\left(x+\frac{1}{2}y\right)^2+\frac{3}{4}y^2\right\}\left\{\left(x-\frac{1}{2}y\right)^2+\frac{3}{4}y^2\right\},\\
&=\left\{\left(x+\frac{1}{2}y\right)^2-\left(\frac{\sqrt{3}}{2}yi\right)^2\right\}\left\{\left(x-\frac{1}{2}y\right)^2-\left(\frac{\sqrt{3}}{2}yi\right)^2\right\},\\
&=\left\{x+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)y\right\}\left\{x+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)y\right\}\left\{x+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)y\right\}\\
&\qquad\left\{x+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)y\right\}.
\end{aligned}$$

Here the student should observe that, if resolution into *quadratic* factors only is required, it can be effected with real coefficients; but, if the resolution be carried to *linear* factors, complex coefficients have to be introduced.

Example 2.

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$
$$=\{x+y\}\left\{x+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)y\right\}\left\{x+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)y\right\}.^*$$

Example 3.

$$x^4+y^4=(x^2+y^2)^2-2x^2y^2$$
$$=(x^2+y^2)^2-(\sqrt{2}xy)^2$$
$$=(x^2+\sqrt{2}xy+y^2)(x^2-\sqrt{2}xy+y^2).$$

Again

$$x^2+\sqrt{2}xy+y^2=\left(x+\frac{\sqrt{2}}{2}y\right)^2+\frac{2}{4}y^2$$
$$=\left(x+\frac{\sqrt{2}}{2}y\right)^2-\left(\frac{\sqrt{2}}{2}iy\right)^2$$
$$=\left\{x+\frac{\sqrt{2}}{2}(1+i)y\right\}\left\{x+\frac{\sqrt{2}}{2}(1-i)y\right\}.$$

The similar resolution for $x^2-\sqrt{2}xy+y^2$ will be obtained by changing the sign of $\sqrt{2}$. Hence, finally,

$$x^4+y^4$$
$$=\left\{x+\frac{\sqrt{2}}{2}(1+i)y\right\}\left\{x+\frac{\sqrt{2}}{2}(1-i)y\right\}\left\{x-\frac{\sqrt{2}}{2}(1+i)y\right\}\left\{x-\frac{\sqrt{2}}{2}(1-i)y\right\}.$$

Example 4.

$$x^{12}-y^{12}=(x^6)^2-(y^6)^2$$
$$=(x^6-y^6)(x^6+y^6)$$
$$=\{(x^2)^3-(y^2)^3\}\{(x^2)^3+(y^2)^3\}$$
$$=(x^2-y^2)(x^4+x^2y^2+y^4)(x^2+y^2)(x^4-x^2y^2+y^4)$$
$$=(x+y)(x-y)(x+iy)(x-iy)(x^4+x^2y^2+y^4)(x^4-x^2y^2+y^4),$$

where the last two factors may be treated as in Example 1.

Example 5.

$$2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$$
$$=4b^2c^2-(a^2-b^2-c^2)^2$$
$$=(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)$$
$$=\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}$$
$$=(a+b-c)(a-b+c)(b+c+a)(b+c-a).$$

---

* The student should observe that the decomposition $x^2+y^2+xy=(x+y+\sqrt{xy})(x+y-\sqrt{xy})$, which is often given by beginners when they are asked to factorise x^2+y^2+xy, although it is a true algebraical identity, is no solution of the problem of factorisation in the ordinary sense, inasmuch as the two factors contain $\sqrt{xy}$, and are therefore not rational integral functions of x and y.

RESULTS OF THE APPLICATION OF THE REMAINDER THEOREM.

§ 11.] It may be well to call the student's attention once more to the use of the theorem of remainders in factorisation. *For every value* a *of* x *that we can find which causes the integral function* $f(x)$ *to vanish we have a factor* $x-a$ *of* $f(x)$.

It is needless, after what has been shown in chap. v., §§ 13-16, to illustrate this point further.

It may, however, be useful, although at this stage we cannot prove all that we are to assert, to state what the ultimate result of the rule just given is as regards the factorisation of integral functions of one variable. If $f(x)$ be of the nth degree, its coefficients being any given numbers, real or imaginary, rational or irrational, it is shown in the chapter on Complex Numbers that there exist n values of x (called the roots of the equation $f(x)=0$) for which $f(x)$ vanishes. These values will in general be all different, but two or more of them may be equal, and one or all of them may be complex numbers.

If, however, the coefficients of $f(x)$ be all real, then there will be an even number of complex roots, and it will be possible to arrange them in pairs of the form $\lambda \pm \mu i$.

It is not said that algebraical expressions for these roots in terms of the coefficients of $f(x)$ can always be found; but, if these coefficients be numerically given, the values of the roots can always be approximately calculated.

From this it follows that $f(x)$ *can in all cases be resolved into* n *linear* * *factors, the coefficients of which may or may not be all real.*

If the coefficients of $f(x)$ *be all real, then it can be resolved into a product of* p *linear and* q *quadratic factors, the coefficients in all of which are real numbers which may in all cases be calculated approximately. We have, of course,* $p+2q=n$, *and either* p *or* q *may be zero.*

The student will find, in §§ 1-10 above, illustrations of these statements in particular cases; but he must observe that the

* "Linear" is used here, as it often is, to mean "of the 1st degree."

general problem of factorising an integral function of the nth degree is coextensive with that of completely solving an equation of the same degree. When either problem is solved the solution of the other follows.

FACTORISATION OF FUNCTIONS OF MORE THAN ONE VARIABLE.

§ 12.] *When the number of variables exceeds unity, the problem of factorisation of an integral function (excepting special cases, such as homogeneous functions of two variables) is not in general soluble, at least in ordinary algebra.*

To establish this it is sufficient to show the insolubility of the problem in a particular case.

Let us suppose that x^2+y^2+1 is resolvable into a product of factors which are integral in x and y, that is, that

$$x^2+y^2+1=(px+qy+r)(p'x+q'y+r'),$$

then

$$\begin{aligned}x^2+y^2+1=&pp'x^2+qq'y^2+rr'\\&+(pq'+p'q)xy+(pr'+p'r)x\\&+(qr'+q'r)y.\end{aligned}$$

Since this is, by hypothesis, an identity, we have

$$\begin{array}{lr|lr} pp'=1 & (1) & pq'+p'q=0 & (4)\\ qq'=1 & (2) & pr'+p'r=0 & (5)\\ rr'=1 & (3) & qr'+q'r=0 & (6).\end{array}$$

First, we observe that, on account of the equations (1) (2) (3), none of the six quantities $p\ q\ r\ p'\ q'\ r'$ can be zero ; and further, $p'=\frac{1}{p}$, $q'=\frac{1}{q}$, $r'=\frac{1}{r}$. Hence, as logical consequences of our hypothesis, we have from (4) (5) and (6)—

$$\frac{p}{q}+\frac{q}{p}=0 \qquad (7)$$

$$\frac{p}{r}+\frac{r}{p}=0 \qquad (8)$$

$$\frac{q}{r}+\frac{r}{q}=0 \qquad (9) ;$$

and, from these again, if we multiply by pq, rp, and qr respectively, we get

$$p^2+q^2=0 \qquad (10)$$

$$p^2+r^2=0 \qquad (11)$$

$$q^2+r^2=0 \qquad (12).$$

Now from (11) and (12) by subtraction we derive

$$p^2-q^2=0 \qquad (13) ;$$

and from (10) and (13) by addition

$$2p^2=0\ ;$$

from this it follows that $p=0$, which is in contradiction with the equation (1). Hence the resolution in this case is impossible.

§ 13.] Nevertheless, it may happen *in particular cases* that the resolution spoken of in last article is possible, even when the function is not homogeneous. This is obvious from the truth of the inverse statement that, if we multiply together two integral functions, no matter of how many variables, the result is integral.

One case is so important in the applications of algebra to geometry, that we give an investigation of the necessary and sufficient condition for the resolvability.

Consider the general function of x and y of the 2nd degree, and write it

$$F=ax^2+2hxy+by^2+2gx+2fy+c.$$

We observe, in the first place, that, if it be possible to resolve F into two linear factors, then we must have

$$\begin{aligned}F&=(\sqrt{a}x+ly+m)\,(\sqrt{a}x+l'y+m'),\\&=[\sqrt{a}x+\{\tfrac{1}{2}(l+l')+\tfrac{1}{2}(l-l')\}y+\tfrac{1}{2}(m+m')+\tfrac{1}{2}(m-m')]\\&\qquad\times[\sqrt{a}x+\{\tfrac{1}{2}(l+l')-\tfrac{1}{2}(l-l')\}y+\tfrac{1}{2}(m+m')-\tfrac{1}{2}(m-m')],\\&=\{\sqrt{a}x+\tfrac{1}{2}(l+l')y+\tfrac{1}{2}(m+m')\}^2-\{\tfrac{1}{2}(l-l')y+\tfrac{1}{2}(m-m')\}^2.\end{aligned}$$

Hence, when F is resolvable into two linear factors, it must be expressible in the form L^2-M^2, where L is a linear function of x and y, and M a linear function of y alone; and, conversely, when F is expressible in this form, it is resolvable, namely, into $(L+M)(L-M)$.

Let us now seek for the relation among the coefficients of F which is necessary and sufficient to secure that F be expressible in the form L^2-M^2.

1st. Let $a\neq 0$, then

$$\begin{aligned}F&=a[x^2+2(hy+g)x/a+(by^2+2fy+c)/a],\\&=a[\{x+(hy+g)/a\}^2-\{(hy+g)^2-a(by^2+2fy+c)\}/a^2],\\&=a[\{x+(hy+g)/a\}^2-\{(h^2-ab)y^2+2(gh-af)y+(g^2-ac)\}/a^2].\end{aligned}$$

Hence the necessary and sufficient condition that F be expressible in the form L^2-M^2 is that $(h^2-ab)y^2+2(gh-af)y+(g^2-ac)$ be a complete square as regards y. For this, by § 7, it is necessary and sufficient that

$$4(gh-af)^2-4(h^2-ab)(g^2-ac)=0\ ;$$

that is,

$$-a\{abc+2fgh-af^2-bg^2-ch^2\}=0.$$

Now, since $a\neq 0$, this condition reduces to

$$abc+2fgh-af^2-bg^2-ch^2=0 \qquad (1).$$

2nd. If $a=0$, but $b\neq 0$, we may arrive at the same result by first arranging F according to powers of y, and proceeding as before.

3rd. If $a=0$, $b=0$, and $h\neq 0$, the present method fails altogether, but F now reduces to

$$F=2hxy+2gx+2fy+c,$$

and it is evident, since x^2 and y^2 do not occur, that if this be resolvable into linear factors the result must be of the form $2h(x+p)(y+q)$. We must therefore have

$$2g=2hq,$$
$$2f=2hp,$$
$$c=2hpq.$$

Now the first two of these give $fg=h^2pq$, that is, $2hpq=\frac{2fg}{h}$; whence using the third,

$$ch=2fg,$$

or, since $h\neq 0$, $$2fgh-ch^2=0 \qquad (2);$$
but this is precisely what (1) reduces to when $a=0$, $b=0$, so that in this third case the condition is still the same.

Moreover, it is easy to see that when (2) is satisfied the resolution is possible, being in fact

$$2hxy+2gx+2fy+c=2h\left(x+\frac{f}{h}\right)\left(y+\frac{g}{h}\right) \qquad (3),$$

which is obviously an identity if $c=2fg/h$.

4th. If $a=0$, $b=0$, $h=0$, F reduces to $2gx+2fy+c$. In this case we may hold that F is resolvable, it being now in fact itself a linear factor. It is interesting to observe that in this case also the condition (1) is satisfied.

Returning to the most general case, where a does not vanish, we observe that, when the condition (1) is satisfied, we have, provided $h^2-ab\neq 0$,

$$\sqrt{\{(h^2-ab)y^2+2(gh-af)y+(g^2-ac)\}}=\sqrt{h^2-ab}\left(y+\frac{gh-af}{h^2-ab}\right),$$

so that the required resolution is

$$\begin{aligned}F=a&\left\{x+\frac{h+\sqrt{h^2-ab}}{a}y+\frac{g}{a}+\frac{gh-af}{a(h^2-ab)}\sqrt{h^2-ab}\right\}\\ \times&\left\{x+\frac{h-\sqrt{h^2-ab}}{a}y+\frac{g}{a}-\frac{gh-af}{a(h^2-ab)}\sqrt{h^2-ab}\right\}\end{aligned} \qquad (4).$$

To the coefficients in the factors various forms may be given by using the relation (1); but they will not be rational functions unless h^2-ab be a complete square, and they will be imaginary unless h^2-ab is positive.

If $h^2-ab=0$, then (1) gives $(gh-af)^2=0$, that is, $gh-af=0$; and the required resolution is

$$F=a\left\{x+\frac{h}{a}y+\frac{g}{a}+\frac{\sqrt{g^2-ac}}{a}\right\}\left\{x+\frac{h}{a}y+\frac{g}{a}-\frac{\sqrt{g^2-ac}}{a}\right\} \qquad (5).$$

The distinction between these cases is of fundamental importance in the analytical theory of curves of the 2nd degree.

The function $abc+2fgh-af^2-bg^2-ch^2$, whose vanishing is the condition for the resolvability of the function of the 2nd degree, is called the *Discriminant* of that function.

It should be noticed that, if

$$\begin{aligned}F&\equiv ax^2+2hxy+by^2+2gx+2fy+c\\&\equiv(lx+my+n)(l'x+m'y+n') \qquad (6),\end{aligned}$$

then

$$ax^2+2hxy+by^2\equiv(lx+my)(l'x+m'y),$$

so that the terms of the 1st degree in the factors of F are simply the factors of $ax^2+2hxy+by^2$. We have therefore merely to find, *if possible*, values for n and n' which will make the identity (6) complete.

Example. To factorise $3x^2+2xy-y^2+2x-2y-1$. We have $3x^2+2xy-y^2\equiv(3x-y)(x+y)$. Hence, if the factorisation be possible, we must have

$$3x^2+2xy-y^2+2x-2y-1\equiv(3x-y+n)(x+y+n') \qquad (7).$$

Therefore, we must have

$$n+3n'=2 \qquad (8),$$
$$n-n'=-2 \qquad (9),$$
$$nn'=-1 \qquad (10).$$

Now, from (8) and (9), we get $n=-1$, and $n'=+1$. Since these values also satisfy (10), the factorisation is possible, and we have

$$3x^2+2xy-y^2+2x-2y-1\equiv(3x-y-1)(x+y+1).$$

It should be noticed that the resolvability of

$$F=ax^2+2hxy+by^2+2gx+2fy+c$$

carries with it the resolvability of the homogeneous function of three variables having the same coefficients, namely,

$$F=ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

as is at once seen by writing x/z, y/z, in place of x and y.

Exercises XI.

Factorise the following functions:—

(1.) $(a+b)^2+(a+c)^2-(c+d)^2-(b+d)^2$. (2.) $4a^2b^2-(a^2+b^2-c^2)^2$.

(3.) $(a^2-2b^2-c^2)^2-4(b^2-c^2)^2$. (4.) $(5x^2-11x+12)^2-(4x^2-15x+6)^2$.

(5.) $\{x^2-(\beta+\gamma)x+\beta\gamma\}^2-(x-\gamma)^2(x-\alpha)^2$. (6.) x^6-y^6.

(7.) x^8-y^8. (8.) $x^2+6xy+9y^2-4$. (9.) $2x^2+3x-2$.

(10.) $x^2+6x-16$. (11.) $x^2-10x+18$. (12.) x^2+x-30.

(13.) $x^2+14x+56$. (14.) x^2+4x+7. (15.) $2x^2+5x-12$.

(16.) $x^2+2x\sqrt{(p+q)}+2q$. (17.) $x^2-2bx/(b+c)+(b-c)/(b+c)$.

(18.) $(x^2+pq)^2-(p+q)^2x^2$. (19.) $ab(x^2-y^2)+xy(a^2-b^2)$.

(20.) $pq(x+y)^2-(p+q)(x^2-y^2)+(x-y)^2$. (21.) $x^3-15x^2+71x-105$.

(22.) x^3-14x^2+148x. (23.) $x^3-13x^2+54x-72$.

(24.) x^3-8x^2+x-8. (25.) $x^3+3px^2+(3p^2-q^2)x+p(p^2-q^2)$.

(26.) $(p+q)x^3+(p-q)x^2-(p+q)x-(p-q)$.

(27.) $x^3-(1+p+p^2)x^2+(p+p^2+p^3)x-p^3$.

(28.) $x^4-(a+b)x^3+(a^2b+ab^2)x-a^2b^2$.

(29.) $x^6+x^5a+x^4a^2-x^2a^4-xa^5-a^6$.

(30.) $(1+x)^2(1+y^2)-(1+y)^2(1+x^2)$. (31.) $x^4+x^2y^2+y^4$.

(32.) Assuming $x^4+y^4=(x^2+pxy+y^2)(x^2+qxy+y^2)$, determine p and q.

(33.) Factorise $x^4+y^4-2(x^2+y^2)+1$.

(34.) Determine r and s in terms of a, p, and q in order that x^2-a^2 may be a factor in $x^4+px^3+qx^2+rx+s$.

Factorise

(35.) $(x^{m+n})^2-(x^m a^n)^2-(x^n a^m)^2+(a^{m+n})^2$.

(36.) $(x^2+a^2)^2(x^4+a^2x^2+a^4)-(x^8+x^4a^4+a^8)$.

(37.) $xy^2-2xy-y^2+x+2y-1$. (38.) $2x^2+xy+7x+3y+3$.

(39.) $2x^2+xy-3y^2-x-4y-1$. (40.) $xy+7x+3y+21$.

(41.) $x^2-2y^2-3z^2+7yz+2zx+xy$.

(42.) Determine λ so that $(x+6y-1)(6x+y-1)+\lambda(3x+2y+1)(2x+3y+1)$ may be resolvable into two linear factors.

(43.) Find an equation to determine λ so that $ax^2+by^2+2hxy+2gx+2fy+c+\lambda xy$ may be resolvable into two linear factors; and find the value of λ when $c=0$.

(44.) Find the condition that $(\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z)-(\alpha'' x+\beta'' y+\gamma'' z)^2$ break up into two linear factors.

(45.) If $(x+p)(x+2q)+(x+2p)(x+q)$ be a complete square in x, then $9p^2-14pq+9q^2=0$.

(46.) If $(x+b)(x+c)+(x+c)(x+a)+(x+a)(x+b)$ be a complete square in x, show that $a=b=c$.

Factorise

(47.) $a^3+b^3+c^3-3abc$. (48.) $x^3+3axy+y^3-a^3$.

(49.) $(x-x^2)^3+(x^2-1)^3+(1-x)^3$.

Factorise the following functions of x, y, z:—*

(50.) $\Sigma(y^2+x^2)(z^2+x^2)(y-z)$.

(51.) $\Sigma(x^3+y^3)(x-y)$. (52.) $\Sigma x^4(y^2-z^2)$. (53.) $(\Sigma x)^3-\Sigma x^3$.

(54.) Simplify $\{\Sigma(x^2+y^2-z^2)(x^2+z^2-y^2)\}/\Pi(x\pm y\pm z)$.

(55.) Show that $\Sigma(y^m z^n-y^n z^m)$ and $\Sigma x^n(y^m z^p-y^p z^m)$ are each exactly divisible by $(y-z)(z-x)(x-y)$.

(56.) Show that $nx^{n+1}-(n+1)x^n+1$ is exactly divisible by $(x-1)^2$.

(57.) Show that $\Sigma x^2(y+z-x)^3$ is exactly divisible by $\Sigma x^2-2\Sigma yz$.

(58.) Show that $(x+y+z)^{2n+1}-x^{2n+1}-y^{2n+1}-z^{2n+1}$ is exactly divisible by $(y+z)(z+x)(x+y)$.

(59.) $(y-z)^{2n+1}+(z-x)^{2n+1}+(x-y)^{2n+1}$ is exactly divisible by $(y-z)(z-x)(x-y)$.

(60.) If n be of the form $6m-1$, then $(y-z)^n+(z-x)^n+(x-y)^n$ is exactly divisible by $\Sigma x^2-\Sigma xy$; and, if n be of the form $6m+1$, the same function is exactly divisible by $(\Sigma x^2-\Sigma xy)^2$.

(61.) Prove directly that $xy-1$ cannot be resolved into a product of two linear factors.

(62.) If a and b be not zero, it is impossible so to determine p and q that $x+py+qz$ shall be a factor of $x^3+ay^3+bz^3$.

---

* Regarding the meaning of Σ in (50), (51), &c., see the footnote on p. 84.

CHAPTER VIII.

Rational Fractions.

§ 1.] *By a rational algebraical fraction is meant simply the quotient of any integral function by any other integral function.*

Unless it is otherwise stated it is to be understood that we are dealing with functions of a single variable x.

If in the rational fraction A/B the degree of the numerator is greater than or equal to the degree of the denominator, the fraction is called an *improper fraction*, if less, a *proper fraction*.

GENERAL PROPOSITIONS REGARDING PROPER AND IMPROPER FRACTIONS.

§ 2.] *Every improper fraction can be expressed as the sum of an integral function and a proper fraction; and, conversely, the sum of an integral function and a proper fraction may be exhibited as an improper fraction.*

For if in $\frac{A_m}{B_n}$ the degree m of A_m be greater than the degree n of B_n, then, by the division-transformation (chap. v.), we obtain

$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R}{B_n},$$

which proves the first part of our statement, since Q_{m-n} is integral, and the degree of R is $< n$.

Again, if P_p be any integral function whatever, and A_m/B_n a proper fraction (that is, $m < n$), then

$$P_p + \frac{A_m}{B_n} = \frac{P_p B_n + A_m}{B_n},$$

which is an improper fraction, since the degree of the numerator, namely, $n+p$, is $> n$.

Examples of these transformations have already been given under division.

It is important to remark that, *if two improper fractions be equal, then the integral parts and the properly fractional parts must be equal separately.*

For let
$$\frac{A_m}{B_n} = Q_{m-n} + \frac{R}{B_n},$$
and
$$\frac{A'_{m'}}{B'_{n'}} = Q'_{m'-n'} + \frac{R'}{B'_{n'}},$$
by the above transformation.

Then, if
$$\frac{A_m}{B_n} = \frac{A'_{m'}}{B'_{n'}},$$
we have
$$Q_{m-n} + \frac{R}{B_n} = Q'_{m'-n'} + \frac{R'}{B'_{n'}}.$$
Hence
$$Q_{m-n} - Q'_{m'-n'} = \frac{R'B_n - RB'_{n'}}{B_n B'_{n'}}.$$

Now, since the degrees of R' and R are less than n' and n respectively, the degree of the numerator on the right-hand side of this last equation is less than $n+n'$. Hence, unless $Q_{m-n} - Q'_{m'-n'} = 0$, we have an integral function equal to a proper fraction, which is impossible (see chap. v., § 1). We must therefore have

$$Q_{m-n} = Q'_{m'-n'}, \text{ and consequently } \frac{R}{B_n} = \frac{R'}{B'_{n'}}.$$

N.B.—From this of course it follows that $m-n = m'-n'$.

As an example, consider the improper fraction $(x^3+2x^2+3x+4)/(x^2+x+1)$, and let us multiply both numerator and denominator by x^2+2x+1; we thus obtain the fraction

$$(x^5+4x^4+8x^3+12x^2+11x+4)/(x^4+3x^3+4x^2+3x+1),$$

which, by chap. iii., § 2, must be equal to the former fraction. Now transform each of these by the long-division transformation, and we obtain respectively

$$x+1+\frac{x+3}{x^2+x+1},$$

and
$$x+1+\frac{x^3+5x^2+7x+3}{x^4+3x^3+4x^2+3x+1}.$$

The integral parts of these are equal; and the fractional parts are also equal (see next section).

The sum of two proper algebraical fractions is a proper algebraical fraction.

After what has been given above, the proof of this proposition will present no difficulty. The proposition is interesting as an instance, if any were needed, that *fraction* in the algebraical sense is a totally different conception from fraction in the arithmetical sense; for it is not true in arithmetic that the sum of two proper fractions is always a proper fraction; for example, $\frac{3}{4}+\frac{1}{2}=\frac{5}{4}$, which is an improper fraction.

§ 3.] Since by chap. iii., § 2, we may divide both numerator and denominator of a fraction by the same divisor, if the numerator and denominator of a rational fraction have any common factors, we can remove them. Hence *every rational fraction can be so simplified that its numerator and denominator are algebraically prime to each other; when thus simplified the fraction is said to be at "its lowest terms."*

The common factors, when they exist, may be determined by inspection (for example, by completely factorising both numerator and denominator by any of the processes described in chap. vii.); or, in the last resort, by the process for finding the G.C.M., which will either give us the common factor required, or prove that there is none.

Example 1.
$$\frac{x^3+5x^2+7x+3}{x^4+3x^3+4x^2+3x+1}.$$

By either of the processes of chap. vi. the G.C.M. will be found to be x^2+2x+1. Dividing both numerator and denominator by this factor, we get, for the lowest terms of the given fraction,
$$\frac{x+3}{x^2+x+1}.$$

The simplification might have been effected thus. Observing that both numerator and denominator vanish when $x=-1$, we see that $x+1$ is a common factor. Removing this factor we get
$$\frac{x^2+4x+3}{x^3+2x^2+2x+1}.$$

Here numerator and denominator both vanish when $x=-1$, hence there is the common factor $x+1$. Removing this we get

$$\frac{x+3}{x^2+x+1}.$$

It is now obvious that numerator and denominator are prime to each other; for the only possible common factor is $x+3$, and this does not divide the denominator, which does not vanish when $x=-3$.

§ 4.] The student should note the following conclusion from the above theory, partly on account of its practical usefulness, partly on account of its analogy with a similar proposition in arithmetic.

If two rational fractions, P/Q, P'/Q', be equal, and P/Q be at its lowest terms, then $P'=\lambda P$, $Q'=\lambda Q$, where λ is an integral function of x, which will reduce to a constant if P'/Q' be also at its lowest terms.

To prove this, we observe that

$$\frac{P'}{Q'}=\frac{P}{Q},$$

whence

$$P'=\frac{Q'P}{Q},$$

that is, $Q'P/Q$ must be integral, that is, $Q'P$ must be divisible by Q; but P is prime to Q, therefore by chap. VI., § 12, $Q'=\lambda Q$, where λ is an integral function of x. We now have

$$P'=\frac{\lambda QP}{Q}=\lambda P;$$

so that $P'=\lambda P$, $Q'=\lambda Q$.

If P'/Q' be at its lowest terms, P' and Q' can have no common factor; so that in this case λ must be a constant, which may of course happen to be unity.

DIRECT OPERATIONS WITH RATIONAL FRACTIONS.

§ 5.] The general principles of operation with fractions have already been laid down; all that the student has now to learn is the application of his knowledge of the properties of integral functions to facilitate such operation in the case of rational fractions. The most important of these applications is the use of the G.C.M. and the L.C.M., and of the dissection of functions by factorisation.

No general rules can be laid down for such transformations as we proceed to exemplify in this paragraph. But the following pieces of general advice will be found useful.

Never make a step that you cannot justify by reference to the fundamental laws of algebra. Subject to this restriction, make the freest use of your judgment as to the order and arrangement of steps.

Take the earliest opportunity of getting rid of redundant members of a function, unless you see some direct reason to the contrary.

Cultivate the use of brackets as a means of keeping composite parts of a function together, and do not expand such brackets until you see that something is likely to be gained thereby, inasmuch as it may turn out that the whole bracket is a redundant member, in which case the labour of expanding is thrown away, and merely increases the risk of error.

Take a good look at each part of a composite expression, and be guided in your treatment by its construction, for example, by the factors you can perceive it to contain, by its degree, and so on.

Avoid the unthinking use of mere rules, such as that for long division, that for finding the G.C.M., &c., as much as possible; and use instead processes of inspection, such as dissection into factors; and general principles, such as the theorem of remainders. In other words, use the head rather than the fingers. But, if you do use a rule involving mechanical calculation, be patient, accurate, and systematically neat in the working. It is well known to mathematical teachers that quite half the failures in algebraical exercises arise from arithmetical inaccuracy and slovenly arrangement.

Make every use you can of general ideas, such as homogeneity and symmetry, to shorten work, to foretell results without labour, and to control results and avoid errors of the grosser kind.

Example 1. Express as a single fraction in its simplest form—

$$\frac{2x^3+4x^2+3x+4}{x^2+1}-\frac{2x^3+4x^2-3x-2}{x^2-1}=\text{F say.}$$

Transform each fraction by division, then

$$\begin{aligned}\text{F}&=(2x+4)+\frac{x}{x^2+1}-(2x+4)-\frac{-x+2}{x^2-1},\\&=\frac{x(x^2-1)-(-x+2)(x^2+1)}{x^4-1},\\&=\frac{2x^3-2x^2-2}{x^4-1},\\&=\frac{2(x^3-x^2-1)}{x^4-1}.\end{aligned}$$

Example 2. Express as a single fraction

$$\text{F}=\frac{1}{x^3-3x^2+3x-1}-\frac{1}{x^3-x^2-x+1}-\frac{1}{x^4-2x^3+2x-1}-\frac{1}{x^4-2x^3+2x^2-2x+1}.$$

We have

$$\begin{aligned}
x^3-3x^2+3x-1 &= (x-1)^3\,;\\
x^3-x^2-x+1 &= x^3+1-x(x+1)=(x+1)\,(x^2-x+1-x),\\
&=(x+1)\,(x-1)^2\,;\\
x^4-2x^3+2x-1 &= x^4-1-2x(x^2-1),\\
&=(x^2-1)\,(x-1)^2,\\
&=(x-1)^3(x+1)\,;\\
x^4-2x^3+2x^2-2x+1 &= (x^2+1)^2-2x(x^2+1),\\
&=(x^2+1)\,(x-1)^2.
\end{aligned}$$

Whence

$$\begin{aligned}
\mathrm{F} &= \frac{1}{(x-1)^3}-\frac{1}{(x+1)\,(x-1)^2}-\frac{1}{(x-1)^3\,(x+1)}-\frac{1}{(x^2+1)\,(x-1)^2},\\
&=\frac{(x+1)-(x-1)}{(x+1)\,(x-1)^3}-\frac{(x^2+1)+(x-1)\,(x+1)}{(x-1)^3\,(x+1)\,(x^2+1)},\\
&=\frac{2}{(x+1)\,(x-1)^3}-\frac{2x^2}{(x-1)^3\,(x+1)\,(x^2+1)},\\
&=2\frac{x^2+1-x^2}{(x+1)\,(x^2+1)\,(x-1)^3},\\
&=\frac{2}{(x^4-1)\,(x-1)^2},\\
&=\frac{2}{x^6-2x^5+x^4-x^2+2x-1}.
\end{aligned}$$

Example 3.

$$\begin{aligned}
&\left(\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}\right)\times\left(\frac{x+y}{x-y}+\frac{x^3+y^3}{x^3-y^3}\right),\\
&=\left(\frac{x-y}{x+y}\right)\left(1-\frac{x^2+xy+y^2}{x^2-xy+y^2}\right)\left(\frac{x+y}{x-y}\right)\left(1+\frac{x^2-xy+y^2}{x^2+xy+y^2}\right),\\
&=\left(\frac{-2xy}{x^2-xy+y^2}\right)\times\left(\frac{2(x^2+y^2)}{x^2+xy+y^2}\right),\\
&=-\frac{4xy(x^2+y^2)}{x^4+x^2y^2+y^4}.
\end{aligned}$$

Example 4.

$$\begin{aligned}
\mathrm{F} &= \frac{2}{b-c}+\frac{b-c}{(c-a)\,(a-b)}+\frac{2}{c-a}+\frac{c-a}{(a-b)\,(b-c)}+\frac{2}{a-b}+\frac{a-b}{(b-c)\,(c-a)},\\
&=\frac{2(c-a)\,(a-b)+(b-c)^2+2(a-b)\,(b-c)+(c-a)^2+2(b-c)\,(c-a)+(a-b)^2}{(b-c)\,(c-a)\,(a-b)}.\\
&=\frac{\{(b-c)+(c-a)+(a-b)\}^2}{\&\text{c.}},\\
&=\frac{0^2}{\&\text{c.}}=\frac{0}{(b-c)\,(c-a)\,(a-b)}=0,
\end{aligned}$$

it being of course supposed that the denominator does not vanish.

Example 5.

$$\begin{aligned} F &= \frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)}, \\ &= \frac{-a^3(b-c) - b^3(c-a) - c^3(a-b)}{(b-c)(c-a)(a-b)}. \end{aligned}$$

Now we observe that when $b=c$ the numerator of F becomes 0, hence $b-c$ is a factor; by symmetry $c-a$ and $a-b$ must also be factors. Hence the numerator is divisible by $(b-c)(c-a)(a-b)$. Since the degree of the numerator is the 4th, the remaining factor, owing to the symmetry of the expression, must be $Pa+Pb+Pc$. Comparing the coefficients of a^3b in

$$-a^3(b-c) - b^3(c-a) - c^3(a-b)$$

and $$P(a+b+c)(b-c)(c-a)(a-b),$$

we see that $$P = +1.$$

Hence, finally, $$F = a+b+c.$$

Example 6.

$$\begin{aligned} F &= \frac{a^2+pa+q}{(a-b)(a-c)(x-a)} + \frac{b^2+pb+q}{(b-a)(b-c)(x-b)} + \frac{c^2+pc+q}{(c-a)(c-b)(x-c)}, \\ -F &= \frac{(b-c)(a^2+pa+q)(x-b)(x-c) + \&c. + \&c.}{(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)}, \\ &= \frac{(b-c)(a^2+pa+q)\{x^2-(b+c)x+bc\} + \&c. + \&c.}{\&c.}. \end{aligned}$$

Now, collect the coefficients of x^2, x, and the absolute term in the numerator, observing that the two &c.'s stand for the result of exchanging a and b and a and c respectively in the first term. We have in the coefficient of x^2 a part independent of p and q, namely,

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = -(b-c)(c-a)(a-b) \qquad (1).$$

The parts containing p and q respectively are

$$\{a(b-c)+b(c-a)+c(a-b)\}p = 0$$

and $$\{(b-c)+(c-a)+(a-b)\}q = 0.$$

The coefficient of x^2 therefore reduces to (1).

Next, in the coefficient of x we have the three parts,

$$\begin{aligned} &-\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\} = 0, \\ &-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}p \\ &\qquad = -(b-c)(c-a)(a-b)p \qquad (2), \end{aligned}$$

and $$-\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\}q = 0.$$

Finally, in the absolute term,

$$\begin{aligned} &abc\{a(b-c)+b(c-a)+c(a-b)\} = 0, \\ &abc\{(b-c)+(c-a)+(a-b)\}p = 0, \\ &\{bc(b-c)+ca(c-a)+ab(a-b)\}q \\ &\qquad = -(b-c)(c-a)(a-b)q \qquad (3). \end{aligned}$$

Hence, removing the common factor $(b-c)(c-a)(a-b)$, which now appears both in numerator and denominator, and changing the sign on both sides, we have

$$F=\frac{x^2+px+q}{(x-a)(x-b)(x-c)}.$$

The student should observe here the constant use of the identities on pp. 81-83, and the abbreviation of the work by two-thirds, effected by taking advantage of the principle of symmetry. In actual practice the greater part of the reasoning above written would of course be conducted mentally.

INVERSE METHOD OF PARTIAL FRACTIONS.

§ 6.] Since we have seen that a sum of rational fractions can always be exhibited as a single rational fraction, it is naturally suggested to inquire how far we can decompose a given rational fraction into others (usually called "*partial fractions*") having denominators of lower degrees.

Since we can always, by ordinary division, represent (and that in one way only) an improper fraction as the sum of an integral function and a proper fraction, we need only consider the latter kind of fraction.

The fundamental theorem on which the operation of dissection into "partial fractions" depends is the following:—

If A/PQ *be a rational proper fraction whose denominator contains two integral factors,* P *and* Q, *which are algebraically prime to each other, then we can always decompose* A/PQ *into the sum of two proper fractions,* $P'/P+Q'/Q$.

Proof.—Since P and Q are prime to each other, we can (see chap. vi., § 11) always find two integral functions, L and M, such that

$$LP+MQ=1 \qquad (1).$$

Multiply this identity by A/PQ, and we obtain

$$\frac{A}{PQ}=\frac{AL}{Q}+\frac{AM}{P} \qquad (2).$$

In general, of course, the degrees of AL and AM will be higher than those of Q and P respectively. If this be so, transform AL/Q and AM/P by division into $S+Q'/Q$ and $T+P'/P$, so that

S, T, Q′, and P′ are integral, and the degrees of P′ and Q′ less than those of P and Q respectively. We now have

$$\frac{A}{PQ} = S + T + \frac{P'}{P} + \frac{Q'}{Q} \qquad (3),$$

where S + T is integral, and P′/P + Q′/Q a proper fraction. But the left-hand side of (3) is a proper fraction. Hence S + T must vanish identically, and the result of our operations will be simply

$$\frac{A}{PQ} = \frac{P'}{P} + \frac{Q'}{Q} \qquad (4).$$

which is the transformation required.

To give the student a better hold of the above reasoning, we work out a particular case.

Consider the fraction

$$F = \frac{x^4+1}{(x^3+3x^2+2x+1)(x^2+x+1)}.$$

Here $\quad A = x^4+1, \quad P = x^3+3x^2+2x+1, \quad Q = x^2+x+1.$

Carrying out the process for finding the G.C.M. of P and Q, we have

$$\begin{array}{l}
1+1+1)1+3+2+1(1+2 \\
\qquad\qquad\quad 2+1+1 \\
\qquad\qquad\qquad -1-1)1+1+1(-1+0 \\
\qquad\qquad\qquad\qquad\qquad 0+1 \\
\qquad\qquad\qquad\qquad\qquad\ \ +1
\end{array}$$

whence, denoting the remainders by R_1 and R_2,

$$P = (x+2)Q + R_1, \quad Q = -xR_1 + R_2.$$

From these successively we get

$$\begin{aligned}
R_1 &= P - (x+2)Q, \\
1 = R_2 &= Q + xR_1, \\
&= Q + xP - x(x+2)Q, \\
1 \quad &= (-x^2-2x+1)Q + xP \qquad (1).
\end{aligned}$$

In this case, therefore,

$$M = -x^2-2x+1, \quad L = x.$$

Multiplying now by A/PQ on both sides of (1), we obtain (putting in the actual values of P and Q in the present case)

$$\begin{aligned}
F &= \frac{(x^4+1)(-x^2-2x+1)}{x^3+3x^2+2x+1} + \frac{(x^4+1)x}{x^2+x+1}; \\
&= \frac{-x^6-2x^5+x^4-x^2-2x+1}{x^3+3x^2+2x+1} + \frac{x^5+x}{x^2+x+1};
\end{aligned}$$

or, carrying out the two divisions,

$$= -x^3+x^2-1+\frac{x^2+2}{x^3+3x^2+2x+1}+x^3-x^2+1+\frac{-1}{x^2+x+1},$$

or, seeing that the integral part vanishes, as it ought to do,

$$F=\frac{x^2+2}{x^3+3x^2+2x+1}+\frac{-1}{x^2+x+1},$$

which is the required decomposition of F into partial fractions.

COR. *If* P, Q, R, S, . . . *be integral functions of x which are prime to each other, then any proper rational fraction* A/PQRS . . . *can be decomposed into a sum of proper fractions,* $P'/P+Q'/Q+R'/R+S'/S+\ldots$

This can be proved by repeated applications of the main theorem.

§ 7.] Having shown *a priori* the possibility of decomposition into partial fractions, we have now to examine the special cases that occur, and to indicate briefer methods of obtaining results which we know must exist.

We have already stated that it may be shown that every integral function B may be resolved into prime factors with real coefficients, which belong to one or other of the types $(x-a)^r$, $(x^2+\beta x+\gamma)^s$.

1st. Take the case where there is a single, not repeated, factor, $x-a$. Then the fraction $F=A/B$ may be written

$$F=\frac{A}{(x-a)Q}$$

say, where $x-a$ and Q are prime to each other. Hence, by our general theorem, we may write

$$F=\frac{P'}{x-a}+\frac{Q'}{Q} \qquad (1),$$

each member being a proper fraction.

In this case the degree of P′ must be zero, that is, P′ is a constant.

It may be determined by methods similar to those used in chap. v., § 21. See below, Example 1.

P′ determined, we go on to decompose the proper fraction Q′/Q, by considering the other factors in its denominator.

2nd. Suppose there is a repeated factor $(x-a)^r$; say $B=(x-a)^r Q$, where Q does not contain the factor $x-a$. We may, by the general principle, write

$$F=\frac{P'}{(x-a)^r}+\frac{Q'}{Q}.$$

P' is now an integral function, whose degree is less than r; hence, by chap. v., § 21, we may put it into the form

$$P'=a_0+a_1(x-a)+\ .\ .\ .\ +a_{r-1}(x-a)^{r-1},$$

and therefore write

$$F=\frac{a_0}{(x-a)^r}+\frac{a_1}{(x-a)^{r-1}}+\ .\ .\ .\ +\frac{a_{r-1}}{x-a}+\frac{Q'}{Q} \qquad (2),$$

where $a_0, a_1, \ldots, a_{r-1}$ are constants to be determined. See below, Example 2.

3rd. Let there be a factor $(x^2+\beta x+\gamma)^s$, so that

$$B=(x^2+\beta x+\gamma)^s Q,$$

Q being prime to $x^2+\beta x+\gamma$. Now, we have

$$F=\frac{P'}{(x^2+\beta x+\gamma)^s}+\frac{Q'}{Q}.$$

P' is in this case an integral function of degree $2s-1$ at most. We may therefore write, see chap. v., § 21,

$$\begin{aligned}P'=(a_0+b_0x)&+(a_1+b_1x)(x^2+\beta x+\gamma)\\&+(a_2+b_2x)(x^2+\beta x+\gamma)^2\\&\ \vdots\\&+(a_{s-1}+b_{s-1}x)(x^2+\beta x+\gamma)^{s-1}.\end{aligned}$$

We thus have

$$F=\frac{a_0+b_0x}{(x^2+\beta x+\gamma)^s}+\frac{a_1+b_1x}{(x^2+\beta x+\gamma)^{s-1}}+\ .\ .\ .\ +\frac{a_{s-1}+b_{s-1}x}{x^2+\beta x+\gamma}+\frac{Q'}{Q} \quad (3);$$

where the $2s$ constants a_0, b_0, &c., have to be determined by any appropriate methods. See Examples 3 and 4.

In the particular case where $s=1$, we have, of course, merely

$$F=\frac{a_0+b_0x}{x^2+\beta x+\gamma}+\frac{Q'}{Q} \qquad (4).$$

By operating successively in the way indicated we can decompose every rational fraction into a sum of partial fractions, each of which belongs to one or other of the two types $p_r/(x-a)^r$, $(a_s+b_sx)/(x^2+\beta x+\gamma)^s$, *where* a, β, γ, p_r, a_s, b_s *are all real constants, and* r *and* s *positive integers.*

It is important to remark that each such partial fraction has a separate and independent existence, and that if necessary or convenient the constant or constants belonging to it can be determined quite independently of the others.

Cor. *If* P *be an integral function of* x *of the* n*th degree, and* $a, a, \ldots, a; \beta, \beta, \ldots, \beta; \gamma, \gamma, \ldots, \gamma, \ldots$ *constants not less than* $n+1$ *in number,* r *of which are equal to* a, s *equal to* β, t *equal to* $\gamma, \ldots$, *then we can always express* P *in the form*

$$P=\Sigma\{a_0+a_1(x-a)+\ldots+a_{r-1}(x-a)^{r-1}\}(x-\beta)^s(x-\gamma)^t\ldots,$$

where $a_0, a_1, \ldots, a_{r-1}, \ldots$ *are constants. In particular, if* $r=1$, $s=1$, $t=1, \ldots$, *we have*

$$P=\Sigma a_0(x-\beta)(x-\gamma)\ldots$$

These theorems follow at once, if we consider the fraction $P/(x-a)^r(x-\beta)^s(x-\gamma)^t\ldots$

There is obviously a corresponding theorem where $x-a$, $x-\beta$, $x-\gamma$ are replaced by any integral functions which are prime to each other, and the sum of whose degrees is not less than $n+1$.

§ 8.] We now proceed to exemplify the practical carrying out of the above theoretical process; and we recommend the student to study carefully the examples given, as they afford a capital illustration of the superior power of general principles as contrasted with "rule of thumb" in Algebra.

Example 1. It is required to determine the partial fraction, corresponding to $x-1$, in the decomposition of

$$(4x^4-16x^3+17x^2-8x+7)/(x-1)(x-2)^2(x^2+1).$$

We have

$$F=\frac{4x^4-16x^3+17x^2-8x+7}{(x-1)(x-2)^2(x^2+1)}=\frac{p}{x-1}+\frac{Q'}{(x-2)^2(x^2+1)} \qquad (1),$$

and we have to find the constant p.

From the identity (1), multiplying both sides by $(x-1)(x-2)^2(x^2+1)$, we deduce the identity

$$4x^4-16x^3+17x^2-8x+7=p(x-2)^2(x^2+1)+Q'(x-1) \qquad (2).$$

Now (2) being true for all values of x, must hold when $x=1$; in this case it becomes

$$4=2p, \text{ that is, } p=2.$$

Hence the required partial fraction is $2/(x-1)$.

If it be required to determine also the integral function Q', this can be done at once by putting $p=2$ in (2), and subtracting $2(x-2)^2(x^2+1)$ from both sides. We thus obtain

$$2x^4-8x^3+7x^2-1=Q'(x-1) \qquad (3).$$

This being an identity, the left-hand side *must be divisible by* $x-1$.* It is so in point of fact; and, after carrying out the division, we get

$$2x^3-6x^2+x+1=Q' \qquad (4),$$

which determines Q'.

The student may verify for practice that we do actually have

$$\frac{4x^4-16x^3+17x^2-8x+7}{(x-1)(x-2)^2(x^2+1)}=\frac{2}{x-1}+\frac{2x^3-6x^2+x+1}{(x-2)^2(x^2+1)}.$$

Example 2. Taking the same fraction as in Example 1, to determine the group of partial fractions corresponding to $(x-2)^2$.

1°. We have now

$$\frac{4x^4-16x^3+17x^2-8x+7}{(x-1)(x-2)^2(x^2+1)}=\frac{a_0}{(x-2)^2}+\frac{a_1}{(x-2)}+\frac{Q'}{(x-1)(x^2+1)} \qquad (1),$$

whence

$$4x^4-16x^3+17x^2-8x+7=a_0(x-1)(x^2+1)+a_1(x-2)(x-1)(x^2+1) + Q'(x-2)^2 \qquad (2).$$

In the identity (2) put $x=2$, and we get

$$-5=5a_0, \text{ that is, } a_0=-1.$$

Putting now $a_0=-1$ in (2), subtracting $(-1)(x-1)(x^2+1)$ from both sides and dividing both sides by $x-2$, we have

$$4x^3-7x^2+2x-3=a_1(x-1)(x^2+1)+Q'(x-2) \qquad (3).$$

Put $x=2$ in this last identity, and there results

$$+5=5a_1, \text{ that is, } a_1=+1.$$

The group of partial fractions required is therefore

$$-1/(x-2)^2+1/(x-2).$$

If required, Q' may be determined as in Example 1 by means of (3).

2°. Another good method for determining a_0 and a_1 depends on the use of "continued division."

If we put $x=y+2$ on both sides of (1), we have the identity

$$\frac{4(y+2)^4-16(y+2)^3+17(y+2)^2-8(y+2)+7}{(y+1)y^2\{(y+2)^2+1\}}=\frac{a_0}{y^2}+\frac{a_1}{y}+\frac{Q''}{(y+1)\{(y+2)^2+1\}},$$

* If it is not, then there has been a mistake in the working.

that is,

$$\frac{-5-4y+\&c.}{5y^2+9y^3+\&c.}=\frac{a_0}{y^2}+\frac{a_1}{y}+\frac{Q''}{(1+y)(3+4y+y^2)} \qquad (4).$$

Now, by chap. v., § 20, the expansion of a rational fraction in descending powers of $1/y$ and ascending powers of y is unique. Hence, if we perform the operation of ascending continued division on the left, the first two terms must be identical with $a_0/y^2+a_1/y$; for $Q''/(1+y)(3+4y+y^2)$ will obviously furnish powers of y merely.

We have

$$\begin{array}{r|l} -5-4+\ .\ .\ . & \ 5+9+\ .\ .\ . \\ +5+\ .\ .\ . & -1+1+\ .\ .\ . \end{array}$$

therefore $a_0=-1$, $a_1=+1$.

The number of coefficients which we must calculate in the numerator and denominator on the left depends of course on the number of coefficients to be determined on the right.

Example 3. Lastly, let us determine the partial fraction corresponding to x^2+1 in the above fraction.

We must now write

$$\frac{4x^4-16x^3+17x^2-8x+7}{(x-1)(x-2)^2(x^2+1)}=\frac{ax+b}{x^2+1}+\frac{Q'}{(x-1)(x-2)^2} \qquad (1).$$

1°. Whence, multiplying by $(x-1)(x-2)^2$,

$$\frac{4x^4-16x^3+17x^2-8x+7}{x^2+1}=\frac{(ax+b)(x-1)(x-2)^2}{x^2+1}+Q' \qquad (2):$$

whence

$$\begin{aligned} 4x^2-16x+13+\frac{8x-6}{x^2+1}&=(ax+b)\left(x-5+\frac{7x+1}{x^2+1}\right)+Q', \\ &=(ax+b)(x-5)+\frac{7ax^2+(7b+a)x+b}{x^2+1}+Q', \\ &=(ax+b)(x-5)+7a+\frac{(7b+a)x+(b-7a)}{x^2+1}+Q' \qquad (3). \end{aligned}$$

Now the proper fractions on the two sides of (3) must be equal—that is, we must have the identity

$$(7b+a)x+(b-7a)=8x-6,$$

therefore $\qquad 7b+a=8, \quad b-7a=-6.$

Multiplying these two equations by 7 and by 1 and adding, we get

$$50b=50, \text{ that is, } b=1.$$

Either of them then gives $a=1$, hence the required partial fraction is

$$(x+1)/(x^2+1).$$

2°. Another method for obtaining this result is as follows.

Remembering that $x^2+1=(x+i)(x-i)$ (see chap. vii.), we see that x^2+1 vanishes when $x=i$.

Now we have

$$\begin{aligned} 4x^4-16x^3+17x^2-8x+7&=(ax+b)(x-1)(x-2)^2+Q'(x^2+1) \\ &=(ax+b)(x^3-5x^2+8x-4)+Q'(x^2+1) \qquad (4). \end{aligned}$$

Put in this identity $x=i$, and observe that

$$i^4=i^2\times i^2=(-1)\times(-1)=+1,$$
$$i^3=i^2\times i=(-1)\times i=-i;$$

and we have
$$8i-6=(ai+b)(7i+1),$$
$$=(7b+a)i+(b-7a);$$

whence
$$(7b+a-8)i=-b+7a-6,$$

an equality which is impossible * unless both sides are zero, hence

$$7b+a-8=0, \quad -b+7a-6=0,$$

from which a and b may be determined as before.

3°. Another method of finding a and b might be used in the present case.

We suppose that the partial fractions corresponding to all the factors except x^2+1 have already been determined. We can then write

$$F=\frac{2}{x-1}-\frac{1}{(x-2)^2}+\frac{1}{x-2}+\frac{ax+b}{x^2+1} \qquad (5).$$

From this we obtain the identity

$$\begin{aligned}4x^4-16x^3&+17x^2-8x+7\\ &=2(x-2)^2(x^2+1)-(x-1)(x^2+1)+(x-1)(x-2)(x^2+1)\\ &\qquad+(ax+b)(x-1)(x-2)^2;\end{aligned}$$

whence

$$x^4-4x^3+3x^2+4x-4=(ax+b)(x-1)(x-2)^2;$$

and, dividing by $(x-1)(x-2)^2$,

$$x+1=ax+b.$$

This being of course an identity, we must have

$$a=1, \quad b=1.$$

Another process for finding the constants in all the partial fractions depends on the method of equating coefficients (see chap. v., § 16), and leads to their determination by the solution of an equal number of simultaneous equations of the 1st degree.

The following simple case will sufficiently illustrate this method.

Example 4.

To decompose $(3x-4)/(x-1)(x-2)$ into partial fractions.

We have

$$\frac{3x-4}{(x-1)(x-2)}=\frac{a}{x-1}+\frac{b}{x-2},$$

therefore
$$3x-4=a(x-2)+b(x-1),$$
$$=(a+b)x-(2a+b).$$

Hence, since this last equation is an identity, we have

$$a+b=3, \quad 2a+b=4.$$

Hence, solving these equations for a and b (see chap. xvi.), we find $a=1$, $b=2$.

---

* For no real multiple (differing from zero) of the imaginary unit can be a real quantity. See above, chap. vii., § 6. The student should recur to this case again after reading the chapter on Complex Numbers.

Example 5. We give another instructive example. To decompose

$$F=\frac{x^2+px+q}{(x-a)(x-b)(x-c)},$$

we may write

$$\frac{x^2+px+q}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c} \qquad (1),$$

where A, B, C are constants.

Now

$$x^2+px+q=A(x-b)(x-c)+B(x-c)(x-a)+C(x-a)(x-b) \qquad (2).$$

Herein put $x=a$, and there results

$$a^2+pa+q=A(a-b)(a-c);$$

whence

$$A=\frac{a^2+pa+q}{(a-b)(a-c)}.$$

By symmetry

$$B=\frac{b^2+pb+q}{(b-a)(b-c)},$$

$$C=\frac{c^2+pc+q}{(c-a)(c-b)}.$$

We have therefore

$$\frac{x^2+px+q}{(x-a)(x-b)(x-c)}$$

$$=\frac{a^2+pa+q}{(a-b)(a-c)(x-a)}+\frac{b^2+pb+q}{(b-c)(b-a)(x-b)}+\frac{c^2+pc+q}{(c-a)(c-b)(x-c)} \qquad (3),$$

an identity already established above, § 5, Example 6. It may strike the student as noteworthy that it is more easily established by the inverse than by the direct process. The method of partial fractions is in point of fact a fruitful source of complicated algebraical identities.

Exercises XII.

Express the following as rational fractions at their lowest terms.

(1.) $(x^3+2x^2-x+6)/(x^4-x^2+4x-4)$.

(2.) $(9x^3+53x^2-9x-18)/(4x^2+44x+120)$.

(3.) $\dfrac{x^4+2x^3-2x-1}{x^4+x^3-3x^2-5x-2}-\dfrac{x^4+x^3-3x^2-5x-2}{x^4+2x^3-2x-1}$.

(4.) $(3x^3-x^2-x-1)/(3x^3+5x^2+3x+1)+(x^3+3x^2+5x+3)/(x^3+x^2+x-3)$.

(5.) $(x^6-2x^3+1)/(x^2-2x+1)+(x^6+2x^3+1)/(x^2+2x+1)$.

(6.) $(6x^3+13ax^2-9a^2x-10a^3)/(9x^3+12ax^2-11a^2x-10a^3)$.

(7.) $(1-a^2)/\{(1+ax)^2-(a+x)^2\}$.

(8.) $\{(w+x+z)(w+x)-y(y+z)\}/\{(w+x+z)(w+z)-y(x+y)\}$.

(9.) $\dfrac{1}{(1-x)(1-x^2)^2}\Big/\left\{\dfrac{1}{(1-x)^2}-\dfrac{1}{(1-x)(1-x^2)}+\dfrac{1}{(1-x^2)^2}\right\}$.

(10.) $\{(al+bm)^2+(am-bl)^2\}/\{(ap+bq)^2+(aq-bp)^2\}$.

(11.) $\dfrac{\{px^2+(k-s)x+r\}^2-\{px^2+(k+s)x+r\}^2}{\{px^2+(k+t)x+r\}^2-\{px^2+(k-t)x+r\}^2}$.

(12.) $\dfrac{x}{2x-2y}+\dfrac{x-y}{2y-2x}$. (13.) $\dfrac{1}{a-b-(a-b)x}+\dfrac{1}{a+b+(a+b)x}$.

(14.) $1/(a-2b-1/(a-2b-1/(a-2b)))$.

(15.) $1/(6x+6)-1/(2x-2)+4/(3-3x^2)$.

(16.) $\dfrac{x^3-y^3}{x^4-y^4}-\dfrac{x-y}{x^2-y^2}-\frac{1}{2}\left\{\dfrac{x+y}{x^2+y^2}-\dfrac{1}{x+y}\right\}$.

(17.) $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\Big/\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

(18.) $\dfrac{6x}{3x-2}-\dfrac{30x^2+4x}{9x^2+4}+\dfrac{4x}{3x+2}$.

(19.) $\dfrac{2}{x}+\dfrac{1}{(x+1)^3}-\dfrac{2}{(x+1)^2}-\dfrac{2}{(x+1)}$.

(20.) $\dfrac{1}{24(x-1)}+\dfrac{5}{8(x+1)}+\dfrac{1}{4(x+1)^2}-\dfrac{1}{2(x+1)^3}-\dfrac{2x+1}{3(x^2+x+1)}$.

(21.) $\dfrac{1}{(x+1)^2(x+2)^2}-\dfrac{1}{(x+2)^2}+\dfrac{2}{x+1}-\dfrac{2}{x+2}$.

(22.) $(a+b)/(x+a)+(a-b)/(x-a)-2a(x+b)/(x^2+a^2)$.

(23.) $\{(x-y)/(x+y)\}+\{(x-y)/(x+y)\}^2+\{(x-y)/(x+y)\}^3$.

(24.) $\left(\dfrac{x^3-3x+2}{x^3+2x^2+2x+1}\right)\times\left(\dfrac{x^2+2x+1}{x^3-5x+4}\right)$.

(25.) $\left(\dfrac{a^2+x^2}{2ax}+1\right)\times\dfrac{ax^2}{a^3+x^3}\div\dfrac{4a(a+x)}{a^2-ax+x^2}$.

(26.) $\dfrac{x+y}{x^3-y^3}+\dfrac{x-y}{x^3+y^3}-2\dfrac{x^2-y^2}{x^4+x^2y^2+y^4}$.

(27.) $\left\{\dfrac{\dfrac{1}{x^2}+\dfrac{1}{y^2}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}-\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}}\right\}\div\left\{\dfrac{8}{\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)\left(\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-2\right)}\right\}$.

(28.) $\dfrac{1}{2a(a-c)(x-a)}+\dfrac{1}{2a(a+c)(x+a)}+\dfrac{1}{(c^2-a^2)(x+c)}$.

(29.) $\left\{1+\dfrac{20}{x+1}-\dfrac{180}{x+2}+\dfrac{420}{x+3}-\dfrac{280}{x+4}\right\}\times\left\{1-\dfrac{20}{x-1}+\dfrac{180}{x-2}-\dfrac{420}{x-3}+\dfrac{280}{x-4}\right\}$.

(30.) $\{(xy-1)^2+(x+y-2)(x+y-2xy)\}/\{(xy+1)^2-(x+y)^2\}$.

(31.) $(1+y^3+z^3-3yz)/(1+y+z)$.

(32.) $\{a(a+2b)+b(b+2c)+c(c+2a)\}/\{a^2-b^2-c^2-2bc\}$.

(33.) $\dfrac{(a+b)^3+(b+c)^3-(a+2b+c)^3}{(a+b)(b+c)(a+2b+c)}$.

(34.) $\dfrac{x^6+a^6}{(x^6+a^6)(x^2-a^2)+a^2x^2(x^4-a^4)}+\dfrac{a^2x^2}{x^6-a^6-a^2x^2(x^2-a^2)}$.

(35.) $\dfrac{a^2+(2ac-b^2)x^2+c^2x^4}{a^2+2abx+(2ac+b^2)x^2+2bcx^3+c^2x^4}\times\dfrac{a^2+(ac-b^2)x^2-bcx^3}{a^2+(ac-b^2)x^2+bcx^3}$.

(36.) $\dfrac{x^2+y^2+x+y-xy+1}{x-y-1}+\dfrac{x^2+y^2+x-y+xy+1}{x+y-1}$.

(37.) $\dfrac{(x^5-10x^3y^2+5xy^4)^2+(5x^4y-10x^2y^3+y^5)^2}{(x^3-3xy^2)^2+(3x^2y-y^3)^2}$.

(38.) $\dfrac{(b+c)^2+2(b^2-c^2)+(b-c)^2}{(b^4-2b^2c^2+c^4)\left\{\dfrac{1}{(b-c)^2}+\dfrac{2}{b^2-c^2}+\dfrac{1}{(b+c)^2}\right\}}$.

(39.) $\Sigma(b^2+c^2-a^2)/(a-b)(a-c)$.

(40.) $(\Sigma x)(\Sigma x^2)/xyz-\Sigma(y+z)/x$.

(41.) $\Sigma(b+c)/(c-a)(a-b)$.

(42.) $\Sigma bc(a+h)/(a-b)(a-c)$.

(43.) $\Sigma(b^2+bc+c^2)/(a-b)(a-c)$.

(44.) $\{\Pi(1-x^2)+\Pi(x-yz)\}/(1-xyz)$.

(45.) $\{\Sigma(b+c)^3-3\Pi(b+c)\}/\{\Sigma a^3-3abc\}$.

(46.) $\dfrac{1-x}{1+x}+\dfrac{x-y}{x+y}+\dfrac{y-1}{y+1}+\dfrac{(1-x)(x-y)(y-1)}{(1+x)(x+y)(y+1)}$.

(47.) $\dfrac{(y-z)^2+(z-x)^2+(x-y)^2}{(y-z)(z-x)(x-y)}+2\left(\dfrac{1}{y-z}+\dfrac{1}{z-x}+\dfrac{1}{x-y}\right)$.

(48.) $\dfrac{b-c}{x-a}+\dfrac{c-a}{x-b}+\dfrac{a-b}{x-c}+\dfrac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}$.

(49.) $\Sigma(a+p)(a+q)/(a-b)(a-c)(a+h)$.

(50.) $\Sigma a^2/(a-b)(a-c)(h-a)$.

(51.) $\Sigma a^2/(a^2-b^2)(a^2-c^2)(h^2+a^2)$.

(52.) $\Sigma(y^2+z^2-x^2)/yz(x-y)(x-z)$.

(53.) $\dfrac{a(b-c)^3+b(c-a)^3+c(a-b)^3+(b^2-c^2)(b-c)+(c^2-a^2)(c-a)+(a^2-b^2)(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

(54.) $\dfrac{\{(x+y)^2+(y+z)^2\}\{(z+x)^2+(x+w)^2\}}{\{(x+y)(z+x)+(y+z)(x+w)\}^2+\{(x+y)(x+w)-(y+z)(z+x)\}^2}$.

Prove the following identities :—

(55.) $\Sigma a^3/(a-b)(a-c)=\Sigma a$.

(56.) $c(u^2-v)=au(1-uv)$, $\quad c(v^2-u)=bv(1-uv)$,

where $\quad u=(ab-c^2)/(bc-a^2)$, $\quad v=(ab-c^2)/(ca-b^2)$.

(57.) $\sum\limits_{abcd}(a+\alpha)(a+\beta)(a+\gamma)/a(a-b)(a-c)(a-d)=-\alpha\beta\gamma/abcd$.

(58.) $\dfrac{(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3}{(b-c)^3+(c-a)^3+(a-b)^3}=\Pi(b+c)$.

(59.) $\dfrac{(ab-cd)(a^2-b^2+c^2-d^2)+(ac-bd)(a^2+b^2-c^2-d^2)}{(a^2-b^2+c^2-d^2)(a^2+b^2-c^2-d^2)+4(ab-cd)(ac-bd)}$

$$=\frac{(b+c)(a+d)}{(b+c)^2+(a+d)^2}.$$

(60.) $\dfrac{a^5(c-b)+b^5(a-c)+c^5(b-a)}{(c-b)(a-c)(b-a)}=-\Sigma a^3-\Sigma bc^2-abc$.

(61.) $\{\Sigma(y-z)^8\}/\{\Sigma(y-z)^2\}-4\Pi(y-z)^2=\{\Sigma x^2-\Sigma yz\}^3$.

Decompose the following into sums of partial fractions :—

(62.) $(x^2-1)/(x-2)(x-3)$.

(63.) $x^2/(x-1)(x-2)(x-3)$.

(64.) $30x^5/(x^2-1)(x^2-4)$.

(65.) $(x^2+4)/(x+1)^2(x-2)(x+3)$.

(66.) $(x^2-2)/(x^3-1)$. (67.) $(x^2+x+1)/(x+1)(x^2+1)$.
(68.) $(2x-3)/(x-1)(x^2+1)^2$. (69.) $1/(x-a)(x-b)(x^2-2px+q)$, $p^2<q$.
(70.) $(1+x+x^2)/(1-x-x^4+x^5)$. (71.) $18/(x^4+4x+3)$.
(72.) $(x+3)/(x^4-1)$. (73.) $1/(x^8+x^7-x^4-x^3)$.

(74.) Express $(3x^2+x+1)/(x^8-1)$ as the sum of two rational fractions whose denominators are x^4-1 and x^4+1.

(75.) Expand $1/(3-x)(2+x)$ in a series of ascending powers of x, using partial fractions and continued division.

(76.) Expand in like manner $1/(1-x)^2(1+x^2)$.

(77.) Show that

$$\sum_{abcd} (b+c+d)/(b-a)(c-a)(d-a)(x-a) = (x-a-b-c-d)/(x-a)(x-b)(x-c)(x-d).$$

CHAPTER IX.

Further Application to the Theory of Numbers.

ON THE VARIOUS WAYS OF REPRESENTING INTEGRAL AND FRACTIONAL NUMBERS.

§ 1.] The following general theorem lies at the root of the theory of the representation of numbers by means of a systematic scale of notation :—

Let $r_1, r_2, r_3, \ldots, r_n, r_{n+1}, \ldots$ *denote an infinite series of integers* restricted in no way except that each is to be greater than* 1, *then any integer* N *may be expressed in the finite form*—

$$N = p_0 + p_1r_1 + p_2r_1r_2 + p_3r_1r_2r_3 + \ldots + p_nr_1r_2\ldots r_n,$$

where $p_0<r_1$, $p_1<r_2$, $p_2<r_3$, $\ldots$, $p_n<r_{n+1}$. *When* $r_1, r_2, r_3, \ldots$ *are given, this can be done in one way only.*

For, divide N by r_1, the quotient being N_1 and the remainder p_0; divide N_1 by r_2, the quotient being N_2 and the remainder p_1, and so on until the last quotient, say p_n, is less than the next number in the series which falls to be taken as divisor. Then, of course, the process stops. We now have

$$\begin{aligned}
N &= p_0 + N_1r_1 \quad (p_0<r_1) && (1),\\
N_1 &= p_1 + N_2r_2 \quad (p_1<r_2) && (2),\\
N_2 &= p_2 + N_3r_3 \quad (p_2<r_3) && (3),\\
&\ldots\ldots\\
N_{n-1} &= p_{n-1} + p_nr_n (p_{n-1}<r_n) && (n).
\end{aligned}$$

* In this chapter, unless the contrary is distinctly implied, every letter used denotes a positive integral number.

From (1), using (2), we get

$$N = p_0 + r_1(p_1 + N_2 r_2),$$
$$= p_0 + p_1 r_1 + r_1 r_2 N_2.$$

Thence, using (3),

$$N = p_0 + p_1 r_1 + p_2 r_1 r_2 + r_1 r_2 r_3 N_3,$$

and so on.

Thus we obtain finally

$$N = p_0 + p_1 r_1 + p_2 r_1 r_2 + p_3 r_1 r_2 r_3 + \ldots + p_n r_1 r_2 \ldots r_n \qquad (A).$$

Again, the resolution is possible in one way only. For suppose we also had

$$N = p_0' + p_1' r_1 + p_2' r_1 r_2 + p_3' r_1 r_2 r_3 + \ldots + p_n' r_1 r_2 \ldots r_n \qquad (B),$$

then, equating (A) and (B), and dividing both sides by r_1, we should have

$$\frac{p_0}{r_1} + (p_1 + p_2 r_2 + p_3 r_2 r_3 + \ldots + p_n r_2 r_3 \ldots r_n)$$
$$= \frac{p_0'}{r_1} + (p_1' + p_2' r_2 + p_3' r_2 r_3 + \ldots + p_n' r_2 r_3 \ldots r_n) \qquad (C).$$

But the two brackets on the right and left of (C) contain integers, and p_0/r_1 and p_0'/r_1 are, by hypothesis, each a proper fraction. Hence we must have $p_0/r_1 = p_0'/r_1$; that is,

$$p_0 = p_0',$$

$$p_1 + p_2 r_2 + p_3 r_2 r_3 + \ldots + p_n r_2 r_3 \ldots r_n$$
$$= p_1' + p_2' r_2 + p_3' r_2 r_3 + \ldots + p_n' r_2 r_3 \ldots r_n \qquad (D).$$

Proceeding now with (D) as we did before with (C), we shall prove $p_1 = p_1'$; and so on. In other words, the two expressions (A) and (B) are identical.

Example. Let $N = 719$, and let the numbers $r_1, r_2, r_3, \ldots$ be the natural series 2, 3, 4, 5, . . . Carrying out the divisions indicated above, we have

$$\begin{array}{r l}
2)\underline{719} & \\
3)\underline{359} & \ldots 1 \\
4)\underline{119} & \ldots 2 \\
5)\underline{29} & \ldots 3 \\
5 & \ldots 4.
\end{array}$$

Hence $p_0=1,\ p_1=2,\ p_2=3,\ p_3=4,\ p_4=5$;
and we have $719=1+2\times 2+3\times 2.3+4\times 2.3.4+5\times 2.3.4.5$.

§ 2.] There is a corresponding proposition for resolving a fraction, namely, $r_1, r_2, \ldots, r_n$, &c., being as before,

Any proper fraction A/B *can be expressed in the form*

$$\frac{A}{B}=\frac{p_1}{r_1}+\frac{p_2}{r_1r_2}+\frac{p_3}{r_1r_2r_3}+\ldots+\frac{p_n}{r_1r_2\ldots r_n}+F,$$

where $p_1<r_1,\ p_2<r_2,\ \ldots,\ p_n<r_n$; *and* F *is either zero or can be made as small as we please by taking a sufficient number of the integers* $r_1, r_2, \ldots, r_n$. *When* $r_1, r_2, \ldots, r_n, \ldots$ *are given, this resolution can be effected in one way only.*

The reader will have no difficulty in deducing this proposition from that of last paragraph. It may also be proved thus:—

$$\frac{A}{B}=\frac{Ar_1}{Br_1}=\frac{Ar_1/B}{r_1}.$$

Now we may put Ar_1/B into the form p_1+q_1/B, where $q_1<B$. We then have

$$\frac{A}{B}=\frac{p_1+q_1/B}{r_1},$$

where $p_1<r_1$, since, by hypothesis, $A<B$.

Hence

$$\frac{A}{B}=\frac{p_1}{r_1}+\frac{1}{r_1}\cdot\frac{q_1}{B} \qquad (1).$$

Treating the proper fraction q_1/B in the same way as we treated A/B, we have

$$\frac{q_1}{B}=\frac{p_2}{r_2}+\frac{1}{r_2}\cdot\frac{q_2}{B},$$

where $$p_2<r_2,\ q_2<B \qquad (2).$$

Similarly,

$$\frac{q_2}{B}=\frac{p_3}{r_3}+\frac{1}{r_3}\cdot\frac{q_3}{B},$$

where $$p_3<r_3,\ q_3<B,\ \&c. \qquad (3).$$

And, finally,

$$\frac{q_{n-1}}{B} = \frac{p_n}{r_n} + \frac{1}{r_n} \cdot \frac{q_n}{B},$$

where $$p_n < r_n, \; q_n < B \qquad (n).$$

Now, using equations (1), (2), . . ., (n) in turn, we deduce successively

$$\begin{aligned} \frac{A}{B} &= \frac{p_1}{r_1} + \frac{p_2}{r_1 r_2} + \frac{q_2}{r_1 r_2 B}, \\ &= \frac{p_1}{r_1} + \frac{p_2}{r_1 r_2} + \frac{p_3}{r_1 r_2 r_3} + \frac{q_3}{r_1 r_2 r_3 B}, \\ &= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= \frac{p_1}{r_1} + \frac{p_2}{r_1 r_2} + \frac{p_3}{r_1 r_2 r_3} + \ldots + \frac{p_n}{r_1 r_2 \ldots r_n} \\ &\qquad + \frac{q_n}{r_1 r_2 \ldots r_n B} \qquad \text{(A)}, \end{aligned}$$

where $p_1 < r_1$, $p_2 < r_2$, . . ., $p_n < r_n$, $q_n < B$.

It appears therefore that $F = q_n / r_1 r_2 \ldots r_n B$, which can clearly be made as small as we please by sufficiently increasing the number of factors in its denominator. This of course involves a corresponding increase in the number of the terms of the preceding series.

It may happen, of course, that q_n vanishes, and then $F = 0$. We leave it as an exercise for the student to prove that this case occurs when $r_1 r_2 \ldots r_n$ is a multiple of B, and that if A/B be at its lowest terms it cannot occur otherwise. He ought also to find little difficulty in proving that the resolution is unique when $r_1, r_2, \ldots, r_n, \ldots$ are given.

Example 1. Let A/B=444/576, and let the numbers r_1, r_2, &c., be 2, 4, 6, 8, &c.

We find $$\frac{444}{576} = \frac{1}{2} + \frac{2}{2 \cdot 4} + \frac{1}{2 \cdot 4 \cdot 6}.$$

Example 2. A/B=11/13, r_1, r_2, &c., being 2, 3, 4, 5, 6, . . ., &c.

$$\frac{11}{13} = \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{0}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \times 13}.$$

Since r_1, r_2, &c., are arbitrary, we may so choose them that the numerators p_1, p_2, &c., shall each be unity. We thus have a process for decomposing any fraction into a sum of others with unit numerators.

Example 3.

$$
\begin{array}{ccc}
\begin{array}{r}
11 \\
2\times \\
\hline
13)22(1 \\
13 \\
\hline
9 \\
2\times \\
\hline
13)18(1 \\
13 \\
\hline
5
\end{array}
&
\begin{array}{r}
5 \\
3\times \\
\hline
13)15(1 \\
13 \\
\hline
2 \\
7\times \\
\hline
13)14(1 \\
13 \\
\hline
1
\end{array}
&
\begin{array}{r}
1 \\
13\times \\
\hline
13)13(1 \\
13 \\
\hline
0
\end{array}
\end{array}
$$

Whence

$$\frac{11}{13}=\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2.2.3}+\frac{1}{2.2.3.7}+\frac{1}{2.2.3.7.13}.$$

Here we have chosen at each step the least multiplier possible. When this is done, it may be shown that the successive remainders diminish down to zero, the successive multipliers increase, and the process may be brought to an end. If this restriction on the multiplier be not attended to, the resolution may be varied in most cases to a considerable extent. Since, however, we always divide by the same divisor B, there are only B possible remainders, namely, $0, 1, 2, \ldots, \overline{B-1}$; hence after $B-1$ operations at most the remainder must recur if the operation has not terminated by the occurrence of a zero.

Example 4. Thus we have

$$\frac{2}{3}=\frac{1}{2}+\frac{1}{2.3};$$

also

$$=\frac{1}{2}+\frac{1}{2.4}+\frac{1}{2.4^2}+\ldots+\frac{1}{2.4^n}+\frac{1}{2.4^n.3}.$$

Example 5.

$$\frac{7}{29}=\frac{1}{5}+\frac{1}{5.5}+\frac{1}{5.5.29};$$

also

$$=\frac{1}{5}+\frac{1}{5.6}+\frac{1}{5^2.6}+\frac{1}{5^2.6^2}+\frac{1}{5^3.6^2}+\frac{1}{5^3.6^3}+\frac{1}{5^4.6^3}+\frac{1}{5^4.6^4}+\&c.;$$

also

$$=\frac{1}{6}+\frac{1}{6.3}+\frac{1}{6.3.3}+\frac{1}{6.3.3.29};$$

and so on.

§ 3.] The most important practical case of the proposition in § 1 is that where $r_1, r_2, \ldots$ are all equal, say each $=r$. Then we have this result—

Every integer N can be expressed, and that in one way only, in the form

$$p_n r^n + p_{n-1} r^{n-1} + \ldots + p_1 r + p_0,$$

where $p_0, p_1, \ldots, p_n$ are each $< r$.

In other words, detaching the coefficients, and agreeing that their position shall indicate the power of r which they multiply, and that apposition shall indicate addition (and not multiplication as usual), we see that, r being any integer whatever chosen

as the *radix* of a *scale of notation*, any integer whatever may be represented in the form $p_n p_{n-1} \dots p_1 p_0$; where each of the letters or *digits* $p_0, p_1, \dots, p_n$ must have some one of the integral values $0, 1, 2, 3, \dots, \overline{r-1}$.

For example, if $r=10$, any integer may be represented by $p_n p_{n-1} \dots p_1 p_0$ where $p_0, p_1, \dots, p_n$ have each some one of the values 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The process of § 1 at once furnishes us with a rule for finding successively the digits $p_0, p_1, p_2, \dots$, namely, *Divide the given integer* N *by the chosen radix r, the remainder will be p_0; divide the integral quotient of last division by r, the remainder will be p_1, and so on.*

Usually, of course, the integer N will be given expressed in some particular scale, say the ordinary one whose radix is 10; and it will be required to express it in some other scale whose radix is given. In that case the operations will be carried on in the given scale.

The student will of course perceive that all the rules of ordinary decimal arithmetic are applicable to arithmetic in any scale, the only difference being that, in the scale of 7 say, there are only 7 digits, 0, 1, 2, 3, 4, 5, 6, and that the "carriages" go by 7's and not by 10's.

If the radix of the scale exceeds 10, new symbols must of course be invented to represent the digits. In the scale of 12, for example, digits must be used for 10 and 11, say τ for 10 and ϵ for 11.

Example 1. To convert 136991 (radix 10) into the scale of 12.

$$\begin{array}{rl}
12)\underline{136991} & \\
12)\underline{11415} & \dots \epsilon \\
12)\underline{951} & \dots 3 \\
12)\underline{79} & \dots 3 \\
6 & \dots 7
\end{array}$$

The result is 6733ϵ.

Example 2. To convert 6733ϵ (radix 12) into the scale of τ.

$$\begin{array}{rl}
\tau)\underline{6733\epsilon} & \\
\tau)\underline{7\epsilon17} & \dots 1 \\
\tau)\underline{961} & \dots 9 \\
\tau)\underline{\epsilon4} & \dots 9 \\
\tau)\underline{11} & \dots 6 \\
1 & \dots 3
\end{array}$$

The result is 136991.

Although this method is good practice, the student may very probably prefer the following:—

$$6733\epsilon \text{ (radix 12) means}$$
$$6\times 12^4+7\times 12^3+3\times 12^2+3\times 12+11.$$

Using the process of chap. v., § 13, Example 1, we have

$$\begin{array}{rrrrr} 6+ & 7+ & 3+ & 3+ & 11 \\ & +72+ & 948+ & 11412+ & 136980 \\ \hline 6+ & 79+ & 951+ & 11415+ & 136991. \end{array}$$

§ 4.] From one point of view the simplest scale of notation would be that which involves the fewest digits. In this respect the binary scale possesses great advantages, for in it every digit is either 0 or 1. For example, 365 expressed in this scale is 101101101. All arithmetical operations then reduce to the addition of units. The counterbalancing disadvantage is the enormous length of the notation when the numbers are at all large.

With any radix whatever we can dispense with the latter part of the digits allowable in that scale provided we allow the use of negative digits. For let the radix be r, then whenever, on dividing by r, the positive remainder p is greater than $r/2$, we can add unity to the quotient and take $-(r-p)$ for a negative remainder, where of course $r-p<r/2$. For example, 3978362 (radix 10) might be written $40\bar{2}\bar{2}4\bar{4}2$, where $\bar{2}$ stands for -2; so that in fact $40\bar{2}\bar{2}4\bar{4}2$ stands for $4\cdot10^6+0\cdot10^5-2\cdot10^4-2\cdot10^3+4\cdot10^2-4\cdot10+2$.

Example 1. Work out the product of 1698 and 314 in the binary scale.

$$\begin{array}{rr} 1698= & 11010100010 \\ 314= & 100111010 \\ \hline & 11010100010 \\ & 11010100010 \\ & 11010100010 \\ & 11010100010 \\ & 110101000100 \\ \hline & 10000010001010110100 \end{array}$$

10000010001010110100 (=533172 radix 10).

Example 2. Express 1698 and 314 in the scale of 5, using no digit greater than 3, and work out the product of the two transformed numbers.

$$\begin{array}{lcr}
5)\underline{1698} & & \\
\;\;5)\underline{339} & \ldots & 3 \\
\;\;\;\;5)\underline{68} & \ldots & \bar{1} \\
\;\;\;\;5)\underline{13} & \ldots & 3 \\
\;\;\;\;\;\;\;\;2 & \ldots & 3
\end{array}
\qquad\qquad
\begin{array}{lcr}
5)\underline{314} & & \\
\;\;5)\underline{63} & \ldots & \bar{1} \\
\;\;5)\underline{12} & \ldots & 3 \\
\;\;\;\;\;\;2 & \ldots & 2
\end{array}$$

$$\begin{array}{r}
233\bar{1}3 \\
\underline{223\bar{1}} \\
\bar{2}\bar{3}\bar{3}1\bar{3} \\
131\bar{1}\bar{1}\bar{1} \\
1021\bar{1}1 \\
\underline{1021\bar{1}1} \\
12\bar{1}1\bar{2}1\bar{3}0\bar{3}
\end{array} \quad *$$

The student may verify that $12\bar{1}1\bar{2}1\bar{3}0\bar{3}$ (radix 5) $=533172$ (radix 10).

Example 3. Show how to weigh a weight of 315 lbs.: first, with a series of weights of 1 lb., 2 lbs., 2^2 lbs., 2^3 lbs., &c., there being one of each kind; second, with a series of weights of 1 lb., 3 lbs., 3^2 lbs., 3^3 lbs., &c., there being one only of each kind.

First. Express 315 in the binary scale. We have

$$315 = 100111011,$$
$$315 = 1 + 2 + 2^3 + 2^4 + 2^5 + 2^8.$$

Hence we must put in one of the scales of the balance the weights 1 lb., 2 lbs., 2^3 lbs., 2^4 lbs., 2^5 lbs., and 2^8 lbs.

Second. Express 315 in the ternary scale, using no digit greater than unity. We have

$$315 = 110\bar{1}00.$$

Hence over against the given weight we must put the weights 3^4 lbs. and 3^5 lbs.; and on the same side as the given weight the weight 3^2 lbs.

§ 5.] If we specialise the proposition of § 2 by making $r_1 = r_2 = \ldots = r_n$, each $= r$ say, we have the following:—*Every proper fraction* A/B *can be expressed, and that in one way only, in the form*—

$$\frac{A}{B} = \frac{p_1}{r} + \frac{p_2}{r^2} + \frac{p_3}{r^3} + \ldots + \frac{p_n}{r^n} + F;$$

where $p_1, p_2, \ldots, p_n$ are each $< r$, and F *either is zero, or can be made as small as we please by sufficiently increasing n.*

If r be the radix of any particular scale of notation, the fraction

$$\frac{p_1}{r} + \frac{p_2}{r^2} + \ldots + \frac{p_n}{r^n}$$

---

* The arrangement of the multiplication in Examples 1 and 2 is purposely varied, because, although it is of no consequence here, sometimes the one order is more convenient, sometimes the other. A similar variety is introduced in § 6, Examples 1 and 2.

is usually called a radix fraction. We may detach the coefficients and place them in apposition, just as in the case of integers, a point being placed first to indicate fractionality.*
Thus we may write

$$\frac{A}{B} = \cdot p_1 p_2 p_3 \ldots p_n,$$

where p_1 in the first place after the radix point stands for p_1/r, p_2 in the second place stands for p_2/r^2, and so on.

Since the digits $p_1 p_2 p_3 \ldots p_n$ are the integral part of the quotient obtained by dividing Ar^n by B, the radix fraction cannot terminate unless Ar^n is a multiple of B for some value of n. Hence, if we suppose A/B reduced to its lowest terms, so that A is prime to B, we see that the radix fraction cannot terminate unless the prime factors of B (see chap. iii., § 10) be powers of prime factors which occur in r. For example, since $r = 10 = 2 \times 5$, no vulgar fraction can reduce to a terminating decimal fraction unless its denominator be of the form $2^m 5^n$.

In all cases, however, where the radix fraction does not terminate, its digits must repeat in a cycle of not more than $B - 1$ figures; for in the course of the division no more than $B - 1$ different remainders can occur (if we exclude 0), and as soon as one of the remainders recurs the figures in the quotient begin to recur.

Example 1. To express 2/3 as a radix fraction in the scale of 10 to within 1/100000th—

$$\frac{2}{3} = \frac{200000}{3 \times 10^5} = \frac{66666 + \frac{2}{3}}{10^5},$$

$$= \frac{6 \times 10^4 + 6 \times 10^3 + 6 \times 10^2 + 6 \times 10 + 6}{10^5} + \frac{2/3}{100000},$$

$$= \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \frac{6}{10^5} + \mathrm{F},$$

where $$\mathrm{F} = \frac{2/3}{100000} < \frac{1}{100000}.$$

* Napier of Merchiston was apparently the first who used the modern form of the notation for decimal fractions. The idea of the regular progression of decimals is older. Stevin fully explains its advantages in his *Arithmétique* (1585); and germs of the idea may be traced much farther back. According to those best qualified to judge, Napier was the first who fully appreciated the

In other words, we have to the required degree of accuracy

$$\frac{2}{3} = \cdot 66666.$$

It is obvious from the repetition of the figures that if we take n 6's after the point we shall have the value of 2/3 correct within $1/10^n$th of its value.

Example 2. Let the fraction be 5/64. Since $64 = 2^6$ this fraction ought to be expressible as a terminating decimal. We have in fact

$$\frac{5}{64} = \frac{5000000}{64 \times 10^6} = \frac{78125}{10^6},$$
$$= \cdot 078125.$$

Example 3. To express 2/3 as a radix fraction in the scale of 2 to within $1/2^6$th.

$$\frac{2}{3} = \frac{2 \times 2^6}{3 \times 2^6} = \frac{128/3}{2^6} = \frac{42 + 2/3}{2^6}.$$

Neglecting $\frac{2/3}{2^6}$, which is $< \frac{1}{2^6}$, and expressing 42 in the scale of 2, we have

$$\frac{2}{3} = \frac{101010}{2^6} = \cdot 101010 \text{ (radix 2)}.$$

§ 6.] When a fraction is given expressed as a radix fraction in any scale, and it is required to express it as a radix fraction in some other scale, the following process is convenient.

Let ϕ be the fraction expressed in the old scale, r the new radix, and suppose

$$\phi = \frac{p_1}{r} + \frac{p_2}{r^2} + \frac{p_3}{r^3} + \ldots,$$

then

$$r\phi = p_1 + \frac{p_2}{r} + \frac{p_3}{r^2} + \ldots$$
$$= p_1 + \phi_1 \text{ say.}$$

Now ϕ_1 is a proper fraction, hence p_1 is the integral part of $r\phi$.

Again

$$r\phi_1 = p_2 + \frac{p_3}{r} + \ldots$$
$$= p_2 + \phi_2 \text{ say.}$$

So that p_2 is the integral part of $r\phi_1$, and so on.

It is obvious that a vulgar fraction in any scale of notation must transform into a vulgar fraction in any other; and we shall

---

operational use of the decimal point; and in his *Constructio* (written long before his death, although not published till 1619) it is frequently used. See Glaisher, Art. "Napier," *Encyclopædia Britannica*, 9th ed.; also Rae's recent translation of the *Constructio*, p. 89.

show in a later chapter (see Geometrical Progression) that every repeating radix fraction can be represented by a vulgar fraction. Hence it is clear that every fraction which is a terminating or a repeating radix fraction in any scale can be represented in any other scale by a radix fraction which either terminates or else repeats. It is not, however, true that a terminating radix fraction always transforms into a terminating radix fraction or a repeater into a repeater. Non-terminating non-repeating radix fractions transform, of course, into non-terminating non-repeating radix fractions, otherwise we should have the absurdity that a vulgar fraction can be transformed into a non-terminating non-repeating radix fraction.

It is obvious that all the rules for operating with decimal fractions apply to radix fractions generally.

Example 1. Reduce 3·168 and 11·346 to the scale of 7, and multiply the latter by the former in that scale; the work to be accurate to 1/1000th throughout.

The required degree of accuracy involves the 5th place after the radical point in the scale of 7.

·168	·346
7	7
1)·176	2)·422
7	7
1)·232	2)·954
7	7
1)·624	6)·678
7	7
4)·368	4)·746
7	7
2)·576	5)·222
3·168 = 3·11142 (radix 7).	11·346 = 14·22645.

$$\begin{array}{r} 14\cdot\dot{2}\dot{2}\dot{6}\dot{4}\dot{5} \\ 3\cdot11142 \\ \hline 46\cdot01601 \\ 1\cdot42265 \\ 14227 \\ 1423 \\ 632 \\ 32 \\ \hline 50\cdot64146 \end{array}$$

On account of the duodecimal division of the English foot into 12 inches, the duodecimal scale is sometimes convenient in mensuration.

Example 2. Find the number of square feet and inches in a rectangular carpet, whose dimensions are $21'\ 3\frac{1}{2}''$ by $13'\ 11\frac{3}{4}''$. Expressing these lengths in feet and duodecimals of a foot, we have

$$21'\ 3\tfrac{1}{2}''=19\cdot 36.$$
$$13'\ 11\tfrac{3}{4}''=11\cdot\epsilon 9.$$

If, following Oughtred's arrangement, we reverse the multiplier, and put the unit figure under the last decimal place which is to be regarded, the calculation runs thus—

$$\begin{array}{r} 19\cdot 36 \\ 9\epsilon 11 \\ \hline 19360 \\ 1936 \\ 1763 \\ 13\epsilon \\ \hline 209\cdot 78 \end{array}$$

$$209 \text{ (radix 12)}=288+9=297 \text{ (radix 10) feet.}$$
$$\cdot 78 \text{ (radix 12)}=7\times 12+8=92 \text{ square inches.}$$

Hence the area is 297 feet 92 inches.

§ 7.] *If a number N be expressed in the scale of r, and if we divide N and the sum of its digits by $r-1$, or by any factor of $r-1$, the remainder is the same in both cases.*

Let $\quad N = p_0 + p_1 r + p_2 r^2 + \ldots + p_n r^n.$

Hence $\quad N - (p_0 + p_1 + \ldots + p_n) = p_1(r-1) + p_2(r^2-1) + \ldots$
$$+ p_n(r^n - 1) \qquad (1).$$

Now, m being an integer, $r^m - 1$ is divisible by $r-1$ (see chap. v., § 17). Hence every term on the right is divisible by $r-1$, and therefore by any factor of $r-1$. Hence, ρ being $r-1$, or any factor of it, and μ some integer, we have

$$N - (p_0 + p_1 + \ldots + p_n) = \mu\rho \qquad (2).$$

Suppose now that the remainder, when N is divided by ρ, is σ, so that $N = \nu\rho + \sigma$. Then (2) gives

$$p_0 + p_1 + \ldots + p_n = (\nu - \mu)\rho + \sigma \qquad (3),$$

which shows that when $p_0 + p_1 + \ldots + p_n$ is divided by ρ the remainder is σ.

Cor 1. *In the ordinary scale, if we divide any integer by 9 or by 3, the remainder is the same as the remainder we obtain by dividing the sum of its digits by 9 or by 3.*

For example, $31692 \div 9$ gives for remainder 3, and so does $(3+1+6+9+2) \div 9$.

Cor. 2. *It also follows that the sum of the digits of every multiple of 9 or 3 must be a multiple of 9 or 3.* For example,

$$\begin{array}{rr} 2 \times 9 = 18 & 1 + 8 = 9 \\ 13 \times 9 = 117 & 1 + 1 + 7 = 9 \\ 128 \times 9 = 1152 & 1 + 1 + 5 + 2 = 9 \\ 128 \times 3 = 384 & 3 + 8 + 4 = 15 = 5 \times 3. \end{array}$$

§ 8.] On Cor. 1 of § 7 is founded the well-known method of checking arithmetical calculations called "casting out the nines."

Let $L = MN$; then, if $L = l9 + L'$, $M = m9 + M'$, $N = n9 + N'$, so that L', M', N' are the remainders when L, M, N are divided by 9, we have—

$$\begin{aligned} l9 + L' &= (m9 + M')(n9 + N'), \\ &= mn81 + (M'n + N'm)9 + M'N', \\ &= (mn9 + M'n + N'm)9 + M'N'; \end{aligned}$$

whence it appears that L' and $M'N'$ must have the same remainder when divided by 9. L', M', N' are obtained in accordance with Cor. 1 of § 7 by dividing the sums of the digits in the respective numbers by 9.

Example 1. Suppose we wish to test the multiplication

$$47923 \times 568 = 27220264.$$

To get the remainder when 47923 is divided by 9, proceed thus: $4+7=11$, cast out 9 and 2 is left; $2+9=11$, cast out 9; $2+2+3=7$. The remainder is 7. Similarly from 568 the remainder is 1, and from 27220264, 7. Now $7 \times 1 \div 9$ gives of course the same remainder as $7 \div 9$. There is therefore a strong presumption that the above multiplication is correct. It should be observed, however, that there are errors which this test would not detect; if we replaced the product by 27319624, for instance, the test would still be satisfied, but the result would be wrong.

In applying this test to division, say to the case $L/M = N + P/M$, since we have $L = MN + P$, and therefore $L - P = MN$, we have to cast out the nines from L, P, M, and N, and so obtain L', P', M', and N' say. Then the test is that $L' - P'$ shall be the same as the result of casting out the nines from $M'N'$.

Example 2. Let us test—

$$27220662 \div 568 = 47923 + 398 \div 568,$$

or

$$27220662 = 47923 \times 568 + 398.$$

Here

$$L' - P' = 0 - 2 = -2,$$
$$M'.\,N' = 7 \times 1 = 9 - 2.$$

The test is therefore satisfied.

§ 9.]* The following is another interesting method for expanding any proper fraction A/B in a series of fractions with unit numerators:—

Let $q_1, q_2, q_3, \ldots, q_n$, *and* $r_1, r_2, r_3, \ldots, r_n$, *be the quotients and remainders respectively when* B *is divided by* $A, r_1, r_2, \ldots, r_{n-1}$ *respectively, then*

$$\frac{A}{B} = \frac{1}{q_1} - \frac{1}{q_1q_2} + \frac{1}{q_1q_2q_3} - \ldots + \frac{(-1)^{n-1}}{q_1q_2\ldots q_n} + F \qquad (1),$$

where $F = (-1)^n r_n/q_1q_2\ldots q_nB$, *that is,* F *is numerically less than* $1/q_1q_2\ldots q_n$.

For we have by hypothesis

$$B = Aq_1 + r_1, \text{ therefore } A/B = 1/q_1 - r_1/q_1B \qquad (2),$$
$$B = r_1q_2 + r_2, \text{ therefore } r_1/B = 1/q_2 - r_2/q_2B \qquad (3),$$
$$B = r_2q_3 + r_3, \text{ therefore } r_2/B = 1/q_3 - r_3/q_3B \qquad (4),$$

and so on.

From (2), (3), (4), we have successively

$$\frac{A}{B} = \frac{1}{q_1} - \frac{1}{q_1q_2} + \frac{1}{q_1q_2}\left(\frac{r_2}{B}\right),$$
$$= \frac{1}{q_1} - \frac{1}{q_1q_2} + \frac{1}{q_1q_2q_3} - \frac{1}{q_1q_2q_3}\left(\frac{r_3}{B}\right);$$

and so on.

Since $r_1, r_2, \ldots, r_n$ go on diminishing, it is obvious that, if A and B be integers as above supposed, the process of successive division must come to a stop, the last remainder being 0. Hence

* In his *Essai d'Analyse Numérique sur la Transformation des Fractions* (*Œuvres*, t. vii. p. 313), on which the present chapter is founded, Lagrange attributes the theorem of § 9 to Lambert (1728-1777). Heis, *Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra* (1882), p. 322, has applied series of this character to express incommensurable numbers such as logarithms, square roots, &c. In the same connection see also Sylvester, *American Jour. Math.*, 1880. See also Cyp. Stéphanos, *Bull. Soc. Math. Fr.* 7 (1879), p. 81; G. Cantor, *Zeitsch. f. Math.* 14 (1869), p. 124; J. Lüroth, *Math. Ann.* 21 (1883), p. 411.

every vulgar fraction can be converted into a terminating series of the form

$$\frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \ldots\ldots .$$

Example.

$$\frac{113}{244} = \frac{1}{2} - \frac{1}{2.13} + \frac{1}{2.13.24} - \frac{1}{2.13.24.61}.$$

From this resolution we conclude that $1/2 - 1/2.13$ represents $113/244$ within $1/26$th, and that $1/2 - 1/2.13 + 1/2.13.24$ represents $113/244$ within $1/624$th.

Exercises XIII.

(1.) Express 16935 (scale of 10) in the scale of 7.

(2.) Express 16·935 (scale of 10) in the scale of 7.

(3.) Express $315\cdot\dot{3}\dot{4}$ (scale of 10) in the scale of 11.

(4.) Express $\tau 7 \epsilon 9 \epsilon\epsilon$ (scale of 12) in the scale of 10.

(5.) Express $1\tau 8\epsilon 54$ (scale of 12) in the scale of 9.

(6.) Express 345·361 (scale of 7) in the scale of 3.

(7.) Express 112/315 (scale of 10) as a radix fraction in the scale of 6.

(8.) Express 3169 in the form $p + q3 + r3.5 + s3.5.7 + \&c.$, where $p<3$, $q<5$, $r<7$, &c.

(9.) Express 7/11 in the form $p/2 + q/2.3 + r/2.3.4 + \&c.$, where $p<2$, $q<3$, $r<4$, &c.

(10.) Express 113/304 in the form $p/3 + q/3.5 + r/3^2.5 + s/3^2.5^2 + t/3^3.5^2 + \&c.$, where $p<3$, $q<5$, $r<3$, &c.

(11.) Multiply 31263 by 56341 in the scale of 7.

(12.) Find correct to 4 places $31\cdot3432 \times 1\cdot50323$, both numbers being in the scale of 6.

(13.) Find to 5 places $31\cdot3432 \div 2\cdot67312$, both numbers being in the scale of 12.

(14.) Extract the square root of 365·738 (scale of 9) to 3 places.

(15.) Express 887/1103 in the form $1/q_1 - 1/q_1q_2 + 1/q_1q_2q_3 - \&c.$

(16.) Show how to make up a weight of 35 lbs. by taking single weights of the series 1 lb., 2 lbs., 2^2 lbs., &c.

(17.) With a set of weights of 1 lb., 5 lbs., 5^2 lbs., &c., how can 7 cwt. be weighed? First, by putting weights in one scale only and using any number of equal weights not exceeding four. Second, by putting weights in either scale but not using more than two equal weights.

(18.) Find the area of a rectangle 35 ft. $3\frac{1}{2}$ in. by 23 ft. $6\frac{3}{4}$ in.

(19.) Find the area of a square whose side is 17 ft. 4 in.

(20.) Find the volume of a cube whose edge is 3 ft. $9\frac{1}{4}$ in.

(21.) Find the side of a square whose area is 139 sq. ft. 130 sq. in.

(22.) Expressed in a certain scale of notation, 79 (scale of 10) becomes 142; find the radix of that scale.

(23.) In what scale of notation does 301 represent a square integer?

(24.) A number of 3 digits in the scale of 7 has its digits reversed when expressed in the scale of 9; find the digits.

(25.) If 1 be added to the product of four consecutive integers the result is always a square integer; and in four cases out of five the last digit (in the common scale) is 1, and in the remaining case 5.

(26.) Any integer of four digits in the scale of 10 is divisible by 7, provided its first and last digits be equal, and the hundreds digit twice the tens digit.

(27.) If any integer be expressed in the scale of r, the difference between the sums of the integers in the odd and even places respectively gives the same remainder when divided by $r+1$ as does the integer itself when so divided. Deduce a test of multiplication by "casting out the elevens."

(28.) The difference of any two integers which are expressed in the scale of 10 by the same digits differently arranged is always divisible by 9.

(29.) If a number expressed in the ordinary scale consist of an even number of digits so arranged that those equidistant from the beginning and end are equal, it is divisible by 11.

(30.) Two integers expressed in the ordinary scale are such that one has zeros in all the odd places, the other zeros in all the even places, the remaining digits being the same in both, but not necessarily arranged in the same order. Show that the sum of the two integers is divisible by 11.

(31.) The rule for identifying leap year is that the number formed by the two last digits of the year must be divisible by 4. Show that this is a general criterion for divisibility by 4, and state the corresponding criterion for divisibility by 2^n.

(32.) If the last three digits of an integer be $p_2p_1p_0$, show that the integer will be exactly divisible by 8, provided $p_0+2p_1+4p_2$ be exactly divisible by 8.

(33.) Show that the sum of all the numbers which can be formed with the digits 3, 4, 5 is divisible by the sum of these digits, and generalise the theorem.

(34.) If p/n and $(n-p)/n$, $p<n$, be converted into circulating decimals, find the relation between the figures in their periods.

(35.) If, in converting the proper fraction a/b into a decimal, a remainder equal to $b-a$ occurs, show that half the circulating period has been found, and that the rest of it will be found by subtracting in order from 9 the digits already found. Generalise this theorem.

(36.) In the scale of 11 every integer which is a perfect 5th power ends in one or other of the three digits 0, 1, τ.

(37.) In the scale of 10 the difference between the square of every number of two digits and the square of the number formed by reversing the digits, is divisible by 99.

(38.) A number of six digits whose 1st and 4th, 2nd and 5th, 3rd and 6th digits are respectively the same is divisible by 7, by 11, and by 13.

(39.) Show that the units digit of every integral cube is either the same as that of the cube root or else is the complementary digit. (By the complementary digit to 3 is meant $10-3$, that is, 7.)

(40.) If in the scale of 12 a square integer (not a multiple of 12) ends

with 0, the preceding digit is 3, and the cube of the square root ends with 60.

(41.) If a be such that $a^m + a = r$, then any number is divisible by a^m, provided the first m integers $p_0, p_1, \ldots, p_{m-1}$ of its expression in the scale of r are such that $p_0 + p_1 a + \ldots + p_{m-1} a^{m-1}$ is divisible by a^m.

(42.) The digits of a are added, the digits of this sum added, and so on, till a single digit is arrived at. This last is denoted by $\phi(a)$. Show that $\phi(a+b) = \phi\{\phi(a) + \phi(b)\}$; and that the values of $\phi(8n)$ for $n = 1, 2, \ldots, a$, successively consist of the nine digits continually repeated in descending order.

(43.) A number of 3 digits is doubled by reversing its digits: show that the same holds for the number formed by the first and last digit, and that such a number can be found in only one scale out of three.

CHAPTER X.

Irrational Functions.

GENERALISATION OF THE CONCEPTION OF AN INDEX. INTERPRETATION OF x^0, $x^{p/q}$, x^{-m}.

§ 1.] The definition of an index given in chap. ii., § 1, becomes meaningless if the index be other than a positive integer.

In accordance with the generalising spirit of algebra we agree, however, that the use of indices shall not be restricted to this particular case. We agree, in fact, that no restriction is to be put upon the value of the index, and lay down merely that the use of the indices shall in every case be subject to the laws already derived for positive integral indices. Less than this we cannot do, since these laws were derived from the fundamental laws of algebra themselves, to which every algebraical symbol must be subject.

The question now arises, What signification shall we attribute to x^m in these new cases? We are not at liberty to proceed arbitrarily, and give any meaning we please, for we have already by implication defined x^m, inasmuch as it has been made subject to the general laws laid down for indices.

§ 2.] *Case of $x^{p/q}$ where p and q are any positive integers.* Let z denote the value of $x^{p/q}$, whatever it may be; then, since $x^{p/q}$ is to be subject to the first law of indices, we must have—

$$\begin{aligned} z^q &= z \times z \times z \times \ldots \ q \text{ factors}, \\ &= x^{p/q} \times x^{p/q} \times x^{p/q} \times \ldots \ q \text{ factors}, \\ &= x^{p/q+p/q+p/q+\ldots \ q \text{ terms}}, \\ &= x^p. \end{aligned}$$

In other words, z is such that its qth power is x^p, that is, z is what is called a qth root of x^p, which is usually denoted by $\sqrt[q]{x^p}$.

Hence $$x^{p/q} = \sqrt[q]{x^p}.$$
In particular, if $p = 1$,
$$x^{1/q} = \sqrt[q]{x}.$$

We have now to consider how far an algebraical value of a qth root of every algebraical quantity can be found.

In the case of a real positive quantity k, since z^q passes continuously* through all positive values between 0 and $+\infty$ as z passes through all positive values between 0 and $+\infty$, it is clear that, for some value of z between 0 and $+\infty$, we must have $z^q = k$. In other words, there exists a real positive value of $\sqrt[q]{k}$.

Unless the contrary is stated we shall, when k is positive, take $k^{1/q}$ as standing for this real positive value.

The student should, however, remark that when q is even, $= 2r$ say, there is at least one other real value of $\sqrt[q]{k}$; for, since $(-z)^{2r} = z^{2r}$, if we have found a positive value of z such that $z^{2r} = k$, that value with its sign changed will also satisfy the requirements of the problem.

Next let k be a negative quantity. If q be odd, then, since z^q passes through all values from $-\infty$ to 0 as z passes through all values from $-\infty$ to 0, there must be some one real negative value of z, such that $z^q = k$. In other words, if q be odd, there is a real negative value of $\sqrt[q]{k}$.

If q be even, then, since every even power of a real quantity (no matter whether $+$ or $-$) is positive, there is no real value of z. Hence, if k be negative and q even, $\sqrt[q]{k}$ is imaginary. This case must be left for future discussion.

It will be useful, however, for the student to know that ultimately it will be proved that $\sqrt[q]{k}$ has in every case q different values, expressions for which, in the form of complex numbers, can be found. Of these values one, or at most two, may be real, as indicated above (see chap. xii.)

* For a fuller discussion of the point here involved see chaps. xv. and xxv.

Only in the case where k is the pth power of a rational quantity can $\sqrt[p]{k}$ be rational.

Example.

If $k = +h^{2p}$,

$\sqrt[2p]{k}$ has two real values, $+h$ and $-h$.

If $k = +h^{2p+1}$,

$\sqrt[2p+1]{k}$ has one real value, $+h$.

If $k = -h^{2p+1}$,

$\sqrt[2p+1]{k}$ has one real value, $-h$.

In all that follows in this chapter, we shall restrict the radicand, k, to be positive; we shall regard only the real positive value of the qth root of k; and this (which is called the PRINCIPAL VALUE OF THE ROOT) *is what we understand to be the meaning of $k^{1/q}$.*

The theory of fractional indices could (as in the first edition of this volume) be extended so as to cover the case of a negative radicand, but only so far as the order of the root is odd. The practical advantage gained by this extension is not worth the trouble which it causes by complicating the demonstrations. We think it better also, from a scientific and educational, as well as from a practical point of view, to consider the radication of negative radicands as a particular case of the radication of complex radicands (see chap. xii., § 19).

§ 3.] We have now to show that the meaning just suggested for $x^{p/q}$ is consistent with all the Laws of Indices laid down in chap. ii. The simplest way of doing this is to re-prove these laws for the newly defined symbol $x^{p/q}$.

We remark in the first place that it is necessary to prove only I. (a), II., and III. (a); because, as has been shown in chap. ii., I. (β) can be deduced from I. (a), and III. (β) from III. (a), without any appeal to the definition of x^m.

To prove I. (a), consider $x^{p/q}$ and $x^{r/s}$, where p, q, r, s are positive integers, and let

$$z = x^{p/q} x^{r/s}.$$

Then, since $x^{p/q}$ and $x^{r/s}$ are, by hypothesis, each real and positive, z is also real and positive. Also

$$\begin{aligned} z^{qs} &= (x^{p/q} x^{r/s})^{qs}, \\ &= (x^{p/q})^{qs}(x^{r/s})^{qs}, \\ &= \{(x^{p/q})^q\}^s\{(x^{r/s})^s\}^q, \end{aligned}$$

all by the laws for positive integral indices, regarding which there is no question.

Now, by the meanings assigned to $x^{p/q}$ and $x^{r/s}$, we have $(x^{p/q})^q = x^p$ and $(x^{r/s})^s = x^r$. Hence

$$\begin{aligned} z^{qs} &= (x^p)^s(x^r)^q, \\ &= x^{ps}x^{qr}, \\ &= x^{ps+qr}, \end{aligned}$$

by the laws for positive integral indices.

It now follows that z is the qsth root of x^{ps+qr}; and, since z is real and positive, it must be that qsth root which we denote by $x^{(ps+qr)/qs}$. Therefore

$$z = x^{(ps+qr)/qs},$$

that is to say,

$$z = x^{p/q+r/s}.$$

The proof is easily extended to any number of factors.

To prove Law II., consider $(x^{p/q})^{r/s}$, where p, q, r, s are positive integers,

and let $$z = (x^{p/q})^{r/s}.$$

Then, since, by hypothesis, $x^{p/q}$ is real and positive, therefore $(x^{p/q})^{r/s}$, that is z, is real and positive. Also

$$\begin{aligned} z^{qs} &= [(x^{p/q})^{r/s}]^{qs}, \\ &= [\{(x^{p/q})^{r/s}\}^s]^q, \\ &\qquad \text{by laws for positive integral indices;} \\ &= [(x^{p/q})^r]^q, \\ &\qquad \text{by definition of a fractional index;} \\ &= (x^{p/q})^{qr}, \\ &= [(x^{p/q})^q]^r, \\ &\qquad \text{by laws of positive integral indices;} \\ &= [x^p]^r, \\ &\qquad \text{by definition of a fractional index;} \\ &= x^{pr}, \\ &\qquad \text{by laws of positive integral indices.} \end{aligned}$$

Hence z is a qsth root of x^{pr}, and, since z is real and positive, we must have

$$z = x^{pr/qs},$$

that is, $$z = x^{(p/q)(r/s)}.$$

Lastly, to prove Law III. (α), let

$$z = x^{p/q} y^{p/q}.$$

Then, since, by hypothesis, $x^{p/q}$ and $y^{p/q}$ are each real and positive, z is real and positive. Also

$$\begin{aligned} z^q &= (x^{p/q} y^{p/q})^q, \\ &= (x^{p/q})^q (y^{p/q})^q, \\ &\qquad \text{by laws for positive integral indices;} \\ &= x^p y^p, \\ &\qquad \text{by definition for a fractional index;} \\ &= (xy)^p, \\ &\qquad \text{by laws for positive integral indices.} \end{aligned}$$

Hence z is a qth root of $(xy)^p$; and, since z is real and positive, we must have

$$z = (xy)^{p/q}.$$

The proof is obviously applicable where there is any number of factors, x, y, . . .

§ 4.] Although it is not logically necessary to give separate proofs of Laws I. (β) and III. (β), the reader should as an exercise construct independent proofs of these laws for himself.

It should be noticed that in last paragraph we have supposed both the indices p/q and r/s to be fractions. The case where either is an integer is met by supposing either $q = 1$ or $s = 1$; the only effect on the above demonstrations is to simplify some of the steps.

§ 5.] Before passing on to another case it may be well to call attention to paradoxes that arise if the strict limitation as to sign of $x^{p/q}$ be departed from.

By the interpretation of a fractional index

$$x^{4/2} = \sqrt[2]{x^4} = \pm x^2.$$

But

$$x^{4/2} = x^2,$$

which is right if we take $x^{4/2}$ to stand for the positive value of $\sqrt[2]{x^4}$; but leads to the paradox $x^2 = -x^2$ if we admit the negative value.

A similar difficulty would arise in the application of the law,

$$(x^m)^n = x^{mn} = (x^n)^m;$$

for example, $$(4^{\frac{1}{2}})^2 = (4^2)^{\frac{1}{2}}$$
would lead to $$(\pm 2)^2 = \pm 4,$$
that is, $$4 = \pm 4,$$
if both values were admitted. Such difficulties are always apt to arise with $x^{p/q}$ where the fraction p/q is not at its lowest terms.

The true way out of all such difficulties is to define and discuss x^n as a continuously varying function of n, which is called the exponential function. In the meantime fractional indices are introduced merely as a convenient notation in dealing with quantities which are (either in form or in essence) irrational; and for such purposes the limited view we have given will be sufficient.

§ 6.] *Case of* x^0. This case arises naturally as the extreme case of Law I. (β), when $n = m$; for, if we are to maintain that law intact, we must have, provided $x \neq 0$,*

$$x^{m-m} = x^m / x^m,$$
that is, $$x^0 = 1.$$
This interpretation is clearly consistent with Law I. (α), for
$$x^m \times x^0 = x^{m+0}$$
simply means
$$x^m \times 1 = x^m,$$
which is true, whatever the interpretation of x^m may be.

Again, $$x^{m0} = (x^m)^0,$$
that is $$x^0 = (x^m)^0,$$
simply means $1 = 1$ by our interpretation;
and $$x^{m0} = (x^0)^m,$$
or $$x^0 = (x^0)^m,$$
gives $$1 = 1^m,$$
which is right, even if m be a positive fraction, provided we adopt the properly restricted interpretation of a fractional index given above. The interpretation is therefore consistent with II. The interpretation $x^0 = 1$ is also consistent with III. (α), for
$$x^0 y^0 = (xy)^0$$
simply means $$1 \times 1 = 1.$$

* This provision is important since the form 0^0 is indeterminate (see chap. xxv.)

§ 7.] *Case of x^{-m}*, where m is any real positive (or signless) number, and $x \neq 0$.

Let $z = x^{-m}$, then, since $x^m \neq 0$, we have

$$z = x^{-m} \times x^m \div x^m,$$
$$= x^{-m+m}/x^m,$$

if Law I. (α) is to hold for negative indices. Whence

$$z = x^0/x^m,$$
$$= 1/x^m$$

by last paragraph. In other words, x^{-m} is the reciprocal of x^m.

As an example of the reconciliation of this with the other laws, let us prove I. (α), say that

$$x^{-m}x^{-n} = x^{-m-n}.$$

By our definition, we have

$$x^{-m}x^{-n} = (1/x^m)(1/x^n),$$
$$= 1/x^m x^n,$$
$$= 1/x^{m+n},$$

the last step by the laws already established for all positive indices;

$$= x^{-(m+n)},$$

by definition of a negative index.

Hence $$x^{-m}x^{-n} = x^{-m-n}.$$

In like manner we could show that

$$x^m x^{-n} = x^{m-n}.$$

The verification of the other laws may be left as an exercise.

§ 8.] The student should render himself familiar with the expression of the results of the laws of indices in the equivalent forms with radicals; and should also, as an exercise, work out demonstrations of these results without using fractional indices at all.

For example, he should prove directly that

$$\sqrt[p]{x}\sqrt[q]{x} = \sqrt[pq]{x^{p+q}} \qquad (1);$$
$$\sqrt[s]{\{\sqrt[q]{x^p}\}^r} = \sqrt[qs]{x^{pr}} = \sqrt[q]{\{\sqrt[s]{x^r}\}^p} \qquad (2);$$
$$\sqrt[p]{x}\sqrt[p]{y}\sqrt[p]{z} = \sqrt[p]{(xyz)} \qquad (3);$$
$$\sqrt[n]{x^m}/\sqrt[n]{y^m} = \sqrt[n]{(x/y)^m} \qquad (4).$$

EXAMPLES OF OPERATION WITH IRRATIONAL FORMS.

§ 9.] Beyond the interpretations $x^{p/q}$, x^0, x^{-m}, the student has nothing new to learn, so far as mere manipulation is concerned, regarding fractional indices and irrational expressions in general. Still some practice will be found necessary to acquire the requisite facility. We therefore work out a few examples of the more commonly occurring transformations. In some cases we quote at each step the laws of algebra which are appealed to; in others we leave it as an exercise for the student to supply the omission of such references.

Example 1.

To express $A\sqrt[m]{B}$ in the form $\sqrt[m]{P}$.

$$\begin{aligned} A\sqrt[m]{B} &= AB^{1/m} = (A^m)^{1/m}B^{1/m}, \text{ by law of indices II.,} \\ &= (A^mB)^{1/m}, \text{ by law of indices III. } (\alpha), \\ &= \sqrt[m]{(A^mB)}. \end{aligned}$$

Example 2.

$$\sqrt[m]{A} = \sqrt[mp]{A^p};$$

for

$$\begin{aligned} \sqrt[m]{A} &= A^{1/m} = A^{p/mp}, \\ &= \sqrt[mp]{A^p}. \end{aligned}$$

Example 3.

$$\begin{aligned} \sqrt[m]{x^{pm+q}} &= x^{(pm+q)/m}, \\ &= x^{p+q/m}, \\ &= x^p \times x^{q/m}, \text{ by law of indices I. } (\alpha), \\ &= x^p\sqrt[m]{x^q}. \end{aligned}$$

Example 4.

To express $\sqrt[q]{x^p}/\sqrt[s]{y^r}$ as the root of a rational function of x and y.

$$\begin{aligned} \sqrt[q]{x^p}/\sqrt[s]{y^r} &= x^{p/q}/y^{r/s} = x^{ps/qs}/y^{qr/qs}, \\ &= (x^{ps})^{1/qs}/(y^{qr})^{1/qs}, \\ &= (x^{ps}/y^{qr})^{1/qs}, \\ &= \sqrt[qs]{(x^{ps}/y^{qr})}. \end{aligned}$$

Example 5.

$$\begin{aligned} \sqrt{32} &= \sqrt{(16 \times 2)}, \\ &= \sqrt{16} \times \sqrt{2}, \\ &= 4 \times \sqrt{2}. \end{aligned}$$

Example 6.

$$\begin{aligned}
&2 \times \sqrt{2} \times \sqrt[4]{2} \times \sqrt[8]{4} \\
&\quad = 2 \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times 2^{\frac{2}{8}}, \\
&\quad = 2^{1+\frac{1}{2}+\frac{1}{4}+\frac{2}{8}}, \\
&\quad = 2^{\frac{16}{8}}, \\
&\quad = 2^2, \\
&\quad = 4.
\end{aligned}$$

Example 7.

$$\begin{aligned}
(a^2 - x^2)^{m/2n} &= \{a^2(1 - x^2/a^2)\}^{m/2n} \\
&= (a^2)^{m/2n}(1 - x^2/a^2)^{m/2n}, \\
&= a^{m/n}(1 - x^2/a^2)^{m/2n}.
\end{aligned}$$

Example 8.

$$\begin{aligned}
&\sqrt{(yx + x^2)} \times \sqrt{(yz + zx)} \\
&\quad = \sqrt{\{x(y + x)\}} \times \sqrt{\{z(y + x)\}}, \\
&\quad = \sqrt{x} \times \sqrt{(y + x)} \times \sqrt{z} \times \sqrt{(y + x)}, \\
&\quad = \sqrt{(xz)} \times \{\sqrt{(y + x)}\}^2, \\
&\quad = (y + x) \times \sqrt{(xz)}.
\end{aligned}$$

Example 9.

$$\begin{aligned}
&\sqrt{240} + \sqrt{40} \\
&\quad = \sqrt{(16 \times 3 \times 5)} + \sqrt{(4 \times 2 \times 5)}, \\
&\quad = \sqrt{16}\sqrt{3}\sqrt{5} + \sqrt{4}\sqrt{2}\sqrt{5}, \\
&\quad = \sqrt{5}(4\sqrt{3} + 2\sqrt{2}).
\end{aligned}$$

Example 10.

$$\begin{aligned}
&(\sqrt{3} + 2\sqrt{2} + 3\sqrt{6})(\sqrt{3} - 2\sqrt{2} + 3\sqrt{6}) \\
&\quad = (\sqrt{3} + 3\sqrt{6})^2 - (2\sqrt{2})^2, \\
&\quad = (\sqrt{3})^2 + 6\sqrt{3}\sqrt{6} + (3\sqrt{6})^2 - (2\sqrt{2})^2, \\
&\quad = 3 + 6\sqrt{(3 \times 6)} + 3^2 \times 6 - 2^2 \times 2, \\
&\quad = 49 + 6\sqrt{18}, \\
&\quad = 49 + 18\sqrt{2}.
\end{aligned}$$

Example 11.

$$\begin{aligned}
&\{\sqrt{(1 - x)} + \sqrt{(1 + x)}\}^4 \\
&\quad = \{(1 - x)^{\frac{1}{2}} + (1 + x)^{\frac{1}{2}}\}^4, \\
&\quad = (1 - x)^2 + (1 + x)^2 \\
&\qquad + 4(1 - x)^{\frac{3}{2}}(1 + x)^{\frac{1}{2}} + 4(1 - x)^{\frac{1}{2}}(1 + x)^{\frac{3}{2}} \\
&\qquad\qquad + 6(1 - x)(1 + x), \\
&\quad = 8 - 4x^2 + 4(1 - x)^{\frac{1}{2}}(1 + x)^{\frac{1}{2}}(1 - x + 1 + x) \\
&\quad = 8 - 4x^2 + 8\sqrt{(1 - x^2)}.
\end{aligned}$$

Example 12.

$$\begin{aligned}
&\sqrt{\left(\frac{x + y}{x - y}\right)} + \sqrt{\left(\frac{x - y}{x + y}\right)} \\
&\quad = \frac{\{\sqrt{(x + y)}\}^2 + \{\sqrt{(x - y)}\}^2}{\sqrt{\{(x - y)(x + y)\}}},
\end{aligned}$$

$$=\frac{x+y+x-y}{\sqrt{(x^2-y^2)}},$$
$$=\frac{2x}{\sqrt{(x^2-y^2)}}.$$

Example 13.

$$(x^{\frac{3}{4}}-x^{\frac{1}{4}}+x^{-\frac{1}{4}}-x^{-\frac{3}{4}})\times(x^{\frac{1}{2}}+1+x^{-\frac{1}{2}})$$
$$\begin{aligned}=x^{\frac{5}{4}}-x^{\frac{3}{4}}+x^{\frac{1}{4}}-x^{-\frac{1}{4}}&\\ +x^{\frac{3}{4}}-x^{\frac{1}{4}}+x^{-\frac{1}{4}}-x^{-\frac{3}{4}}&\\ +x^{\frac{1}{4}}-x^{-\frac{1}{4}}+x^{-\frac{3}{4}}-x^{-\frac{5}{4}}&,\\ =x^{\frac{5}{4}}+x^{\frac{1}{4}}-x^{-\frac{1}{4}}-x^{-\frac{5}{4}}&.\end{aligned}$$

Example 14.

Show that

$$S=\sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}+\sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}=\sqrt{(a+\sqrt{b})}.$$

We have

$$S^2=\frac{a+\sqrt{(a^2-b)}}{2}+\frac{a-\sqrt{(a^2-b)}}{2}+2\sqrt{\left[\frac{\{a+\sqrt{(a^2-b)}\}\{a-\sqrt{(a^2-b)}\}}{4}\right]},$$
$$=a+\sqrt{[a^2-\{\sqrt{(a^2-b)}\}^2]},$$
$$=a+\sqrt{b}.$$

Hence, extracting the square root, we have

$$S=\sqrt{(a+\sqrt{b})}.$$

RATIONALISING FACTORS.

§ 10.] Given certain irrationals, say $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, we may consider rational, and it may be also integral, functions of these. For example, $l\sqrt{p}+m\sqrt{q}+n\sqrt{r}$, and $l(\sqrt{p})^2+m\sqrt{(pq)}+n(\sqrt{q})^2$, are integral functions of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, of the 1st and 2nd degrees respectively, provided l, m, n do not contain $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$. Again, $(l\sqrt{p}+m\sqrt{q})/(l\sqrt{q}+m\sqrt{r})$ is a rational, but not integral, function of these irrationals. $\sqrt{(l\sqrt{p}+m\sqrt{q})}$, on the other hand, is an irrational function of $\sqrt{p}$ and $\sqrt{q}$.

The same ideas may also be applied to higher irrationals, such as $p^{1/m}$, $q^{1/n}$, &c.

§ 11.] Confining ourselves for the present to quadratic irrationals, we shall show that every rational function of a given set of quadratic irrationals, $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, &c., can be

reduced to a linear integral function of the square roots of p, q, r, and of their products, pq, pr, qr, pqr, &c.

This reduction is effected mainly by means of *rationalising factors*, whose nature and use we proceed to explain.

If P *be any integral function of certain given irrationals, and* Q *another integral function of the same, such that the product* QP *is rational so far as the given irrationals are concerned, then* Q *is called a rationalising factor of* P *with respect to the given irrationals.*

It is, of course, obvious that, if one rationalising factor, Q, has been obtained, we may obtain as many others as we please by multiplying Q by any *rational* factor.

§ 12.] *Case of Monomials.*

1°. Suppose we have only quadratic irrational forms to deal with, say two such, namely, $p^{\frac{1}{2}}$ and $q^{\frac{1}{2}}$.

Then the most general monomial integral function of these is

$$I = A(p^{\frac{1}{2}})^{2m+1}(q^{\frac{1}{2}})^{2n+1},$$

where A is rational. There is no need to consider even indices, since $(p^{\frac{1}{2}})^{2m} = p^m$ is rational.

Now I reduces to

$$I = (Ap^m q^n)p^{\frac{1}{2}}q^{\frac{1}{2}},$$

where the part within brackets is rational.

Hence a rationalising factor is $p^{\frac{1}{2}}q^{\frac{1}{2}}$, for we have

$$Ip^{\frac{1}{2}}q^{\frac{1}{2}} = (Ap^m q^n)pq,$$

which is rational.

Example. A rationalising factor of $16 \,.\, 2^{\frac{3}{2}} \,.\, 3^{\frac{5}{2}} \,.\, 5^{\frac{1}{2}}$ is $2^{\frac{1}{2}}3^{\frac{1}{2}}5^{\frac{1}{2}}$, that is, $(30)^{\frac{1}{2}}$.

2°. Suppose we have the irrationals $p^{1/s}$, $q^{1/t}$, $r^{1/u}$, say, and consider

$$I = Ap^{l/s}q^{m/t}r^{n/u},*$$

which is the most general monomial integral function of these.

A rationalising factor clearly is

$$p^{1-l/s}q^{1-m/t}r^{1-n/u},$$

or

$$p^{(s-l)/s}q^{(t-m)/t}r^{(u-n)/u}.$$

* Where of course $l<s$, $m<t$, $n<u$, for if they were not they could be reduced by a preliminary process like that in case 1°.

Example.

$$\mathrm{I}=31\,.\,3^{\frac{7}{6}}\,.\,5^{\frac{3}{5}}\,.\,7^{\frac{3}{2}},$$
$$=31\,.\,3^{1+\frac{1}{6}}\,.\,5^{\frac{3}{5}}\,.\,7^{1+\frac{1}{2}},$$
$$=(31\,.\,3\,.\,7^{1})\,.\,3^{\frac{1}{6}}\,.\,5^{\frac{3}{5}}\,.\,7^{\frac{1}{2}}.$$

A rationalising factor is $3^{\frac{5}{6}}\,.\,5^{\frac{2}{5}}\,.\,7^{\frac{1}{2}}$.

§ 13.] *Case of Binomials.*

1°. The most general form when only quadratic irrationals are concerned is $a\sqrt{p}+b\sqrt{q}$, where a and b are rational; for, if we suppose p a complete square, this reduces to the more special form $\mathrm{A}+\mathrm{B}\sqrt{q}$, where A and B are rational.

A rationalising factor clearly is $a\sqrt{p}-b\sqrt{q}$. For, if

$$\mathrm{I}=a\sqrt{p}+b\sqrt{q},$$
$$\mathrm{I}(a\sqrt{p}-b\sqrt{q})=(a\sqrt{p})^2-(b\sqrt{q})^2,$$
$$=a^2p-b^2q,$$

which is rational.

The two forms $a\sqrt{p}+b\sqrt{q}$ and $a\sqrt{p}-b\sqrt{q}$ are said to be conjugate to each other with reference to $\sqrt{q}$, and we see that any binomial integral function of quadratic irrationals is rationalised by multiplying it by its conjugate.

2°. Let us consider the forms $ap^{\alpha/\gamma}\pm bq^{\beta/\delta}$, to which binomial integral functions of given irrationals can always be reduced.* Let

$$x=ap^{\alpha/\gamma},\quad y=bq^{\beta/\delta},$$
$$\mathrm{I}=ap^{\alpha/\gamma}-bq^{\beta/\delta},$$
$$=x-y.$$

Let m be the L.C.M. of the two integers, γ, δ. Now, using the formula established in chap. iv., § 16, we have

$$(x^{m-1}+x^{m-2}y+\ .\ .\ .\ +xy^{m-2}+y^{m-1})\mathrm{I}=x^m-y^m.$$

Here $x^m-y^m=(a^m\,p^{m\alpha/\gamma}-b^m\,q^{m\beta/\delta})$, where $m\alpha/\gamma$ and $m\beta/\delta$ are integers, since m is divisible by both γ and δ, that is, x^m-y^m is rational.

A rationalising factor is therefore $x^{m-1}+x^{m-2}y+\ .\ .\ +xy^{m-2}+y^{m-1}$, in which x is to be replaced by $ap^{\alpha/\gamma}$, and y by $bq^{\beta/\delta}$

* Tartaglia's problem. See *Cossali Storia dell' Algebra* (1797), vol. ii. p. 266.

The form $ap^{\alpha/\gamma} + bq^{\beta/\delta}$ may be treated in like manner by means of formulæ (4) or (5) of chap. iv., § 16.

Example.

$$I = 3.2^{\frac{1}{3}} - 4.3^{\frac{1}{6}}.$$

Here $m = 6,\quad x = 3.2^{\frac{1}{3}},\quad y = 4.3^{\frac{1}{6}}$;

and a rationalising factor is

$$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$$
$$= 3^5.2^{\frac{5}{3}} + 3^4.4.2^{\frac{4}{3}}.3^{\frac{1}{6}} + 3^3.4^2.2.3^{\frac{1}{3}} + 3^2.4^3.2^{\frac{2}{3}}.3^{\frac{1}{2}} + 3.4^4.2^{\frac{1}{3}}.3^{\frac{2}{3}} + 4^5.3^{\frac{5}{6}},$$
$$= 3^5.2.2^{\frac{2}{3}} + 3^4.8.2^{\frac{1}{3}}.3^{\frac{1}{6}} + 3^3.32.3^{\frac{1}{3}} + 3^2.4^3.2^{\frac{2}{3}}.3^{\frac{1}{2}} + 3.4^4.2^{\frac{1}{3}}.3^{\frac{2}{3}} + 4^5.3^{\frac{5}{6}}.$$

§ 14.] *Trinomials with Quadratic Irrationals.* This case is somewhat more complicated. Let

$$I = \sqrt{p} + \sqrt{q} + \sqrt{r};\text{*}$$

and let us first attempt to get rid of the irrational $\sqrt{r}$. This we can do by multiplying by the conjugate of $\sqrt{p} + \sqrt{q} + \sqrt{r}$ with respect to $\sqrt{r}$, namely, $\sqrt{p} + \sqrt{q} - \sqrt{r}$. We then have

$$(\sqrt{p} + \sqrt{q} - \sqrt{r})I = (\sqrt{p} + \sqrt{q})^2 - (\sqrt{r})^2,$$
$$= p + q - r + 2\sqrt{(pq)} \qquad (1).$$

To get rid of $\sqrt{(pq)}$ we must multiply by the conjugate of $p + q - r + 2\sqrt{(pq)}$ with respect to $\sqrt{(pq)}$. Thus finally

$$\{p + q - r - 2\sqrt{(pq)}\}(\sqrt{p} + \sqrt{q} - \sqrt{r})I = (p + q - r)^2 - \{2\sqrt{(pq)}\}^2,$$
$$= p^2 + q^2 + r^2 - 2pq - 2pr - 2qr.$$

Hence a rationalising factor of I is

$$\{p + q - r - 2\sqrt{(pq)}\}(\sqrt{p} + \sqrt{q} - \sqrt{r}),$$

or

$$(\sqrt{p} - \sqrt{q} + \sqrt{r})(\sqrt{p} - \sqrt{q} - \sqrt{r})(\sqrt{p} + \sqrt{q} - \sqrt{r}) \qquad (2).$$

By considering attentively the factor (2) the student will see that *its constituent factors are obtained by taking every possible arrangement of the signs + and − in*

$$+\sqrt{p} \pm \sqrt{q} \pm \sqrt{r},$$

except the arrangement $+\ +\ +$, which occurs in the given trinomial.

---

* This is really the most general form, for $a\sqrt{p} + b\sqrt{q} + c\sqrt{r}$ may be written $\sqrt{(a^2p)} + \sqrt{(b^2q)} + \sqrt{(c^2r)}$.

Example 1. A rationalising factor of

$$\sqrt{2}-\sqrt{3}+\sqrt{5}$$

is $$(\sqrt{2}-\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5}).$$

Example 2. A rationalising factor of

$$1+2\sqrt{3}-3\sqrt{2}$$

is $$(1+2\sqrt{3}+3\sqrt{2})(1-2\sqrt{3}+3\sqrt{2})(1-2\sqrt{3}-3\sqrt{2}).$$

In actual practice it is often more convenient to work out the rationalisation by successive steps, instead of using at once the factor as given by the rule. But the rule is important, because *it is general, and will furnish a rationalising factor for a sum of any number of quadratic irrationals.*

Example 3. A rationalising factor of

$$1+\sqrt{2}-\sqrt{3}+\sqrt{4}$$

is $$\begin{gathered}(1+\sqrt{2}-\sqrt{3}-\sqrt{4})(1+\sqrt{2}+\sqrt{3}+\sqrt{4})(1+\sqrt{2}+\sqrt{3}-\sqrt{4}) \\ \times(1-\sqrt{2}-\sqrt{3}+\sqrt{4})(1-\sqrt{2}-\sqrt{3}-\sqrt{4})(1-\sqrt{2}+\sqrt{3}+\sqrt{4}) \\ \times(1-\sqrt{2}+\sqrt{3}-\sqrt{4}).\end{gathered}$$

Before giving a formal proof of the general truth of this rule, it will be convenient to enunciate one or two general propositions which are of considerable importance, both for future application and for making clear the general character of the operations which we are now discussing.

§ 15.] *Every integral function of a series of square roots, $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, &c., can be expressed as the sum of a rational term and rational multiples of $\sqrt{p}, \sqrt{q}, \sqrt{r}$, &c., and of their products $\sqrt{(pq)}$, $\sqrt{(pr)}$, $\sqrt{(pqr)}$, &c.**

First, let there be only one square root, say $\sqrt{p}$, and consider any rational integral function of $\sqrt{p}$, say $\phi(\sqrt{p})$. Every term of even degree in $\sqrt{p}$ will be rational, and every term of odd degree, such as $A(\sqrt{p})^{2m+1}$ may be reduced to $(Ap^m)\sqrt{p}$, that is, will be a rational multiple of $\sqrt{p}$. Hence, collecting all the even terms together, and all the odd terms together, we have

$$\phi(\sqrt{p})=P+Q\sqrt{p} \qquad (1),$$

where P and Q are rational.

* Such a sum is called a "Linear Form."

Next, suppose the function to contain two square roots, say $\phi(\sqrt{p}, \sqrt{q})$. First of all, proceeding as before, and attending to $\sqrt{p}$ alone, we get

$$\phi(\sqrt{p}, \sqrt{q}) = P + Q\sqrt{p},$$

where P and Q are rational so far as p is concerned, but are irrational as regards q, being each rational integral functions of $\sqrt{q}$. Reducing now each of these with reference to $\sqrt{q}$ we shall obtain, as in (1),

$$P = P' + Q'\sqrt{q}, \quad Q = P'' + Q''\sqrt{q},$$

and, finally,

$$\begin{aligned}\phi(\sqrt{p}, \sqrt{q}) &= P' + Q'\sqrt{q} + (P'' + Q''\sqrt{q})\sqrt{p},\\ &= P' + P''\sqrt{p} + Q'\sqrt{q} + Q''\sqrt{(pq)} \qquad (2),\end{aligned}$$

which proves the proposition for two irrationals.

If there be three, we have now to treat P', P'', Q', Q'' by means of (1), and we shall evidently thereby arrive at the form $A + B\sqrt{p} + C\sqrt{q} + D\sqrt{r} + E\sqrt{(qr)} + F\sqrt{(rp)} + G\sqrt{(pq)} + H\sqrt{(pqr)}$, and so on.

Cor. It follows at once from the process by which we arrived at (1) that

$$\phi(-\sqrt{p}) = P - Q\sqrt{p}.$$

Hence if $\phi(\sqrt{p})$ be any integral function of $\sqrt{p}$, $\phi(-\sqrt{p})$ is a rationalising factor of $\phi(\sqrt{p})$; and, more generally, if $\phi(\sqrt{p}, \sqrt{q}, \sqrt{r}, \ldots)$ be an integral function of $\sqrt{p}, \sqrt{q}, \sqrt{r}, \ldots$, then, if we take any one of them, say $\sqrt{q}$, and change its sign, the product $\phi(\sqrt{p}, \sqrt{q}, \sqrt{r}, \ldots) \times \phi(\sqrt{p}, -\sqrt{q}, \sqrt{r}, \ldots)$ is rational, so far as $\sqrt{q}$ is concerned.

Example 1. If $\phi(x) = x^3 + x^2 + x + 1$, find the values of $\phi(1 + \sqrt{3})$ and $\phi(1 - \sqrt{3})$ and $\phi(1 + \sqrt{3}) \times \phi(1 - \sqrt{3})$.

$$\begin{aligned}\phi(1 + \sqrt{3}) &= (1 + \sqrt{3})^3 + (1 + \sqrt{3})^2 + (1 + \sqrt{3}) + 1,\\ &= 1 + 3\sqrt{3} + 3 . 3 + 3\sqrt{3}\\ &\quad + 1 + 2\sqrt{3} + 3\\ &\quad + 1 + \sqrt{3}\\ &\quad + 1.\\ &= 16 + 9\sqrt{3}\end{aligned}$$

$\phi(1 - \sqrt{3})$ is deduced by writing $-\sqrt{3}$ in place of $+\sqrt{3}$ everywhere in the above calculation. Hence

$$\begin{aligned}\phi(1 - \sqrt{3}) &= 16 - 9\sqrt{3};\\ \phi(1 + \sqrt{3}) \times \phi(1 - \sqrt{3}) &= (16)^2 - (9\sqrt{3})^2,\\ &= 256 - 243,\\ &= 13.\end{aligned}$$

Example 2. Find the value of $x^3+y^3+z^3-xyz$, when $x=\sqrt{q}-\sqrt{r}$, $y=\sqrt{r}-\sqrt{p}$, $z=\sqrt{p}-\sqrt{q}$.

Since
$$\begin{aligned}x+y+z&=\sqrt{q}-\sqrt{r}+\sqrt{r}-\sqrt{p}+\sqrt{p}-\sqrt{q}\\&=0,\end{aligned}$$
we have (chap. iv., § 25, IX.)
$$\begin{aligned}\Sigma x^3-3xyz&=\Sigma x(\Sigma x^2-\Sigma xy),\\&=0.\end{aligned}$$
Therefore
$$\begin{aligned}\Sigma x^3-xyz&=2xyz,\\&=2(\sqrt{q}-\sqrt{r})(\sqrt{r}-\sqrt{p})(\sqrt{p}-\sqrt{q}),\\&=2(q-r)\sqrt{p}+2(r-p)\sqrt{q}+2(p-q)\sqrt{r}.\end{aligned}$$

Example 3. Evaluate $(1+y+z)(1+z+x)(1+x+y)$ when $x=\sqrt{2}$, $y=\sqrt{3}$, $z=\sqrt{5}$.

$$\begin{aligned}&(1+y+z)(1+z+x)(1+x+y)\\&\quad=1+2(x+y+z)+x^2+(y+z)x+yz+\&c.+\&c.\\&\qquad+x(y^2+z^2)+\&c.+\&c.+2xyz,\\&\quad=1+x^2+y^2+z^2+(2+y^2+z^2)x+(2+z^2+x^2)y\\&\qquad+(2+x^2+y^2)z+3yz+3zx+3xy+2xyz,\\&\quad=11+10\sqrt{2}+9\sqrt{3}+7\sqrt{5}+3\sqrt{15}+3\sqrt{10}+3\sqrt{6}+2\sqrt{30}.\end{aligned}$$

§ 16.] We can now prove very easily the general proposition indicated above in § 14.

If P *be the sum of any number of square roots, say* $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, . . ., *a rationalising factor* Q *is obtained for* P *by multiplying together all the different factors that can be obtained from* P *as follows:—Keep the sign of the first term unchanged, and take every possible arrangement of sign for the following terms, except that which occurs in* P *itself.*

For the factors in the product Q × P contain every possible arrangement of the signs of all but the first term. Hence along with the + sign before any term, say that containing $\sqrt{q}$, there will occur every possible variety of arrangement of all the other variable signs; and the same is true for the − sign before $\sqrt{q}$. Hence, if we denote the product of all the factors containing $+\sqrt{q}$ by $\phi(\sqrt{q})$, the product of all those factors that contain $-\sqrt{q}$ will differ from $\phi(\sqrt{q})$ only in having $-\sqrt{q}$ in place of $+\sqrt{q}$, that is, may be denoted by $\phi(-\sqrt{q})$. Hence we may write $Q\times P=\phi(\sqrt{q})\times\phi(-\sqrt{q})$, which, by § 15, Cor. 1, is rational so far as $\sqrt{q}$ is concerned. The like may of course be proved for every one of the irrationals $\sqrt{q}$, $\sqrt{r}$, . . . Also, for every factor in Q × P of the form $\sqrt{p}+k$ there is evidently another of the form

$\sqrt{p}-k$; so that $Q \times P$ is rational as regards $\sqrt{p}$. Hence $Q \times P$ is entirely rational, as was to be shown.

§ 17.] *Every rational function, whether integral or not, of any number of square roots, $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, . . ., can be expressed as the sum of a rational part and rational multiples of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, &c., and of their products $\sqrt{(pq)}$, $\sqrt{(pr)}$, $\sqrt{(qr)}$, $\sqrt{(pqr)}$, &c.**

For every rational function is the quotient of two rational integral functions, say R/P. Let Q be a rationalising factor of P (which we have seen how to find), then

$$\frac{R}{P}=\frac{RQ}{PQ}.$$

But PQ is now rational, and RQ is a rational *integral* function of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, . . ., and can therefore be expressed in the required form. Hence the proposition is established.

Example 1. To express $1/(1+\sqrt{2}+\sqrt{3})$ as a sum of rational multiples of square roots. Rationalising the denominator we obtain by successive steps,

$$\begin{aligned}\frac{1}{1+\sqrt{2}+\sqrt{3}}&=\frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2})^2-(\sqrt{3})^2},\\&=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}},\\&=\frac{\sqrt{2}(1+\sqrt{2}-\sqrt{3})}{2\times 2},\\&=\tfrac{1}{4}(\sqrt{2}+2-\sqrt{6}),\\&=\tfrac{1}{2}+\tfrac{1}{4}\sqrt{2}-\tfrac{1}{4}\sqrt{6}.\end{aligned}$$

Example 2. Evaluate $(x^2-x+1)/(x^2+x+1)$, where $x=\sqrt{3}+\sqrt{5}$.

$$\begin{aligned}\frac{x^2-x+1}{x^2+x+1}&=\frac{9+2\sqrt{15}-\sqrt{3}-\sqrt{5}}{9+2\sqrt{15}+\sqrt{3}+\sqrt{5}},\\&=\frac{(9+2\sqrt{15})^2-2(9+2\sqrt{15})(\sqrt{3}+\sqrt{5})+(\sqrt{3}+\sqrt{5})^2}{(9+2\sqrt{15})^2-(\sqrt{3}+\sqrt{5})^2};\\&=\frac{149-38\sqrt{3}-30\sqrt{5}+38\sqrt{15}}{133+34\sqrt{15}}\\&=\frac{(149-38\sqrt{3}-30\sqrt{5}+38\sqrt{15})(133-34\sqrt{15})}{133^2-34^2\times 15},\\&=\frac{+437+46\sqrt{3}-114\sqrt{5}-12\sqrt{15}}{349}.\end{aligned}$$

* Besides its theoretical interest, the process of reducing a rational function of quadratic irrationals to a linear function of such irrationals is important from an arithmetical point of view; inasmuch as the linear function is in general the most convenient form for calculation. Thus, if it be required to calculate the value of $1/(1+\sqrt{2}+\sqrt{3})$ to six places of decimals, it will be found more convenient to deal with the equivalent form $\frac{1}{2}+\frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{6}$.

GENERALISATION OF THE FOREGOING THEORY.

§ 18.] It may be of use to the student who has already made some progress in algebra to sketch here a generalisation of the theory of §§ 13-17. It is contained in the following propositions:—

I. *Every integral function of* $p^{1/n}$ *can be reduced to the form* $A_0 + A_1 p^{1/n} + A_2 p^{2/n} + \ldots + A_{n-1} p^{(n-1)/n}$, *where* $A_0, A_1, \ldots, A_{n-1}$ *are rational, so far as* $p^{1/n}$ *is concerned.*

After what has been done this is obvious.

II. *Every integral function of* $p^{1/l}$, $q^{1/m}$, $r^{1/n}$, *&c., can be expressed as a linear function of* $p^{1/l}, p^{2/l}, \ldots, p^{(l-1)/l}$; $q^{1/m}, q^{2/m}, \ldots, q^{(m-1)/m}$; $r^{1/n}, r^{2/n}, \ldots, r^{(n-1)/n}$; *&c., and of the products of these quantities, two, three, &c., at a time, namely,* $p^{1/l}\,q^{1/m}$, $p^{2/l}\,q^{1/m}$, *&c., the coefficients of the linear function being rational, so far as* $p^{1/l}$, $q^{1/m}$, $r^{1/n}$, *&c., are concerned.*

Proved (as in § 15 above) by successive applications of I.

III. *A rationalising factor of* $A_0 + A_1 p^{1/n} + A_2 p^{2/n} + \ldots + A_{n-1} p^{(n-1)/n}$ *can always be found.*

We shall prove this for the case $n = 3$, but it will be seen that the process is general.

Let $$P = A_0 + A_1 p^{\frac{1}{3}} + A_2 p^{\frac{2}{3}} \qquad (1),$$
then $$p^{\frac{1}{3}}P = pA_2 + A_0 p^{\frac{1}{3}} + A_1 p^{\frac{2}{3}} \qquad (2),$$
and $$p^{\frac{2}{3}}P = pA_1 + pA_2 p^{\frac{1}{3}} + A_0 p^{\frac{2}{3}} \qquad (3).$$

Let us now put x for $p^{\frac{1}{3}}$, and y for $p^{\frac{2}{3}}$, on the right-hand sides of (1), (2), and (3); we may then write them

$$(A_0 - P) + A_1 x + A_2 y = 0 \qquad (1'),$$
$$(pA_2 - p^{\frac{1}{3}}P) + A_0 x + A_1 y = 0 \qquad (2'),$$
$$(pA_1 - p^{\frac{2}{3}}P) + pA_2 x + A_0 y = 0 \qquad (3'),$$

whence, eliminating x and y, we must have (see chap. xvi., § 8)

$$(A_0 - P)(A_0^2 - pA_1A_2) + (pA_2 - p^{\frac{1}{3}}P)(pA_2^2 - A_0A_1) + (pA_1 - p^{\frac{2}{3}}P)(A_1^2 - A_0A_2) = 0 \qquad (4).$$

Whence

$$\begin{aligned}\{(A_0^2 - pA_1A_2) + (pA_2^2 - A_0A_1)p^{\frac{1}{3}} + (A_1^2 - A_0A_2)p^{\frac{2}{3}}\}P \\ = A_0(A_0^2 - pA_1A_2) + pA_2(pA_2^2 - A_0A_1) + pA_1(A_1^2 - A_0A_2) \\ = A_0^3 + pA_1^3 + p^2A_2^3 - 3pA_0A_1A_2 \qquad (5).\end{aligned}$$

Hence a rationalising factor of P is

$$(A_0^2 - pA_1A_2) + (pA_2^2 - A_0A_1)p^{\frac{1}{3}} + (A_1^2 - A_0A_2)p^{\frac{2}{3}} \qquad (6),$$

and the rationalised result is

$$A_0^3 + pA_1^3 + p^2A_2^3 - 3pA_0A_1A_2 \qquad (7).$$

The reader who is familiar with the elements of the theory of determinants will see from the way we have obtained them that (6) and (7) are the expansions of

$$\begin{vmatrix} 1 & A_1 & A_2 \\ p^{\frac{1}{3}} & A_0 & A_1 \\ p^{\frac{2}{3}} & pA_2 & A_0 \end{vmatrix} \qquad (6'),$$

and

$$\begin{vmatrix} A_0 & A_1 & A_2 \\ pA_2 & A_0 & A_1 \\ pA_1 & pA_2 & A_0 \end{vmatrix} \qquad (7'),$$

and will have no difficulty in writing down the rationalising factor and the result of rationalisation in the general case.

IV. *A rationalising factor can be found for any rational integral function of* $p^{1/l}$, $q^{1/m}$, $r^{1/n}$, . . ., &c., by first rationalising with respect to $p^{1/l}$, then rationalising the result with respect to $q^{1/m}$, and so on.

V. *Every rational function of* $p^{1/l}$, $q^{1/m}$, $r^{1/n}$, *whether integral or not, can be expressed as a linear function of* $p^{1/l}$, $p^{2/l}$, . . ., $p^{(l-1)/l}$; $q^{1/m}$, $q^{2/m}$, . . ., $q^{(m-1)/m}$, *&c.; and of the products* $p^{\alpha/l}q^{\beta/n}r^{\gamma/m}$ . . ., *the coefficients of the function being rational, so far as* $p^{1/l}$, $q^{1/m}$, $r^{1/n}$ *are concerned.*

For every such function has the form P/Q where P and Q are rational and integral functions of the given irrationals; and, if R be a rationalising factor of Q, PR/QR will be of the form required.

Example 1. Show that a rationalising factor of $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$ is

$$\begin{aligned}&(x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}})\\ &\times(x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}+2y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}})\\ &\times(x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}+2z^{\frac{1}{3}}x^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}})\\ &\times(x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}})\,;\end{aligned}$$

and that the result of the rationalisation is $(x+y+z)^3-27xyz$.

Exercises XIV.

Express as roots of rational numbers—

(1.) $2^{\frac{3}{4}}\times 3^{-\frac{1}{2}}\times 4^{\frac{1}{8}}\times 9^{\frac{4}{3}}$. (2.) $\{\sqrt[7]{(6^{\frac{1}{5}})}\}^3\div\{\sqrt[8]{(6^{\frac{1}{7}})}\}^3$.

(3.) $3\sqrt[12]{(\sqrt[5]{7})}-2\sqrt[5]{(\sqrt[12]{7})}$. (4.) $\{\sqrt[7]{(5^2 3^{14})}\}\times\{\sqrt[14]{(15^3)}\}$.

Simplify the following—

(5.) $\{(a^{-2}/b^{-2})\sqrt{(a/b)}\}\times\sqrt[n]{(a^{1/(n-1)}b^{1/(n-1)})}$.

(6.) $(x^{(b+c)/(c-a)})^{1/(a-b)}\times(x^{(c+a)/(a-b)})^{1/(b-c)}\times(x^{(a+b)/(b-c)})^{1/(c-a)}$.

(7.) Show that $\sqrt[n]{(\sqrt[n+1]{x})}/\sqrt[n+1]{(\sqrt[n+2]{x})}=\sqrt[n]{x}\div\sqrt[n+1]{x^2}\times\sqrt[n+2]{x}$.

Simplify—

(8.) $(x^{-1}-y^{-1})/(x^{-\frac{1}{3}}-y^{-\frac{1}{3}})$. (9.) $(x^3-2+x^{-3})/(x-2+x^{-1})$.

(10.) $(x+x^{-1})^2(x^{\frac{1}{2}}-x^{-\frac{1}{2}})^3$.

(11.) $(x^{\frac{1}{2}}-x^{\frac{1}{4}}+1)(x^{\frac{1}{2}}+x^{\frac{1}{4}}+1)(x-x^{\frac{1}{2}}+1)$.

(12.) $(x^{\frac{1}{5}}-x^{-\frac{7}{10}}y^{\frac{1}{2}}+3x^{\frac{2}{5}}y^{\frac{1}{5}})(x^{\frac{1}{2}}-y^{\frac{1}{2}})$.

(13.) $(x^{\frac{4}{3}}+4y^2)/(x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+2y)$.

(14.) $2^{\frac{4}{3}}-1+2^{\frac{1}{3}}/(2^{\frac{1}{3}}-1)+1/(2^{\frac{4}{3}}+1/(2^{\frac{1}{3}}+1))$.

(15.) $(x^{2/n}-2x^{1/n}y^{1/n}+y^{2/n})(x^{1/n}+x^{1/2n}y^{1/2n}+y^{1/n})(x^{1/n}-y^{1/n})$.

(16.) Show that $x/(x^{\frac{1}{3}}-1)-x^{\frac{2}{3}}/(x^{\frac{1}{3}}+1)-1/(x^{\frac{1}{3}}-1)+1/(x^{\frac{1}{3}}+1)=x^{\frac{2}{3}}+2$.

(17.) If $\phi(x)=(a^x-a^{-x})/(a^x+a^{-x})$, $\quad F(x)=2/(a^x+a^{-x})$,

then
$$\phi(x+y)=\{\phi(x)+\phi(y)\}/\{1+\phi(x)\phi(y)\}\,;$$
$$F(x+y)=F(x)F(y)/\{1+\phi(x)\phi(y)\}.$$

(18.) If $x^p/y^q=1$, then $x^{p-m}/y^{q-n}=x^{np/q-m}=y^{n-mq/p}$.

(19.) If $m=a^x$, $n=a^y$, $m^y n^x=a^{2/z}$, then $xyz=1$.

Transform the following into sums of simple irrational terms :—

(20.) $\sqrt{a}/(\sqrt{a}+\sqrt{b})+\sqrt{b}/(\sqrt{a}-\sqrt{b})$.

(21.) $(2\sqrt{5}-3\sqrt{2}+\frac{1}{2}\sqrt{6})^2$.

(22.) $(x+1-\sqrt{2}+\sqrt{3})(x+1+\sqrt{2}-\sqrt{3})(x+1-\sqrt{2}-\sqrt{3})$, arranging according to powers of x.

(23.) $(1/\sqrt{x}+1/\sqrt{a})(x^{\frac{1}{2}}-a^{\frac{1}{2}})/\{(\sqrt{a}+\sqrt{x})^2-(\sqrt{a}-\sqrt{x})^2\}$.

(24.) $\dfrac{\sqrt{(a^2+\{a(1-m)/3\sqrt{m}\}^2)}+a(m-1)/2\sqrt{m}}{\sqrt{(a^2+\{a(1-m)/3\sqrt{m}\}^2)}-a(m-1)/2\sqrt{m}}$.

(25.) $\dfrac{\sqrt{(p/a+x)}+\sqrt{(p/a-x)}}{\sqrt{(p/a+x)}-\sqrt{(p/a-x)}}$, where $ax=\dfrac{2pq}{1+q^2}$.

(26.) $\dfrac{p-a}{x-a}+\dfrac{p-b}{x-b}-\dfrac{p+a}{x+a}-\dfrac{p+b}{x+b}$, where $x=\sqrt{(ab)}$.

(27.) $1/\{\sqrt{(p-q)}+\sqrt{p}+\sqrt{q}\}+1/\{\sqrt{(p-q)}-\sqrt{p}-\sqrt{q}\}+1/(\sqrt{p}-\sqrt{q})$.

(28.) $\sqrt{\left\{\dfrac{1-x-\sqrt{(2x+x^2)}}{1-x+\sqrt{(2x+x^2)}}\right\}}+\sqrt{\left\{\dfrac{1-x+\sqrt{(2x+x^2)}}{1-x-\sqrt{(2x+x^2)}}\right\}}$.

(29.) $(\sqrt{(a+b+c)}+\sqrt{(a-b+c)})^2/(\sqrt{(a+b+c)}-\sqrt{(a-b+c)})^2$.

(30.) $\{(\sqrt{(p-q)}+\sqrt{p}-\sqrt{q})/(\sqrt{(p-q)}-\sqrt{p}-\sqrt{q})\}$.

(31.) $(2x^3-6x+5)/(\sqrt[3]{2}x+\sqrt[3]{4}+1)$.

(32.) $(x^3+3\sqrt[3]{2}x+1)/(x+\sqrt[3]{2}-1)$.

(33.) $\{\sqrt[3]{(a+b)}-\sqrt[3]{(a-b)}\}\{[\sqrt[3]{(a+b)}]^2+[\sqrt[3]{(a-b)}]^2-\sqrt[3]{(a^2-b^2)}\}$.

Show that

(34.) $\left(\dfrac{1+\sqrt{(1-x)}}{1-\sqrt{(1-x)}}\right)^{\frac{1}{2}}+\left(\dfrac{1-\sqrt{(1-x)}}{1+\sqrt{(1-x)}}\right)^{\frac{3}{2}}=\dfrac{2(2-x)}{x^2}\{\sqrt{x}-\sqrt{(x-x^2)}\}$.

(35.) $(\sqrt{(p^2+1)}+\sqrt{(p^2-1)})^{-3}+(\sqrt{(p^2+1)}-\sqrt{(p^2-1)})^{-3}=(p^2-\frac{1}{2})\sqrt{(p^2+1)}$.

(36.) $\sqrt{[\sqrt{\{a^2+\sqrt[3]{(a^4b^2)}\}}+\sqrt{\{b^2+\sqrt[3]{(a^2b^4)}\}}]}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{4}}$.

Express in linear form—

(37.) $(\sqrt[4]{x}-\sqrt[4]{y})/(\sqrt[4]{x}+\sqrt[4]{y})$.

(38.) $(1+\sqrt{3}+\sqrt{5}+\sqrt{7})/(1-\sqrt{3}-\sqrt{5}+\sqrt{7})$.

(39.) $\Sigma(\sqrt{b}+\sqrt{c})/(\sqrt{b}+\sqrt{c}-\sqrt{a})-4\Sigma a(\sqrt{b}+\sqrt{c})/\Pi(\sqrt{b}+\sqrt{c}-\sqrt{a})$.

Rationalise the following:—

(40.) $3\,.\,5^{\frac{1}{3}}-4^{\frac{1}{3}}$.

(41.) $\Sigma\sqrt{(b+c-a)}$.

(42.) $\sqrt{5}+\sqrt{3}+\sqrt{4}-\sqrt{6}$.

(43.) $3\,.\,2^{\frac{2}{3}}+4\,.\,2^{\frac{1}{3}}-1$.

(44.) $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

(45.) $2^{\frac{1}{4}}+2^{\frac{1}{2}}+1$.

(46.) If $u=x\sqrt{(1+y^2)}+y\sqrt{(1+x^2)}$, then $\sqrt{(1+u^2)}=xy+\sqrt{((1+x^2)(1+y^2))}$.

(47.) Show that $\dfrac{2}{\sqrt{(y-z)}+\sqrt{(z-x)}+\sqrt{(x-y)}}$

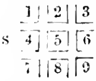
$$=\frac{(y-z)^{\frac{3}{2}}+(z-x)^{\frac{3}{2}}+(x-y)^{\frac{3}{2}}+(y-z)^{\frac{1}{2}}(z-x)^{\frac{1}{2}}(x-y)^{\frac{1}{2}}}{x^2+y^2+z^2-yz-zx-xy}.$$

(48.) If $x=1/(\sqrt{b}+\sqrt{c}-\sqrt{a})$, $y=1/(\sqrt{c}+\sqrt{a}-\sqrt{b})$, $z=1/(\sqrt{a}+\sqrt{b}-\sqrt{c})$, $u=1/(\sqrt{a}+\sqrt{b}+\sqrt{c})$,
then
$$\prod_{xyz}(-x+y+z+u)/(\Sigma x-u)^3=\Pi(b+c-a)/8abc.$$

Historical Note.—The use of exponents began in the works of the German "Cossists," Rudolff (1525) and Stifel (1544), who wrote over the contractions

for the names of the 1st, 2nd, 3rd, . . . powers of the variable, which had been used in the syncopated algebra, the numbers 1, 2, 3, . . . Stifel even states expressly the laws for multiplying and dividing powers by adding and subtracting the exponents, and indicates the use of negative exponents for the reciprocals of positive integral powers. Bombelli (1579) writes $\smile$, $\overset{1}{\smile}$, $\overset{2}{\smile}$, $\overset{3}{\smile}$, . . ., where we should write x^0, x, x^2, x^3, . . . Stevin (1585) uses in a similar way ⓪, ①, ②, ③, . . ., and suggests, although he does not practically use, fractional powers such as $\textcircled{\tfrac{1}{2}}$, $\textcircled{\tfrac{2}{3}}$, which are equivalent to the $x^{\frac{1}{2}}$, $x^{\frac{2}{3}}$, of the present day. Viète (1591) and Oughtred (1631), who were in full possession of a literal calculus, used contractions for the names of the powers, thus, *Aq*, *Ac*, *Aqq*, to signify A^2, A^3, A^4. Harriot (1631) simply repeated the letter, thus, *aa*, *aaa*, *aaaa*, for a^2, a^3, a^4. Herigone (1634) used numbers written after the letter, thus, A, A2, A3, . . . Descartes introduced the modern forms A, A^2, A^3, . . . The final development of the general idea of an index unrestricted in magnitude, that is to say, of an exponential function a^x, is due to Newton, and came in company with his discovery of the general form of the binomial coefficients as functions of the index. He says, in the letter to Oldenburg of 13th June 1676, "Since algebraists write a^2, a^3, a^4, &c., for *aa*, *aaa*, *aaaa*, &c., so I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.a^5}$; and I write a^{-1}, a^{-2}, a^{-3}, &c., for $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{aaa}$, &c."

The sign $\sqrt{}$ was first used by Rudolff; both he and Scheubel (1551) used $w\!\!\sqrt{}$ to denote 4th root, and $ww\!\!\sqrt{}$ to denote cube root. Stifel used both $\sqrt{\mathfrak{z}}$. and $\sqrt{}$ to denote square root, $\sqrt{\mathfrak{zz}}$. to denote 4th root, and so on. Girard (1633) uses the notation of the present day, $\sqrt{}$, $\sqrt[3]{}$, &c. Other authors of the 17th century wrote $\sqrt{2}$:, $\sqrt{3}$:, &c. So late as 1722, in the second edition of Newton's *Arithmetica Universalis*, the usage fluctuates, the three forms $\sqrt{3}$:, $\sqrt{}^{3}$:, $\sqrt[3]{}$ all occurring.

In an incomplete mathematical treatise, entitled *De Arte Logistica*, &c., which was found among the papers of Napier of Merchiston (1550-1617; published by Mark Napier, Edinburgh, 1839), and shows in every line the firm grasp of the great inventor of logarithms, a remarkable system of notation for irrationals is described. Napier takes the figure , and divides it thus .

He then uses , $\llcorner\!\lrcorner$, $\llcorner\_$, &c., which are in effect a new set of symbols for the nine digits 1, 2, 3, &c., as radical signs. Thus $\llcorner\!\lrcorner$10 stands for $\sqrt{10}$, $\llcorner_$10 for $\sqrt[3]{10}$, $_\lrcorner$10 for $\sqrt[10]{10}$, $_\lrcorner$10 for $\sqrt[11]{10}$, $\_\rfloor\,\urcorner$ or $\urcorner$ for $\sqrt[14]{10}$; and so on.

Many of the rules for operating with irrationals at present in use have come, in form at least, from the German mathematicians of the 16th century, more particularly from Scheubel, in whose *Algebræ Compendiosa Facilisque Descriptio* (1551) is given the rule of chap. xi., § 9, for extracting the square root of a binomial surd. In substance many of these rules are doubtless much older (as old as Book X. of Euclid's *Elements*, at least); they were at all events more or less familiar to the contemporary mathematicians of the Italian school, who did so much for the solution of equations by means of radicals, although in symbolism they were far behind their transalpine rivals. See Hutton's *Mathematical Dictionary*, Art. "Algebra."

The process explained at the end of next chapter for extracting the square or

cube root by successive steps is found in the works of the earliest European writers on algebra. for example, Leonardo Fibonacci (*c.* 1200) and Luca Pacioli (*c.* 1500). The first indication of a general method appears in Stifel's *Arithmetica Integra*, where the necessary table of binomial coefficients (see p. 81) is given. It is not quite clear from Stifel's work that he fully understood the nature of the process and clearly saw its connection with the binomial theorem. The general method of root extraction, together with the triangle of binomial coefficients, is given in Napier's *De Arte Logistica*. He indicates along the two sides of his triangle the powers of the two variables (præcedens and succedens) with which each coefficient is associated, and thus gives the binomial theorem in diagrammatic form. His statement for the cube is—"Supplementum triplicationis tribus constat numeris: primus est, duplicati præcedentis triplum multiplicatum per succedens; secundus est, præcedentis triplum multiplicatum per duplicatum succedentis; tertius est, ipsum triplicatum succedentis." In modern notation,

$$(a+b)^3-a^3=3a^2b+3ab^2+b^3.$$

CHAPTER XI.

Arithmetical Theory of Surds.

ALGEBRAICAL AND ARITHMETICAL IRRATIONALITY.

§ 1.] In last chapter we discussed the properties of irrational functions in so far as they depend merely on outward form; in other words, we considered them merely from the algebraical point of view. We have now to consider certain peculiarities of a purely arithmetical nature. Let p denote any *commensurable number*; that is, either an integer, or a proper or improper vulgar fraction with a finite number of digits in its numerator and denominator; or, what comes to the same thing, let p denote a number which is either a terminating or repeating decimal. Then, if n be any positive integer, $\sqrt[n]{p}$ will not be commensurable unless p be the nth power of a commensurable number;* for if $\sqrt[n]{p}=k$, where k is commensurable, then, by the definition of $\sqrt[n]{p}$, $p=k^n$, that is, p is the nth power of a commensurable number.

If therefore p be not a perfect nth power, $\sqrt[n]{p}$ is incommensurable. For distinction's sake $\sqrt[n]{p}$ is then called a *surd number*. In other words, we define a *surd number as the incommensurable root of a commensurable number.*

Surds are classified according to the index, n, of the root to be extracted, as quadratic, cubic, biquadratic or quartic, quintic, . . . n-tic surds.

The student should attend to the last phrase of the definition of a surd; because incommensurable roots might be conceived which do not come under

* This is briefly put by saying that p is a perfect nth power.

the above definition ; and to them the demonstrations of at least some of the propositions in this chapter would not apply. For example, the number e (see the chapter on the Exponential Theorem in Part II. of this work) is incommensurable, and $\sqrt{e}$ is incommensurable ; hence $\sqrt{e}$ is not a surd in the exact sense of the definition. Neither is $\sqrt{(\sqrt{2}+1)}$, for $\sqrt{2}+1$ is incommensurable. On the other hand, $\sqrt{(\sqrt{2})}$, which can be expressed in the form $\sqrt[4]{2}$, does come under that definition, although not as a quadratic but as a *biquadratic* surd.

He should also observe that an algebraically irrational function, say $\sqrt{x}$, may or may not be arithmetically irrational, that is, surd, strictly so called, according to the value of the variable x. Thus $\sqrt{4}$ is not a surd, but $\sqrt{2}$ is.

CLASSIFICATION OF SURDS.

§ 2.] A single surd number, or, what comes to the same, a rational multiple of a single surd, is spoken of as a *simple monomial surd number* ; the sum of two such surds, or of a rational number and a simple monomial surd number, as a *simple binomial surd number*, and so on.

The propositions stated in last chapter amount to a proof of the statement that every rational function of surd numbers can be expressed as a simple surd number, monomial, binomial, trinomial, &c., as the case may be.

§ 3] *Two surds are said to be similar when they can be expressed as rational multiples of one and the same surd ; dissimilar when this is not the case.*

For example, $\sqrt{18}$ and $\sqrt{8}$ can be expressed respectively in the forms $3\sqrt{2}$ and $2\sqrt{2}$, and are therefore similar ; but $\sqrt{6}$ and $\sqrt{2}$ are dissimilar.

Again, $\sqrt[3]{54}$ and $\sqrt[3]{16}$, being expressible in the forms $3\sqrt[3]{2}$ and $2\sqrt[3]{2}$, are each similar to $\sqrt[3]{2}$.

All the surds that arise from the extraction of the same nth root are said to be equiradical.

Thus $p^{\frac{1}{3}}$, $p^{\frac{2}{3}}$, $p^{\frac{4}{3}}$, $p^{\frac{5}{3}}$ are all equiradical with $p^{\frac{1}{3}}$.

There are $n-1$ distinct surds equiradical with $p^{1/n}$, namely, $p^{1/n}$, $p^{2/n}$, . . ., $p^{(n-1)/n}$, and no more.

For, if we consider $p^{m/n}$ where $m>n$, then we have $p^{m/n}=p^{\mu+\nu/n}$, where μ and ν are integers, and $\nu<n$. Hence $p^{m/n}=p^{\mu}p^{\nu/n}=$ a rational multiple of one of the above series.

All the surds equiradical with $p^{1/n}$ are rational functions (namely,

positive integral powers) of $p^{1/n}$; and every rational function of $p^{1/n}$ or of surds equiradical with $p^{1/n}$ may be expressed as a linear function of the $n-1$ distinct surds which are equiradical with $p^{1/n}$, that is, in the form $A_0 + A_1 p^{1/n} + A_2 p^{2/n} + \ldots + A_{n-1} p^{(n-1)/n}$, where A_0, A_1, . . ., A_{n-1} are rational so far as $p^{1/n}$ is concerned.

This is merely a restatement of § 18 of chap. x.

§ 4.] *The product or quotient of two similar quadratic surds is rational, and if the product or quotient of the two quadratic surds is rational they are similar.*

For, if the surds are similar, they are expressible in the forms $A\sqrt{p}$ and $B\sqrt{p}$, where A and B are rational; therefore $A\sqrt{p} \times B\sqrt{p} = ABp$; and $A\sqrt{p}/B\sqrt{p} = A/B$, which proves the proposition, since ABp and A/B are rational.

Again, if $\sqrt{p} \times \sqrt{q} = A$, or $\sqrt{p}/\sqrt{q} = B$, where A and B are rational, then in the one case $\sqrt{p} = (A/q)\sqrt{q}$, in the other $\sqrt{p} = B\sqrt{q}$. But A/q and B are rational. Hence $\sqrt{p}$ and $\sqrt{q}$ are similar in both cases.

The same is not true for surds of higher index than 2, but *the product of two similar or of two equiradical surds is either rational or an equiradical surd.*

INDEPENDENCE OF SURD NUMBERS.

§ 5.] *If p, q, A, B be all commensurable, and none of them zero, and $\sqrt{p}$ and $\sqrt{q}$ incommensurable, then we cannot have*

$$\sqrt{p} = A + B\sqrt{q}.$$

For, squaring, we should have as a consequence,

$$p = A^2 + B^2 q + 2AB\sqrt{q};$$

whence

$$\sqrt{q} = (p - A^2 - B^2 q)/2AB,$$

which asserts, contrary to our hypothesis, that $\sqrt{q}$ is commensurable.

Since every rational function of $\sqrt{q}$ may (chap. x., § 15) be expressed in the form $A + B\sqrt{q}$, the above theorem is equivalent to the following:—

One quadratic surd cannot be expressed as a rational function of another which is dissimilar to it.

Since every rational equation between $\sqrt{p}$ and $\sqrt{q}$ which is not a mere equation between commensurables (for example, $(\sqrt{3})^2 + (\sqrt{2})^2 = 5$) is reducible to the form

$$A\sqrt{(pq)} + B\sqrt{p} + C\sqrt{q} + D = 0,$$

where A, B, C, D are rational; and, since this equation may immediately be reduced to another of the form

$$\sqrt{p} = L + M\sqrt{q},$$

where L and M are rational, it follows that

No rational relation, which is not a mere equation between rational numbers, can subsist between two dissimilar quadratic surds.

§ 6.] *If the quadratic surds $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, $\sqrt{(qr)}$ be dissimilar to each other, then $\sqrt{p}$ cannot be a rational function of $\sqrt{q}$ and $\sqrt{r}$.*

For, if this were so, then we should have

$$\sqrt{p} = A + B\sqrt{q} + C\sqrt{r} + D\sqrt{(qr)},$$

where A, B, C, D are all rational.

Now we cannot, by our hypotheses, have three of the four A, B, C, D equal to zero.

In any other case, we should get on squaring

$$p = \{A + B\sqrt{q} + C\sqrt{r} + D\sqrt{(qr)}\}^2,$$

which would either be a rational equation connecting two dissimilar quadratic surds, which is impossible, as we have just seen; or else an equation asserting the rationality of one of the surds, which is equally impossible.

An important particular case of the above is the following:—

A quadratic surd cannot be the sum of two dissimilar quadratic surds.

It will be a good exercise for the student to prove this directly.

§ 7.] The theory which we have established so far for quadratic surds may be generalised, and also extended to surds whose index exceeds 2. This is not the place to pursue the matter farther, but the reader who has followed so far will find the ideas gained useful in paving the way to an understanding of the delicate researches of Lagrange, Abel, and Galois regarding

the algebraical solution of equations whose degree exceeds the 4th.

§ 8.] It follows as a necessary consequence of §§ 5 and 6 that, if we are led to any equation such as

$$A + B\sqrt{p} + C\sqrt{q} + D\sqrt{(pq)} = 0,$$

where $\sqrt{p}$ and $\sqrt{q}$ are dissimilar surds, then we must have

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 0.$$

One case of this is so important that we enunciate and prove it separately.

If x, y, z, u be all commensurable, and $\sqrt{y}$ and $\sqrt{u}$ incommensurable, and if $x + \sqrt{y} = z + \sqrt{u}$, then must $x = z$ and $y = u$.

For if $x \neq z$, but $= a + z$ say, where $a \neq 0$, then by hypothesis

$$a + z + \sqrt{y} = z + \sqrt{u},$$

whence
$$a + \sqrt{y} = \sqrt{u},$$
$$a^2 + y + 2a\sqrt{y} = u,$$
$$\sqrt{y} = (u - a^2 - y)/2a,$$

which asserts that $\sqrt{y}$ is commensurable. But this is not so. Hence we must have $x = z$; and, that being so, we must also have $\sqrt{y} = \sqrt{u}$, that is, $y = u$.

SQUARE ROOTS OF SIMPLE SURD NUMBERS.

§ 9.] Since the square of every simple binomial surd number takes the form $p + \sqrt{q}$, it is natural to inquire whether $\sqrt{(p + \sqrt{q})}$ can always be expressed as a simple binomial surd number, that is, in the form $\sqrt{x} + \sqrt{y}$, where x and y are rational numbers. Let us suppose that such an expression exists; then

$$\sqrt{(p + \sqrt{q})} = \sqrt{x} + \sqrt{y},$$

whence
$$p + \sqrt{q} = x + y + 2\sqrt{(xy)}.$$

If this equation be true, we must have, by § 8,

$$x + y = p \qquad (1),$$
$$2\sqrt{(xy)} = \sqrt{q} \qquad (2);$$

and, from (1) and (2), squaring and subtracting, we get

$$(x + y)^2 - 4xy = p^2 - q,$$

that is,
$$(x - y)^2 = p^2 - q \qquad (3).$$

Now (3) gives either

$$x-y=+\sqrt{(p^2-q)} \qquad (4),$$

or

$$x-y=-\sqrt{(p^2-q)} \qquad (4^*).$$

Taking, meantime, (4) and combining it with (1), we have

$$(x+y)+(x-y)=p+\sqrt{(p^2-q)} \qquad (5),$$
$$(x+y)-(x-y)=p-\sqrt{(p^2-q)} \qquad (6);$$

whence

$$2x=p+\sqrt{(p^2-q)},$$
$$2y=p-\sqrt{(p^2-q)};$$

that is,

$$x=\tfrac{1}{2}\{p+\sqrt{(p^2-q)}\} \qquad (7),$$
$$y=\tfrac{1}{2}\{p-\sqrt{(p^2-q)}\} \qquad (8).$$

If we take (4*) instead of (4), we simply interchange the values of x and y, which leads to nothing new in the end.

Using the values of (7) and (8) we obtain the following result :—

$$\sqrt{x}=\pm\sqrt{\left\{\frac{p+\sqrt{(p^2-q)}}{2}\right\}}.$$

$$\sqrt{y}=\pm\sqrt{\left\{\frac{p-\sqrt{(p^2-q)}}{2}\right\}}.$$

Since, by (2), $2\sqrt{x}\times\sqrt{y}=+\sqrt{q}$, we must take either the two upper signs together or the two lower.

If we had started with $\sqrt{(p-\sqrt{q})}$, it would have been necessary to choose $\sqrt{x}$ and $\sqrt{y}$ with opposite signs.

Finally therefore we have

$$\sqrt{(p+\sqrt{q})}=\pm\left(\sqrt{\left\{\frac{p+\sqrt{(p^2-q)}}{2}\right\}}+\sqrt{\left\{\frac{p-\sqrt{(p^2-q)}}{2}\right\}}\right) \qquad (9),$$

$$\sqrt{(p-\sqrt{q})}=\pm\left(\sqrt{\left\{\frac{p+\sqrt{(p^2-q)}}{2}\right\}}-\sqrt{\left\{\frac{p-\sqrt{(p^2-q)}}{2}\right\}}\right) \qquad (9^*).$$

The identities (9) and (9*) are certainly true; we have in fact already verified one of them (see chap. x., § 9, Example 14). They will not, however, furnish a solution of our problem, unless the values of x and y are rational. For this it is necessary and sufficient that p^2-q be a positive perfect square, and that p be

positive. Hence *the square root of $p+\sqrt{q}$ can be expressed as a simple binomial surd number, provided p be positive and p^2-q be a positive perfect square.*

Example 1. Simplify $\sqrt{(19-4\sqrt{21})}$.

Let $$\sqrt{(19-4\sqrt{21})}=\sqrt{x}+\sqrt{y}.$$

Then $$x+y=19,$$
$$2\sqrt{x}\sqrt{y}=-4\sqrt{21},$$
$$(x-y)^2=361-336$$
$$=25,$$
$$x-y=+5 \text{ say},$$
$$x+y=19;$$

whence $$x=12, \quad y=7,$$
$$\sqrt{x}=\pm\sqrt{12}, \quad \sqrt{y}=\mp\sqrt{7},$$

so that $$\sqrt{(19-4\sqrt{21})}=\pm(\sqrt{12}-\sqrt{7}).$$

Example 2. To find the condition that $\sqrt{(\sqrt{p}+\sqrt{q})}$ may be expressible in the form $(\sqrt{x}+\sqrt{y})\sqrt[4]{p}$ we have

$$\sqrt{(\sqrt{p}+\sqrt{q})}=\sqrt[4]{p}\times\sqrt{\{1+\sqrt{(q/p)}\}}.$$

Now $\sqrt{\{1+\sqrt{(q/p)}\}}$ will be expressible in the form $\sqrt{x}+\sqrt{y}$, provided $1-q/p$ be a positive perfect square; this, therefore, is the required condition. For example,

$$\sqrt{(5\sqrt{7}+2\sqrt{42})}=\sqrt[4]{7}\times\sqrt{(5+2\sqrt{6})}$$
$$=\pm\sqrt[4]{7}(\sqrt{3}+\sqrt{2}).$$

Example 3. It is obvious that in certain cases $\sqrt{(p+\sqrt{q}+\sqrt{r}+\sqrt{s})}$ must be expressible in the form $\sqrt{x}+\sqrt{y}+\sqrt{z}$, where x, y, z are rational. To find the condition that this may be so, and to determine the values of x, y, z, let

$$\sqrt{(p+\sqrt{q}+\sqrt{r}+\sqrt{s})}=\sqrt{x}+\sqrt{y}+\sqrt{z} \qquad (1),$$

then $$p+\sqrt{q}+\sqrt{r}+\sqrt{s}=x+y+z+2\sqrt{(yz)}+2\sqrt{(zx)}+2\sqrt{(xy)} \qquad (2).$$

Now let us suppose that

$$2\sqrt{(yz)}=\sqrt{q} \qquad (3),$$
$$2\sqrt{(zx)}=\sqrt{r} \qquad (4),$$
$$2\sqrt{(xy)}=\sqrt{s} \qquad (5).$$

From (4) and (5) we have by multiplication

$$4x\sqrt{(yz)}=\sqrt{(rs)};$$

whence, by using (3), $$x=\tfrac{1}{2}\sqrt{(rs/q)} \qquad (6).$$

Proceeding in like manner with y and z, we obtain

$$y=\tfrac{1}{2}\sqrt{(qs/r)} \qquad (7).$$
$$z=\tfrac{1}{2}\sqrt{(qr/s)} \qquad (8).$$

It is further necessary, in order that (2) may hold, that the values (6), (7), (8) shall satisfy the equation

$$x+y+z=p \qquad (9).$$

that is, we must have

$$\sqrt{(rs/q)}+\sqrt{(qs/r)}+\sqrt{(qr/s)}=2p \qquad (10),$$

where the signs throughout must be positive, since x, y, z must all be positive.

Also, since x, y, z must all be rational, we must have

$$\frac{rs}{q}=\alpha^2, \quad \frac{qs}{r}=\beta^2, \quad \frac{qr}{s}=\gamma^2 ;$$

where α, β, γ are positive rational numbers, such that

$$\alpha+\beta+\gamma=2p,$$

whence, in turn, we obtain

$$q=\beta\gamma, \quad r=\gamma\alpha, \quad s=\alpha\beta.$$

ARITHMETICAL METHODS FOR FINDING APPROXIMATE RATIONAL VALUES FOR SURD NUMBERS.

§ 10.] It has already been stated that a rational approximation, as close as we please, can always be found for every surd number. It will be well to give here one method at least by which such approximations can be obtained. We begin with the approximation to a quadratic surd; and we shall afterwards show that all other cases might be made to depend on this.

§ 11.] First of all, we may point out that in every case we may reduce our problem to the finding of the integral part of the square root of an integer. Suppose, for example, we wish to find the square root of 3·689 correct to five places of decimals. Then, since $\sqrt{3{\cdot}689} = \sqrt{36890000000}/10^5$, we have merely to find the square root of the integer 36890000000 correct to the last integral place, and then count off five decimal places.

§ 12.] The following propositions are all that are required for the present purpose:—

I. *The result of subtracting* $(A+B)^2$ *from* N *is the same as the result of first subtracting* A^2, *then* 2AB, *and finally* B^2.

This is obvious, since $(A+B)^2 = A^2 + 2AB + B^2$.

II. *If the first* p *out of the* n *digits of the square root of the integer* N *have been found, so that* $P10^{n-p}$ *is a first approximation to* $\sqrt{N}$, *then the next* $p-1$ *digits will be the first* $p-1$ *digits of the integral part of the quotient* $\{N-(P10^{n-p})^2\}/2P10^{n-p}$, *with a possible error in excess of* 1 *in the last digit.*

Let the whole of the rest of the square root be Q. Then

$$\sqrt{N} = P10^{n-p} + Q,$$

where $$10^{p-1} < P < 10^p,\ Q < 10^{n-p};$$

whence $$N = (P10^{n-p})^2 + 2PQ10^{n-p} + Q^2,$$

$$\frac{N - (P10^{n-p})^2}{2P10^{n-p}} = Q + \frac{Q^2}{2P10^{n-p}} \qquad (1).$$

Now

$$Q^2/2P10^{n-p} < 10^{2(n-p)}/2 \times 10^{p-1}10^{n-p} < 10^{n-2p+1}/2.$$

Hence $Q^2/2P10^{n-p}$ will at most affect the $(n - 2p + 1)$th place, and the error in that place will be at the utmost 5 in excess. Therefore, since Q contains $n - p$ digits, the first $p - 1$ of these will be given by the first $p - 1$ digits of $\{N - (P10^{n-p})^2\}/2P10^{n-p}$ with a possible error in excess of 1 in the last digit.*

§ 13.] In the actual calculation of the square root the first few figures may be found singly by successive trials, Proposition I. being used to find the residues, which must, of course, always be positive. Then Proposition II. may be used to find the succeeding digits in larger and larger groups. The approximation can thus be carried out with great rapidity, as will be seen by the following example:—

Let it be required to find the square root of $N = 68010000000000000$, which, for shortness, we write 6801(14).

Obviously $8(8) < \sqrt{N} < 9(8)$; in other words, $\sqrt{N}$ contains 9 digits, and the first is 8.

Now $N - \{8(8)\}^2 = 401(14)$, which is the first residue. We have now to find the greatest digit x which can stand in the second place, and still leave the square of the part found less than N, that is (by Proposition I.), leave the residue $401(14) - 2 \times 8(8) \times x(7) - \{x(7)\}^2$ positive. It is found by inspection that $x = 2$. Carrying out the subtractions indicated, that is, subtracting $\{16(8) + 2(7)\} \times 2(7) = 162 \times 2(14)$ from 401(14), we have now as residue 7700(12).

---

* The effect of such an error would be to give a negative residue in the process of § 13; so that in practice it would be immediately discovered and rectified. As an example of a case where the error actually occurs, the reader might take the square root of 5558(12), namely, 74551995, and attempt to deduce from 745 the two following digits. He will find by the above rule 52 instead of 51. If it be a question of *the best approximation*, the rule gives here, as always, the best result; but this is not always what is wanted.

The double of the whole of the part of $\sqrt{N}$ now found is 164(7); and we have next to find y as large as may be, so that $7700(12) - \{164(7) + y(6)\} \times y(6)$ shall remain positive. This value of y is seen to be 4. It might, of course, be found (by Proposition II.) by dividing 7700(12) by 164(7), and taking the first figure of the quotient.

The residue is now 112400(10). The process of finding the first four digits in this way may be arranged thus:—

8(8)	6801(14)	8(8)
16(8)	6400(14)	
162(7)	401(14)	+2(7)
164(7)	324(14)	
1644(6)	7700(12)	+4(6)
1648(6)	6576(12)	
16486(5)	112400(10)	+6(5)
16492(5)	98916(10)	
	134840(9)	

We might, of course, continue in the same way, figure by figure, as long as we please; and we might omit the records in brackets of the zeros in each line.

Having, however, already found four figures, we can find three more by dividing the residue 134840(9) by 16492(5), which is the double of 8246(5), the part of $\sqrt{N}$ already found.

16492(5)	134840(9)	817(2)
	131936	
	29040	
	16492	
	125480	
	115444	
	10036000(4)	

The next three digits are therefore 817. 10036000(4) is not the residue; for we have only subtracted from $\sqrt{N}$ as yet $\{8246(5)\}^2$ and $2 \times 8246(5) \times 817(2)$. Subtracting also $\{817(2)\}^2$ we get the true residue, namely, 93685110000. We may now divide this by $2 \times 8246817(2)$, that is, by 1649363400, and thus get the last two figures. We have then

	10036000(4)	
	667489(4)	
1649363400	93685110000	56
	8246817000	
	11216940000	
	9896180400	
	1320759600	

We have now found the whole of the integral part of $\sqrt{6801(14)}$, namely, 824681756.

If it were desired to carry the approximation farther, 8 places after the decimal point could at once be found by dividing the true residue $(1320759600 - 56^2)$ by 2×824681756. If we require no more places than those 8 places, then the residue is of no importance, and we may save labour by adopting the abbreviated method of long division (see Brook Smith's *Arithmetic*, chap. vi., § 153). Thus

$$\begin{array}{r|r|l}
 & 1320759600 & \\
 & 3136 & \\
\cline{2-2}
16\cancel{4}\cancel{9}\cancel{3}\cancel{6}\cancel{3}\cancel{5}\cancel{1}\cancel{2} & 1320756464 & 80076736 \\
 & 1319490810 & \\
\cline{2-2}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{1265654} & \\
\multicolumn{1}{r}{} & \multicolumn{1}{r}{1154554} & \\
\cline{2-2}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{111100} & \\
\multicolumn{1}{r}{} & \multicolumn{1}{r}{98962} & \\
\cline{2-2}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{12138} & \\
\multicolumn{1}{r}{} & \multicolumn{1}{r}{11545} & \\
\cline{2-2}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{593} & \\
\multicolumn{1}{r}{} & \multicolumn{1}{r}{495} & \\
\cline{2-2}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{98} & \\
\multicolumn{1}{r}{} & \multicolumn{1}{r}{98} & \\
\cline{2-2}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{0} &
\end{array}$$

We thus find $\sqrt{6801(14)} = 824681756{\cdot}80076736$. On verifying, the reader will find that in point of fact

$$(824681756{\cdot}80076736)^2 = 680100000000000001{\cdot}82 \ldots$$

It will be a good exercise for him to find out how many decimal places of the square root of a given integer must be found before the square of the approximation ceases to be incorrect in the last integral place.

§ 14.] By continually extracting the square root (that is to say, by extracting the square root, then extracting the square root of the square root, and so on), we may bring any number greater than unity as near unity as we please. In other words, by making n sufficiently great, $N^{1/2^n}$ may be made to differ from 1 by less than any assignable quantity.

For let it be required to make $N^{1/2^n}$ less than $1 + a$, where a is any positive quantity. This will be done if 2^n be made such that $(1 + a)^{2^n} > N$. Now (chap. iv., § 11) $(1 + a)^{2^n} = 1 + 2^n a +$ a series of terms, which are all positive. Hence it will be sufficient if we make $1 + 2^n a > N$, that is, if we make $2^n a > N - 1$, that is,

if we make $2^n > (N-1)/a$, which can always be done, since by making n sufficiently great 2^n may be made to exceed any quantity, however great.

Example. How many times must we extract the square root beginning with 51 in order that the final result may differ from 1 by less than ·001?

We must have

$$2^n > (51-1)/\cdot 001,$$
$$2^n > 50000.$$

Now $$2^{15}=32768, \quad 2^{16}=65536,$$

hence we must make $n=16$.

In other words, if we extract the square root sixteen times, beginning with 51, the result will be less than 1·001.

§ 15.] It follows from § 14 that we can approximate to any surd whatever, say $p^{1/n}$, by the process of extracting the square root. For (see chap. ix., § 2) let $1/n$ be expressed in the binary scale, then we shall have

$$1/n = \alpha/2 + \beta/2^2 + \gamma/2^3 + \ldots + \mu,$$

where each of the numerators $\alpha, \beta, \gamma, \ldots$ is either 0 or 1, and μ is either absolutely 0 or $< 1/2^r$, where r is as great as we choose.

Hence

$$p^{1/n} = p^{\alpha/2+\beta/2^2+\gamma/2^3+\ldots+\mu},$$
$$= p^{\alpha/2} \times p^{\beta/2^2} \times p^{\gamma/2^3} \times \ldots \times p^{\mu} \qquad (1).$$

Now, excepting the last, each of these factors is either 1, or of the form $p^{1/2^s}$, which can be approximated to as closely as we please by continued extraction of the square root. If $\mu=0$, the last factor is 1; and if $\mu < 1/2^r$, since r may be as great as we choose, we can make it differ from 1 by as small a fraction as we choose. It follows therefore that the product on the right hand of (1) may be found in rational terms as accurately as we please.

Example. To find an approximate value of $51^{1/3}$.

We have

$$\frac{1}{3} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \frac{1}{2^{12}} + \frac{1}{2^{14}} + \frac{1}{2^{16}} + \frac{1}{2^{18}} + \frac{1}{2^{20}} + \mu,$$

where $\mu < 1/2^{20}$.

Now we have, correct to the fourth decimal place, the following values:—

$51^{1/2^2}=2{\cdot}67234$,	$51^{1/2^{12}}=1{\cdot}00096$,
$51^{1/2^4}=1{\cdot}27857$,	$51^{1/2^{14}}=1{\cdot}00023$,
$51^{1/2^6}=1{\cdot}06336$,	$51^{1/2^{16}}=1{\cdot}00006$,
$51^{1/2^8}=1{\cdot}01548$,	$51^{1/2^{18}}=1{\cdot}00002$,
$51^{1/2^{10}}=1{\cdot}00385$,	$51^{1/2^{20}}<1{\cdot}00001$.

Hence, multiplying the first nine numbers together, we get

$$51^{1/3}=3{\cdot}70841$$

The correct value is $3{\cdot}708429$ . . .

§ 16.] The method just explained, although interesting in theory, would be very troublesome in practice.

The method given in § 13 for extracting the square root may be easily generalised into a method for extracting an nth root directly, figure by figure, and group by group of figures. The student will be able to establish for himself two propositions, counterparts of I. and II., § 12, and to arrange a process for the economical calculation of the residues. A method of this kind is given in most arithmetical text-books for extracting the cube root, but it is needless to reproduce it here, as the extraction of cube and higher roots, and even of square roots, is now accomplished in practice by means of logarithmic or other tables (see chap. xxi.) Moreover, the extraction of the nth root of a given number is merely a particular case of the numerical solution of an equation of the nth degree, a process for which, called *Horner's Method*, will be given in a later chapter.

Our reason for dwelling on the more elementary methods of this chapter is a desire to cultivate in the mind of the learner exact ideas regarding the nature of approximate calculation—a process which lies at the root of many useful applications of mathematics.

SQUARE ROOT OF AN INTEGRAL FUNCTION OF x.

§ 17.] When an integral function of x is a complete square as regards x, its square root can be found by a method analogous

to that explained in § 12, for finding the square root of a number. Although the method is of little interest, either theoretically or practically, we give a brief sketch of it here, because it illustrates at once the analogy and the fundamental difference between arithmetical and algebraical operations.*

I. We may restate Proposition I. of § 12, understanding now A and B to mean integral functions of x.

II. *If* $F = p_0x^{2n} + p_1x^{2n-1} + \ldots + p_{2n}$, *and if* $\sqrt{F} = (q_0x^n + q_1x^{n-1} + \ldots + q_{n-p+1}x^{n-p+1}) + (q_{n-p}x^{n-p} + \ldots + q_0) = P + Q$, *say; and if we suppose the first* p *terms, namely,* $P = q_0x^n + q_1x^{n-1} + \ldots + q_{n-p+1}x^{n-p+1}$, *of* $\sqrt{F}$ *to be known, then the next* p *terms will be the first* p *terms in the integral part of* $(F - P^2)/2P$.

For we have

$$F = P^2 + 2PQ + Q^2;$$

hence
$$\frac{F - P^2}{2P} = Q + \frac{Q^2}{2P}.$$

Now the degree of the integral part of $Q^2/2P$ is $2(n-p) - n = n - 2p$. Hence $Q^2/2P$ will at most affect the term in x^{n-2p}. Hence $(F - P^2)/2P$ will be identical with Q down to the term in x^{n-2p+1} inclusive. In other words, the first $n - p - (n - 2p) = p$ terms obtained by dividing $F - P^2$ by $2P$ will be the p terms of the square root which follow P.

We may use this rule to obtain the whole of the terms one at a time, the highest being of course found by inspection as the square root of the highest term of the radicand; or we may obtain a certain number in this way, and then obtain the rest by division.†

The process will be understood from the following example,

---

* The method was probably obtained by analogy from the arithmetical process. It was first given by Recorde in *The Whetstone of Witte* (black letter, 1557), the earliest English work on algebra.

† Just as in division, we may, if we please, arrange the radicand according to ascending powers of x. The final result will be the same whichever arrangement be adopted, provided the radicand is a complete square. If this is not the case the operation may be prolonged indefinitely just as in continued division. We leave the learner to discover the meaning of the result obtained in such cases. The full discussion of the matter would require some reference to the theory of infinite series.

in which we first find three of the terms of the root singly, and then deduce the remaining two by division :—

Example.

To find the square root of

$$x^{10}+6x^9+13x^8+4x^7-18x^6-12x^5+14x^4-12x^3+9x^2-2x+1.$$

$$\begin{array}{l|r|l}
 & 1+6+13+4-18-12+14-12+9-2+1 & +1 \\
1 & 1 & \\
\hline
 & 6+13+4-18-12+14-12+9-2+1 & +3 \\
2+3 & 6+9 & \\
\hline
 & 4+4-18-12+14-12+9-2+1 & +2 \\
2+6+2 & 4+12+4 & \\
\hline
 & -8-22-12+14-12+9-2+1 & -4 \\
2+6+4-4 & -8-24-16+16 & \\
\hline
 & 2+4-2-12+9-2+1 & +1-1 \\
2+6+4-8 & 2+6+4-8 & \\
\hline
 & -2-6-4+9-2+1 & \\
 & -2-6-4+8 & \\
\hline
 & 1-2+1 &
\end{array}$$

Hence the square root is $x^5+3x^4+2x^3-4x^2+x-1$; and, since the residue x^2-2x+1 is the square of the two last terms, namely, $x-1$, we see that the radicand is an exact square. Of course we obtain another value of the square root by changing the sign of every coefficient in the above result.

A similar process can be arranged for the extraction of the cube root; but it is needless to pursue the matter further.

§ 18.] The student should observe that in the simpler cases the root can be obtained by inspection; and that in all cases the method of indeterminate coefficients renders any special process for the extraction of roots superfluous. This will be understood from the following example.

Example.

To extract the square root of

$$x^{10}+6x^9+13x^8+4x^7-18x^6-12x^5+14x^4-12x^3+9x^2-2x+1 \quad (1).$$

If the radicand be a complete square, its square root must be of the form

$$x^5+px^4+qx^3+rx^2+sx+t \quad (2).$$

The square of (2) is

$$x^{10}+2px^9+(p^2+2q)x^8+(2pq+2r)x^7+(2pr+q^2+2s)x^6+\ldots \quad (3).$$

Now this must be identical with (1); hence we must have

$$2p=6, \quad p^2+2q=13, \quad 2pq+2r=4, \quad 2pr+q^2+2s=-18.$$

The first of these equations gives $p=3$; p being thus known, the second gives $q=2$; p and q being known, the third gives $r=-4$; and p, q, r being known, the last gives $s=1$. We could now find t in like manner; but it is obvious from the coefficient of x that $t=-1$.

Hence one value of the square root is

$$x^5+3x^4+2x^3-4x^2+x-1.$$

N.B.—The equating of the coefficients of the remaining terms of (1) and (3) will simply give equations that are satisfied by the values of p, q, r, s already found, always supposing that the given radicand is an exact square.

A process exactly similar to the above will furnish the root of an exact cube, an exact 4th power, and so on.

Exercises XV.

Express the following as linear functions of the irrationals involved.

(1.) $1/(\sqrt{11}+\sqrt{3}+\sqrt{14})$. (2.) $\sqrt{12}/(1+\sqrt{2})(\sqrt{6}-\sqrt{3})$.

(3.) $(1-\sqrt{2}+\sqrt{3})/(1+\sqrt{2}+\sqrt{3})+(1-\sqrt{2}-\sqrt{3})/(1+\sqrt{2}-\sqrt{3})$.

(4.) $(3-\sqrt{5})/(\sqrt{3}+\sqrt{5})^2+(3+\sqrt{5})/(\sqrt{3}-\sqrt{5})^2$.

(5.) $\sqrt{5}/(\sqrt{3}+\sqrt{5}-2\sqrt{2})-\sqrt{2}/(\sqrt{3}+\sqrt{2}-\sqrt{5})$.

(6.) $(7-2\sqrt{5})(5+\sqrt{7})(31+13\sqrt{5})/(6-2\sqrt{7})(3+\sqrt{5})(11+4\sqrt{7})$.

(7.) $\sqrt{(25+10\sqrt{6})}$. (8.) $\sqrt{(3/2+\sqrt{2})}$.

(9.) $\sqrt{(123-22\sqrt{2})}$. (10.) $\sqrt{(44\sqrt{2}+12\sqrt{26})}$.

(11.) $\sqrt{\{(8+4\sqrt{10})/(8-4\sqrt{10})\}}$.

(12.) $\sqrt{(7+4\sqrt{3})}+\sqrt{(5-2\sqrt{6})}$.

(13.) $\sqrt{(15-4\sqrt{14})}+1/\sqrt{(15+4\sqrt{14})}$.

(14.) $1/\sqrt{(16+2\sqrt{63})}+1/\sqrt{(16-2\sqrt{63})}$.

(15.) $1/\sqrt{(16\sqrt{3}+6\sqrt{21})}+\sqrt{(16\sqrt{3}-6\sqrt{21})}$.

(16.) Calculate to five places of decimals the value of $\{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}\}/\{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}\}$.

(17.) Calculate to seven places of decimals the value of $\sqrt{(\sqrt{15}+\sqrt{13})}+\sqrt{(\sqrt{15}-\sqrt{13})}$.

Simplify—

(18.) $\sqrt{\{3+\sqrt{(9-p^2)}\}}+\sqrt{\{3-\sqrt{(9-p^2)}\}}$.

(19.) $\sqrt{\{a+b-c+2\sqrt{(b(a-c))}\}}$.

(20.) $\sqrt{\{a^2-2+a\sqrt{(a^2-4)}\}}$.

(21.) $\sqrt{\left\{\left(1-\dfrac{1}{1-x^2}\right)\left(\dfrac{1}{1-1/x^2}-1\right)\right\}}$.

(22.) Show that $\sqrt{\{2+\sqrt{(2-\sqrt{2})}\}}=\sqrt{\left\{\dfrac{2+\sqrt{(2+\sqrt{2})}}{2}\right\}}+\sqrt{\left\{\dfrac{2-\sqrt{(2+\sqrt{2})}}{2}\right\}}$.

(23.) Express in a linear form $\sqrt{(5+\sqrt{6}+\sqrt{10}+\sqrt{15})}$.

(24.) ,, ,, $\sqrt{(25-4\sqrt{3}-12\sqrt{2}+6\sqrt{6})}$.

(25.) If $a^2d=bc$, then $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}$ can always be expressed in the form $(\sqrt{x}+\sqrt{y})(\sqrt{X}+\sqrt{Y})$. Show that this will be advantageous if a^2-b and a^2-c are perfect squares.

(26.) If $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, where a, b, x, y are rational, and $\sqrt{b}$ and $\sqrt{y}$ irrational, then $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$. Hence show that, if $a^2-b=z^3$, where z is rational, and if x be such that $4x^3-3xz=a$, then $\sqrt[3]{(a+\sqrt{b})}$ $=x+\sqrt{(x^2-z)}$.

(27.) Express in linear form $\sqrt[3]{(99-35\sqrt{8})}$.

(28.) ,, ,, $\sqrt[3]{(395+93\sqrt{18})}$.

(29.) ,, ,, $\sqrt[3]{(117\sqrt{2}+74\sqrt{5})}$.

(30.) Show that $\sqrt[3]{(90+34\sqrt{7})}-\sqrt[3]{(90-34\sqrt{7})}=2\sqrt{7}$.

(31.) If $x=\sqrt[3]{(p+q)}+\sqrt[3]{(p-q)}$, and $p^2-q^2=r^3$, show that $x^3-3rx-2p$ $=0$.

(32.) If $py^{\frac{2}{3}}+qy^{\frac{1}{3}}+r=0$, where p, q, r, y are all rational, and $y^{\frac{1}{3}}$ irrational, then $p=0$, $q=0$, $r=0$. Hence show that, if x, y, z be all rational, and $x^{\frac{1}{3}}$, $y^{\frac{1}{3}}$, $z^{\frac{1}{3}}$ all irrational, then neither of the equations $x^{\frac{1}{3}}+y^{\frac{1}{3}}=z$, $x^{\frac{1}{3}}+y^{\frac{1}{3}}=z^{\frac{1}{3}}$ is possible.

(33.) Find, by the full use of the ordinary rule, the value of $\sqrt{10}$ to 5 places of decimals; and find as many more figures as you can by division alone. Use the value of $\sqrt{10}$ thus found to obtain $\sqrt{\cdot 004}$.

Extract the square root of the following:—

(34.) $(yz+zx+xy)^2-4xyz(z+x)$.

(35.) $25x^2+9y^2+z^2+6yz-10zx-30xy$.

(36.) $9x^4+24x^3+10x^2-8x+1$. (37.) $x^4-4x^3+2x^2+4x+1$.

(38.) $4x^4-12x^3y+25x^2y^2-24xy^3+16y^4$.

(39.) $x^6-6x^4+4x^3+9x^2-12x+4$.

(40.) $4x^6-12x^5+5x^4+22x^3-23x^2-8x+16$.

(41.) $27(p+q)^2(p^2+q^2)^2-2(p^2+4pq+q^2)^3$.

(42.) $x^2-2x\sqrt{x}+3x-2\sqrt{x}+1$.

(43.) Extract the cube root of

$$8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27.$$

(44.) Extract the cube root of

$$18(p^3+p^2q+pq^2+q^3)\pm 2\sqrt{3}(5p^3+3p^2q-3pq^2-5q^3).$$

(45.) Show that λ can be determined so that $x^4+6x^3+7x^2-6x+\lambda$ shall be an exact square.

(46.) Find a, b, c, so that $x^6-8x^5+ax^4+bx^3+cx^2-44x+4$ shall be an exact square.

(47.) If $ax^4+bx^3+cx^2$ be subtracted from $(x^2+2x+4)^3$ the remainder is an exact square; find a, b, c.

(48.) If $x^6+ax^5+bx^4+cx^3+dx^2+ex+f$ be an exact square, show that

$$d=\tfrac{5}{64}a^4-\tfrac{3}{8}a^2b+\tfrac{1}{4}b^2+\tfrac{1}{2}ac,$$
$$e=-\tfrac{1}{64}a^5+\tfrac{1}{8}a^3b-\tfrac{1}{8}a^2c-\tfrac{1}{4}ab^2+\tfrac{1}{2}bc,$$
$$f=\tfrac{1}{256}a^6-\tfrac{1}{32}a^4b+\tfrac{1}{16}a^3c+\tfrac{1}{16}a^2b^2-\tfrac{1}{4}abc+\tfrac{1}{4}c^2;$$

And that the square root is

$$x^3+\tfrac{1}{2}ax^2+(-\tfrac{1}{8}a^2+\tfrac{1}{2}b)x+(\tfrac{1}{16}a^3-\tfrac{1}{4}ab+\tfrac{1}{2}c).$$

(49.) $4x^6+12x^5+5x^4-2x^3$ are the first four terms of an exact square; find the remaining three terms.

(50.) If $x^6+3dx^5+ex^4+fx^3+gx^2+hx+k^3$ be a perfect cube, find its cube root; and determine the coefficients e, f, g, h, in terms of d and k.

(51.) Show that

$$b^2(a-b)(c-b)\{(a-b)^2+(c-b)^2\}-ab^2c(a^2+c^2)+b^5(a-b+c)$$

is an exact cube.

(52.) Express $\sqrt{\{1+x+x^2+x^3+\ldots \textit{ad } \infty\}}$ in the form $a+bx+cx^2+\ldots$ as far as the 4th power of x. To how many terms does the square of your result agree with $1+x+x^2+x^3+\ldots$?

(53.) Express, by means of the ordinary rule for extracting the square root, $\sqrt[4]{(1-x)}$ as an ascending series of integral powers of x, as far as the 4th power.

(54.) Express $\sqrt{(x+1)}$ as a descending series of powers of x, calculating six terms of the series.

(55.) Show that Lambert's theorem (chap. ix., § 9) can be used to find rational approximations to surd numbers. Apply it to show that $\sqrt{2}=1+1/2-1/2.5+1/2.5.7-1/2.5.7.197$ approximately; and estimate the error.

CHAPTER XII.

Complex Numbers.

ON THE FUNDAMENTAL NATURE OF COMPLEX NUMBERS.

§ 1.] The attempt to make certain formulæ for factorisation as general as possible has already shown us the necessity of introducing into algebra an imaginary unit i, having the property $i^2 = -1$. It is obvious from its definition that i *cannot be equal to any real quantity*, for the squares of all real quantities are positive. The properties of i as a subject of operation are therefore to be deduced entirely from its definition, and from the general laws of algebra to which, like every other algebraical quantity, it must be subject.

Since i must, when taken along with other algebraical quantities, obey all the laws of algebra, we may consider any real multiples of i, say yi and $y'i$, where y and y' are positive or negative, and we must have $yi = iy$, $yi + y'i = (y + y')i = i(y + y')$, and so on; exactly as if i were a real quantity.

By taking all real multiples of i from $-\infty i$ to $+\infty i$, we have a continuous series of purely imaginary quantity,

$$-\infty i \;.\;.\;.\; -i \;.\;.\;.\; 0i \;.\;.\;.\; +i \;.\;.\;.\; +\infty i \qquad \text{I.,}$$

whose unit is i, and which corresponds to the series of real quantity,

$$-\infty \;.\;.\;.\; -1 \;.\;.\;.\; 0 \;.\;.\;.\; +1 \;.\;.\;.\; +\infty \qquad \text{II.,}$$

whose unit is 1.

No quantity of the series I. (except $0i$) can be equal to any quantity of the series II., for the square of any real multiple of i, say yi, is $y^2i^2 = y^2(-1) = -y^2$, that is, is a negative quantity.

Hence *no purely imaginary quantity except* $0i$ *can be equal to a real quantity.* Since $0i = (+a-a)i = +(ai)-(ai) = 0$, if the same laws are to apply to imaginary as to real quantity, we infer *that* $0i = 0$. Hence 0 is the middle value of the series of purely imaginary, just as it is of the series of real quantity; it is, in fact, the only quantity common to the two series.

Conversely, if $yi = 0$, *we infer that* $y = 0$. For, since $yi = 0$, $yi \times yi = 0$, that is, $-y^2 = 0$; hence $y = 0$.

§ 2.] If we combine, by addition, any real quantity x with a purely imaginary quantity yi, there arises a mixed quantity $x + yi$, to which the name *complex number* is applied.

We may consider the infinite series of complex numbers formed by giving all possible real values to x, and all possible real values to y. We thus have a doubly infinite series of complex quantity. The student should note at the outset this double character of complex quantity, on account of the contrast which thus arises between purely real or purely imaginary quantity on the one hand, and complex quantity on the other. Thus there is only one way of varying z continuously (without repetition of intermediate values) from -1 to $+2$, say, if z is to be always real; and only one way of varying z in like manner from $-i$ to $+2i$, if z is to be always purely imaginary. But there are an infinite number of ways of varying z continuously from $-1+i$ to $2+3i$, say, if there be no restriction upon the nature of z, except that it is to be a complex number.

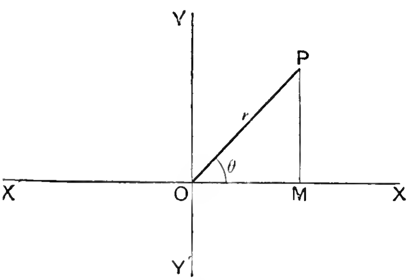


FIG. 1.

This will be best understood if we adopt the diagrammatic method of representing complex numbers introduced by Argand.

Let XOX', YOY' be two rectangular axes. We shall call XOX' the axis of real quantity, or x-axis; and YOY' the axis of purely imaginary quantity, or y-axis. To

represent any complex number $x+yi$ we measure from O (called the origin) a distance OM, containing x units of length, to the right or left according as x is positive or negative; and we draw MP, containing y units of length, upwards or downwards according as y is positive or negative. The point P, or, as is more convenient from some points of view, the "radius vector" OP, is then said to represent the complex number $x+yi$. It is obvious that to every conceivable complex number there corresponds one and only one point in the plane of XX′ and YY′; and, conversely, that to every one of the doubly infinite series of points in that plane there corresponds one and only one complex number. P is often called the *affixe* of $x+yi$, or simply the "Point $x+yi$."

If P lie on the axis XX′, then $y=0$, and the number $x+yi$ is wholly real. If P lie on the axis YY′, then $x=0$, and $x+yi$ is wholly imaginary. Now there is only one way of passing from any point on XX′ to any other point, if we are not to leave the axis, namely, we must pass along the x-axis; and the same is true for the axis YY′. If, however, we are not restricted as to our path, there are an infinity of ways of passing from one point in the plane of XX′ and YY′ to any other point in the same plane. If we draw any continuous curve whatever from P to Q, and imagine a point to travel along it from P to Q, the value of x corresponding to the moving point will vary continuously from the value OM to the value ON, and the value of y in like manner from MP to NQ. Hence there are as many ways of varying $x+yi$ from $\mathrm{OM}+\mathrm{MP}i$ to $\mathrm{ON}+\mathrm{NQ}i$ as there are ways of drawing a continuous curve from P to Q.

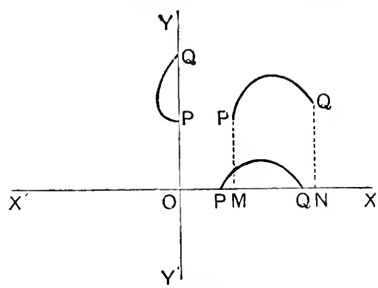


FIG. 2.

Similar remarks apply when P and Q happen, as they may in particular cases, to be both on the x-axis, or both on the y-axis, provided that there is no restriction that the varying quantity shall be always real or always imaginary. There are

many other properties of complex numbers, which are best understood by studying Argand's diagram, and we shall return to it again in this chapter. In the meantime, however, to prevent confusion in the mind of the reader, we shall confine ourselves for a little to purely analytical considerations.

§ 3.] *If* $x + yi = 0$, *then* $x = 0$, $y = 0$.* For it follows from $x + yi = 0$ that $x = -yi$. Hence, if y did not vanish, we should have a real quantity x equal to a purely imaginary quantity $-yi$, which is impossible. We must therefore have $y = 0$; and consequently $x = -0i = 0$.

Cor. *Hence if* $x + yi = x' + y'i$, *then must* $x = x'$ *and* $y = y'$.

For $x + yi = x' + y'i$ gives, if we subtract $x' + y'i$ from both sides,

$$(x - x') + (y - y')i = 0.$$

Hence $$x - x' = 0, \quad y - y' = 0,$$

that is, $$x = x', \qquad y = y'.$$

RATIONAL FUNCTIONS OF COMPLEX NUMBERS.

§ 4.] We have seen that so long as we operate upon real quantities, provided we confine ourselves to the rational operations—addition, subtraction, multiplication, and division, we reproduce real quantities and real quantities only. On the other hand, if we use the irrational operation of root extraction, it becomes necessary, if we are to keep up the generality of algebraical operations, to introduce the imaginary unit i. We are thus led to the consideration of complex numbers. The question now naturally presents itself. "If we operate, rationally or irrationally, in accordance with the general laws of algebra on quantities real or complex as now defined, shall we always reproduce quantities real or complex as now defined; or may it happen that at some stage it will be necessary in the interest of algebraic generality to introduce some new kind of imaginary quantity not as yet imagined?" The answer to this question is that, so far at least as the algebraical operations of addition,

* Here and hereafter in this chapter, when we write the form $x + yi$, it is understood that this denotes a complex number in its simplest form, so that x and y are real.

subtraction, multiplication, division, and root extraction are concerned, no further extension of the conception of algebraic quantity is needed. It is, in fact, one of the main objects of the present chapter to prove that algebraic operations on complex numbers reproduce only complex numbers.

§ 5.] *The sum or product of any number of complex numbers, and the quotient of two complex numbers, may be expressed as a complex number.*

Suppose we have, say, three complex numbers, $x_1 + y_1 i$, $x_2 + y_2 i$, $x_3 + y_3 i$, then

$$(x_1 + y_1 i) + (x_2 + y_2 i) - (x_3 + y_3 i) = (x_1 + x_2 - x_3) + (y_1 + y_2 - y_3)i,$$

by the laws of algebra as already established.

But $x_1 + x_2 - x_3$ and $y_1 + y_2 - y_3$ are real, since $x_1, x_2, x_3, y_1, y_2, y_3$ are so. Hence $(x_1 + x_2 - x_3) + (y_1 + y_2 - y_3)i$ is in the standard form of a complex number. The conclusion obviously holds, however many terms there may be in the algebraic sum.

Again, consider the product of two complex numbers, $x_1 + y_1 i$ and $x_2 + y_2 i$. We have, by the law of distribution,

$$(x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + y_1 y_2 i^2 + x_1 y_2 i + x_2 y_1 i.$$

Hence, bearing in mind the definition of i, we have

$$(x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i,$$

which proves that the product of two complex numbers can be expressed as a complex number.

To prove the proposition for a product of three complex numbers, say for

$$\mathrm{P} = (x_1 + y_1 i)(x_2 + y_2 i)(x_3 + y_3 i),$$

we have merely to apply the law of association, and write

$$\mathrm{P} = \{(x_1 + y_1 i)(x_2 + y_2 i)\}(x_3 + y_3 i).$$

We have already shown that the function within the crooked brackets reduces to a complex number; hence P is the product of two complex numbers. Hence, again, by what we proved above, P reduces to a complex number. In this way we can extend the theorem to a product of any number of complex numbers.

Lastly, consider the quotient of two complex numbers. We have

$$\frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2)^2 - (y_2 i)^2},\text{*}$$

$$= \frac{(x_1 x_2 + y_1 y_2) - (x_1 y_2 - x_2 y_1)i}{x_2^2 + y_2^2},$$

$$= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) - \left(\frac{x_1 y_2 - x_2 y_1}{x_2^2 + y_2^2}\right)i.$$

This proves that the quotient of two complex numbers can always be reduced to a complex number.

Cor. 1. Since every rational function involves only the operations of addition, subtraction, multiplication, and division, it follows from what has just been shown that *every rational function of one or more complex numbers can be reduced to a complex number.*

Cor. 2. *If $\phi(x + yi)$ be any rational function of $x + yi$, having all its coefficients real, and if*

$$\phi(x + yi) = X + Yi,$$

then

$$\phi(x - yi) = X - Yi,$$

X and Y being of course real.

Cor. 3. Still more generally, *if $\phi(x_1 + y_1 i, x_2 + y_2 i, \ldots, x_n + y_n i)$ be any rational function of n complex numbers, having all its coefficients real, and if*

$$\phi(x_1 + y_1 i, \quad x_2 + y_2 i, \quad \ldots, \quad x_n + y_n i) = X + Yi,$$

then

$$\phi(x_1 - y_1 i, \quad x_2 - y_2 i, \quad \ldots, \quad x_n - y_n i) = X - Yi.$$

Cor. 4. *If all the coefficients of the integral function $\phi(z)$ be real, and if $\phi(z)$ vanish when $z = x + yi$, then $\phi(z)$ vanishes when $z = x - yi$.*

* Here we perform an operation which we might describe as "realising" the denominator; it is analogous to the process of rationalising described in chap. x.

For, by Cor. 1, $\phi(x+yi) = X + Yi$ where X and Y are real. Hence, if $\phi(x+yi) = 0$, we have $X + Yi = 0$. Hence, by § 3, $X = 0$ and $Y = 0$. Therefore $\phi(x-yi) = X - Yi = 0 - 0i = 0$.

Cor. 5. *If all the coefficients of the integral function $\phi(z_1, z_2, \ldots, z_n)$ be real, and if the function vanish when $z_1, z_2, \ldots, z_n$ are equal to $x_1 + y_1 i, x_2 + y_2 i, \ldots, x_n + y_n i$ respectively, then the function will also vanish when $z_1, z_2, \ldots, z_n$ are equal to $x_1 - y_1 i, x_2 - y_2 i, \ldots, x_n - y_n i$ respectively.*

Example 1.

$$\begin{aligned} 3(3+2i) - 2(2-3i) + (6+8i) &= 9 + 6i - 4 + 6i + 6 + 8i, \\ &= 11 + 20i. \end{aligned}$$

Example 2.

$$\begin{aligned} (2+3i)(2-5i)(3+2i) &= (2-5i)(6-6+9i+4i), \\ &= (2-5i)13i, \\ &= 26i + 65, \\ &= 65 + 26i. \end{aligned}$$

Example 3.

$$\begin{aligned} &(b+c-ai)(c+a-bi)(a+b-ci) \\ &\quad = \{\Pi(b+c) - \Sigma bc(b+c)\} + \{abc - \Sigma a(a+b)(a+c)\} i, \\ &\quad = 2abc + \{abc - \Sigma a^3 - \Sigma a^2(b+c) - 3abc\} i, \\ &\quad = 2abc - \{a^3 + b^3 + c^3 + (b+c)(c+a)(a+b)\} i. \end{aligned}$$

Example 4.

To show that the values of the powers of i recur in a cycle of 4.

We have $i = i, \quad i^2 = -1, \quad i^3 = i^2 \times i = -i, \quad i^4 = (i^2)^2 = +1,$

$i^5 = i^4 \times i = i, \quad i^6 = i^4 \times i^2 = -1, \quad i^7 = i^4 \times i^3 = -i, \quad i^8 = i^4 \times i^4 = +1;$

and, in general,

$$i^{4n+1} = i, \qquad i^{4n+2} = -1, \qquad i^{4n+3} = -i, \qquad i^{4(n+1)} = +1.$$

Example 5.

$$\frac{3+5i}{2-3i} = \frac{(3+5i)(2+3i)}{4+9} = \frac{6-15+19i}{13} = -\frac{9}{13} + \frac{19}{13}i.$$

Example 6.

$$\begin{aligned} (x+yi)^n &= x^n + {}_nC_1 x^{n-1}(yi) + {}_nC_2 x^{n-2}(yi)^2 + \ldots, \\ &= (x^n - {}_nC_2 x^{n-2}y^2 + {}_nC_4 x^{n-4}y^4 - \ldots) \\ &\quad + ({}_nC_1 x^{n-1}y - {}_nC_3 x^{n-3}y^3 + {}_nC_5 x^{n-5}y^5 - \ldots)i. \end{aligned}$$

In particular

$$(x+yi)^4 = (x^4 - 6x^2y^2 + y^4) + (4x^3y - 4xy^3)i.$$

Example 7.

If $$\phi(z) \equiv \frac{z^2 - z + 1}{z^2 + z + 1},$$

then $$\begin{aligned} \phi(2+3i) &= \frac{(2+3i)^2 - (2+3i) + 1}{(2+3i)^2 + (2+3i) + 1}, \\ &= \frac{-5 + 12i - 2 - 3i + 1}{-5 + 12i + 2 + 3i + 1}, \end{aligned}$$

$$=\frac{-6+9i}{-2+15i}=\frac{3(-2+3i)(-2-15i)}{229},$$

$$=\frac{3}{229}\{4+45-6i+30i\},$$

$$=\frac{147}{229}+\frac{72}{229}i.$$

From this we infer that

$$\phi(2-3i)=\frac{147}{229}-\frac{72}{229}i;$$

a conclusion which the student should verify by direct calculation.

CONJUGATE COMPLEX NUMBERS, NORMS, AND MODULI.

§ 6.] Two complex numbers which differ only in the sign of their imaginary part are said to be *conjugate*. Thus $-3-2i$ and $-3+2i$ are conjugate, so are $-4i$ and $+4i$; and, generally, $x+yi$ and $x-yi$.

Using this nomenclature we may enunciate Cor. 3 of § 5 as follows:—

If the coefficients of the rational function ϕ be real, then the values of

$$\phi(x_1+y_1i,\quad x_2+y_2i,\quad \ldots,\quad x_n+y_ni)$$

and

$$\phi(x_1-y_1i,\quad x_2-y_2i,\quad \ldots,\quad x_n-y_ni),$$

where the values of the variables are conjugate, are conjugate complex numbers.

The reader will readily establish the following:—

The sum and the product of two conjugate complex numbers are real.

Conversely, if both the sum and the product of two complex numbers be real, then either both are real or they are conjugate.

§ 7.] By the *modulus* of the complex number $x+yi$ is meant $+\sqrt{(x^2+y^2)}$. This is usually denoted by $|x+yi|$.*

It is obvious that *a complex number and its conjugate have the same modulus; and that this modulus is the positive value of the square root of their product.*

Examples.

$$|-3+4i|=+\sqrt{\{(-3)^2+4^2\}}=5.$$
$$|-3-4i|=+\sqrt{\{(-3)^2+(-4)^2\}}=5.$$
$$|1+i|=+\sqrt{(1^2+1^2)}=\sqrt{2}.$$

* Formerly by mod $(x+yi)$.

It should be noticed that if $y=0$, that is, if the complex number be wholly real, then the modulus reduces to $+\sqrt{x^2}$, which is simply the value of x taken with the positive sign, or, say, the numerical value of x. For example, $|-3| = +\sqrt{(-3)^2} = +3$. For this reason Continental writers frequently use $|x|$ where x is a real quantity, as an abbreviation for "the numerical value of x." We shall occasionally make use of this convenient contraction.

For reasons that will be understood by referring once more to § 2, the ordinary algebraical ideas of greater and less which apply to real quantities cannot be attached to complex numbers. The reader will, however, find that for many purposes the measure of the "magnitude" of a complex number is its modulus. We cannot at the present stage explain precisely how "magnitude" is here to be understood, but we may remark that, in Argand's diagram, the representative points of all complex numbers whose moduli are less than ρ lie within a circle whose centre is at the origin and whose radius is ρ.

§ 8.] *If a complex number vanish its modulus vanishes; and, conversely, if the modulus vanish the complex number vanishes.*

For if $x+yi=0$, then by § 3, $x=0$ and $y=0$. Hence $\sqrt{(x^2+y^2)}=0$.

Again, if $\sqrt{(x^2+y^2)}=0$, then $x^2+y^2=0$; but, since both x and y are real, both x^2 and y^2 are positive, hence their sum cannot be zero unless each be zero. Therefore $x=0$ and $y=0$.

If two complex numbers are equal their moduli are equal; but the converse is not true.

For, if $x+yi=x'+y'i$, then, by § 3, $x=x'$, $y=y'$. Hence $\sqrt{(x^2+y^2)} = \sqrt{(x'^2+y'^2)}$.

On the other hand, it does not follow from $\sqrt{(x^2+y^2)} = \sqrt{(x'^2+y'^2)}$ that $x=x'$, $y=y'$. Hence the converse is not true.

§ 9.] Provided all the coefficients in $\phi(x+yi)$ be real, we have seen (§ 5, Cor. 2) that if

$$\phi(x+yi) = X + Yi,$$

where X and Y are real, then

$$\phi(x-yi) = X - Yi.$$

Now
$$\begin{aligned} |\phi(x+yi)| = \sqrt{(X^2+Y^2)} &= \sqrt{\{(X+Yi)(X-Yi)\}}, \\ &= \sqrt{\{\phi(x+yi)\quad\phi(x-yi)\}}, \\ &= |\phi(x-yi)| \qquad (1). \end{aligned}$$

In like manner it follows from § 5, Cor. 3, that

$$\begin{aligned}&|\,\phi(x_1+y_1i,\quad x_2+y_2i,\quad \ldots,\quad x_n+y_ni)\,|\\ =&|\,\phi(x_1-y_1i,\quad x_2-y_2i,\quad \ldots,\quad x_n-y_ni)\,|\\ =&+\sqrt{[\phi(x_1+y_1i,\quad x_2+y_2i,\quad \ldots,\quad x_n+y_ni)}\\ &\quad\times\phi(x_1-y_1i,\quad x_2-y_2i,\quad \ldots,\quad x_n-y_ni)] \qquad (2).\end{aligned}$$

The theorems expressed by (1) and (2) are very useful in practice, as will be seen in the examples worked below.

It should be observed that (1) contains certain remarkable particular cases. For example,

$$\begin{aligned}&|\,(x_1+y_1i)\,(x_2+y_2i)\,\ldots\,(x_n+y_ni)\,|\\ &\quad=+\sqrt{[(x_1+y_1i)\,(x_2+y_2i)\,\ldots\,(x_n+y_ni)}\\ &\qquad\qquad\times(x_1-y_1i)\,(x_2-y_2i)\,\ldots\,(x_n-y_ni)],\\ &\quad=+\sqrt{(x_1^2+y_1^2)\,(x_2^2+y_2^2)\,\ldots\,(x_n^2+y_n^2)},\\ &\quad=|\,(x_1+y_1i)\,|\times|\,(x_2+y_2i)\,|\times\,\ldots\,\times|\,(x_n+y_ni)\,| \qquad (3).\end{aligned}$$

In other words, *the modulus of the product of n complex numbers is equal to the product of their moduli.*

Also

$$\left|\frac{x_1+y_1i}{x_2+y_2i}\right|=\frac{\sqrt{(x_1^2+y_1^2)}}{\sqrt{(x_2^2+y_2^2)}}=\frac{|\,x_1+y_1i\,|}{|\,x_2+y_2i\,|} \qquad (4).$$

In other words, *the modulus of the quotient of two complex numbers is the quotient of their moduli.*

§ 10.] The reader should establish the results (3) and (4) of last paragraph directly.

It may be noted that we are led to the following identities :—

$$(x_1^2+y_1^2)\,(x_2^2+y_2^2)=(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2.$$

If we give to x_1, y_1, x_2, y_2 positive integral values, this gives us the proposition that *the product of two integers, each of which is the sum of two square integers, is itself the sum of two square integers*; and the formula indicates how one pair of values of the two integers last mentioned can be found.

Also

$$\begin{aligned}(x_1^2+y_1^2)\,(x_2^2+y_2^2)\,(x_3^2+y_3^2)=&\,(x_1x_2x_3-x_1y_2y_3-x_2y_3y_1-x_3y_1y_2)^2\\ &+(y_1x_2x_3+y_2x_3x_1+y_3x_1x_2-y_1y_2y_3)^2.\end{aligned}$$

This shows that *the product of three sums of two integral squares is the sum of two integral squares*, and shows one way at least of finding the two last-mentioned integers.

Similar results may of course be obtained for a product of any number of factors.

Example 1.

Find the modulus of $(2+3i)(3-2i)(6-4i)$.

$$\begin{aligned}|(2+3i)(3-2i)(6-4i)| &= |(2+3i)|\times|(3-2i)|\times|(6-4i)|,\\ &= \sqrt{(13)}\times\sqrt{(13)}\times\sqrt{(52)},\\ &= 26\sqrt{(13)}.\end{aligned}$$

Example 2.

Find the modulus of $(\sqrt{2}+i\sqrt{3})(\sqrt{3}+i\sqrt{5})/(\sqrt{2}+i\sqrt{5})$.

$$\begin{aligned}&\left|\frac{(\sqrt{2}+i\sqrt{3})(\sqrt{3}+i\sqrt{5})}{\sqrt{2}+i\sqrt{5}}\right|\\ &= \sqrt{\left[\left\{\frac{(\sqrt{2}+i\sqrt{3})(\sqrt{3}+i\sqrt{5})}{\sqrt{2}+i\sqrt{5}}\right\}\times\left\{\frac{(\sqrt{2}-i\sqrt{3})(\sqrt{3}-i\sqrt{5})}{\sqrt{2}-i\sqrt{5}}\right\}\right]},\\ &= \sqrt{\left[\frac{(2+3)(3+5)}{2+5}\right]}=\sqrt{\left(\frac{40}{7}\right)}.\end{aligned}$$

Example 3.

Find the modulus of $\{(\beta+\gamma)+(\beta-\gamma)i\}\{(\gamma+\alpha)+(\gamma-\alpha)i\}\{(\alpha+\beta)+(\alpha-\beta)i\}$.

The modulus is $\sqrt{(\{(\beta+\gamma)^2+(\beta-\gamma)^2\}\{(\gamma+\alpha)^2+(\gamma-\alpha)^2\}\{(\alpha+\beta)^2+(\alpha-\beta)^2\})}$ $=\sqrt{\{8(\beta^2+\gamma^2)(\gamma^2+\alpha^2)(\alpha^2+\beta^2)\}}$.

Example 4.

To represent $26\times20\times34$ as the sum of two integral squares.

Using the formula of § 10 we have

$$\begin{aligned}26\times20\times34 &= (1^2+5^2)(2^2+4^2)(3^2+5^2),\\ &= (1.2.3-1.4.5-2.5.5-3.5.4)^2+(5.2.3+4.3.1+5.1.2-5.4.5)^2,\\ &= 124^2+48^2.\end{aligned}$$

§ 11.] *The modulus of the sum of n complex numbers is never greater than the sum of their moduli, and is in general less.*

This may be established directly; but an intuitive proof will be obtained immediately from Argand's diagram.

§ 12.] We have seen already that, when $PQ=0$, then either $P=0$ or $Q=0$, provided P and Q be real quantities. It is natural now to inquire whether the same will hold if P and Q be complex numbers.

If P and Q be complex numbers then PQ is a complex number. Also, since $PQ=0$, by § 8, $|PQ|=0$. But $|PQ|$ $=|P|\times|Q|$, by § 10. Hence $|P|\times|Q|=0$. Now $|P|$ and

$|Q|$ are both real, hence either $|P| = 0$ or $|Q| = 0$. Hence, by § 8, either $P = 0$ or $Q = 0$.

We conclude, therefore, that *if* $PQ = 0$, *then either* $P = 0$ *or* $Q = 0$, *whether* P *and* Q *be real quantities or complex numbers.*

DISCUSSION OF COMPLEX NUMBERS BY MEANS OF ARGAND'S DIAGRAM.

§ 13.] Returning now to Argand's diagram, let us consider the complex number $x + yi$, which is represented by the radius vector OP (Fig. 1). Let OP, which is regarded as a signless magnitude, or, what comes to the same thing, as always having the positive sign, be denoted by r, and let the angle XOP, measured counter-clock-wise, be denoted by θ.

We have seen that if OP represent $x + yi$, then x and y are the projections of r on X'OX and Y'OY respectively. Hence we have, by the geometrical definitions of $\cos\theta$ and $\sin\theta$,

$$r = +\sqrt{(x^2 + y^2)} \qquad (1),$$

$$x/r = \cos\theta, \quad y/r = \sin\theta, \qquad (2).$$

From (1) it appears that r, that is OP, is the modulus of the complex number. The equations (2) uniquely determine the angle θ, provided we restrict it to be less than 2π, and agree that it is always to be measured counter-clock-wise from OX.* We call θ the *amplitude* of the complex number. It follows from (2) that every complex number can be expressed in terms of its modulus and amplitude; for we have

$$x + yi = r(\cos\theta + i\sin\theta) \qquad (3).$$

This new form, which we may call the *normal form*, possesses many important advantages.

* Sometimes it is more convenient to allow θ to increase from $-\pi$ to $+\pi$; that is, to suppose the radius OP to revolve counter-clock-wise from OX' to OX' again. In either way, the amplitude is uniquely determined when the coefficients x and y of the complex number are given, except in the case of a real negative number, where the amplitude apart from external considerations is obviously ambiguous.

Since two conjugate complex numbers differ only in the sign of the coefficient of i, it follows that the radii vectores which represent them are the images of each other in the axis of x (Fig. 3). Hence two such have the same modulus, as we have already shown analytically; and, if the amplitude of the one be θ, the amplitude of the other will be $2\pi - \theta$. In other words, the amplitudes of two conjugate complex numbers are conjugate angles.

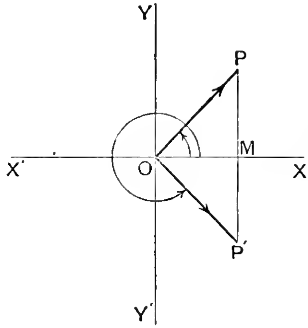


FIG. 3.

Example.

$$-1+i=\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right);$$
$$-1-i=\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right).$$

§ 14.] If OP, OQ′ (Fig. 4) represent the complex numbers $x+yi$ and $x'+y'i$, and if $\overrightarrow{PQ}$ be drawn parallel and equal to $\overrightarrow{OQ'}$, then OQ will represent the sum of $x+yi$ and $x'+y'i$. For the projection of $\overrightarrow{OQ}$ on the x-axis is the algebraic sum of the projections of $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ on the same axis, that is to say, the projection of $\overrightarrow{OQ}$ on the x-axis is $x+x'$. Also the projection of $\overrightarrow{OQ}$ on the y-axis is, by the same reasoning, $y+y'$. Hence OQ represents the complex number $(x+x')+(y+y')i$, which is equal to $(x+yi)+(x'+y'i)$.

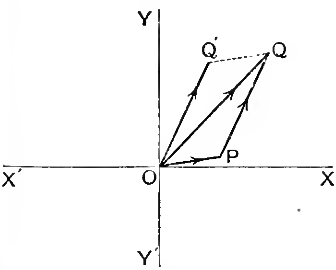


FIG. 4.

By similar reasoning we may show that if OP_1, OP_2, OP_3, OP_4, OP_5, say, represent five complex numbers, and if $\overrightarrow{P_1Q_2}$ be parallel

and equal to $\overrightarrow{OP_2}$, $\overrightarrow{Q_2Q_3}$, parallel and equal to $\overrightarrow{OP_3}$, and so on, then OQ_5 represents the complex number which is the sum of the complex numbers represented by OP_1, OP_2, OP_3, OP_4, OP_5.

This is precisely what is known as the polygon law for compounding vectors. Since OQ_5 is never greater than the perimeter $OP_1Q_2Q_3Q_4Q_5$, and is in general less, *Fig. 5 gives us an intuitive geometrical proof that the modulus of a sum of complex numbers is in general less than the sum of their moduli. It is equally obvious from Fig. 4 that the modulus of the sum of two complex numbers is in general greater than the difference of the moduli.*

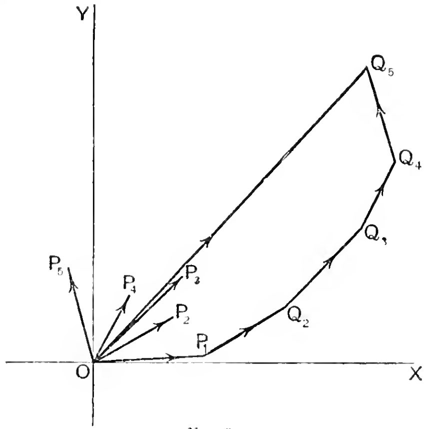


FIG. 5.

"Sum of complex numbers" in these theorems means, of course, algebraic sum.

§ 15.] If we employ the normal form for a complex number, and work out the product of two complex numbers, $r_1(\cos\theta_1+i\sin\theta_1)$ and $r_2(\cos\theta_2+i\sin\theta_2)$, we have

$$\begin{aligned}&r_1(\cos\theta_1+i\sin\theta_1)r_2(\cos\theta_2+i\sin\theta_2)\\&\quad=r_1r_2\{(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)i\},\\&\quad=r_1r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\} \qquad (1).\end{aligned}$$

We thus prove that the product of two complex numbers is a complex number, whose modulus r_1r_2 is the product of the moduli

of the two numbers, a result already established; and we have the new theorem that *the amplitude of the product is, to a multiple of* 2π, *the sum of the amplitudes of the factors.* For we can always find an angle ϕ lying between 0 and 2π, such that $\cos\phi = \cos(\theta_1+\theta_2)$ and $\sin\phi = \sin(\theta_1+\theta_2)$, and we then have $\theta_1+\theta_2 = 2n\pi+\phi$.

This last result is clearly general; for, if we multiply both sides of (1) by an additional factor, $r_3(\cos\theta_3 + i\sin\theta_3)$, we have

$$\begin{aligned} r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)r_3(\cos\theta_3 + i\sin\theta_3) \\ = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}\, r_3(\cos\theta_3 + i\sin\theta_3), \\ = r_1r_2r_3\{\cos(\theta_1+\theta_2+\theta_3) + i\sin(\overline{\theta_1+\theta_2}+\theta_3)\}, \end{aligned}$$

by the case already proved,

$$= r_1r_2r_3\{\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\}.$$

Proceeding in this way we ultimately prove that

$$\begin{aligned} r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)\ldots r_n(\cos\theta_n + i\sin\theta_n) \\ = r_1r_2\ldots r_n\{\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)\} \quad (2). \end{aligned}$$

This result may be expressed in words thus—

The product of n *complex numbers is a complex number whose modulus is the product of the moduli, and whose amplitude is, to a multiple of* 2π, *the sum of the amplitudes of the* n *complex numbers.*

If we put $r_1 = r_2 = \ldots = r_n$, each $= 1$ say, we have

$$\begin{aligned} (\cos\theta_1 + i\sin\theta_1)(\cos\theta_3 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) \\ = \cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n) \quad (3). \end{aligned}$$

This is the most general form of what is known as *Demoivre's Theorem.*

If we put $\theta_1 = \theta_2 = \ldots = \theta_n$, each $= \theta$, then (3) becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (4),$$

which is the usual form of Demoivre's Theorem.* It is an analytical result of the highest importance, as we shall see presently.

* This theorem was first given in Demoivre's *Miscellanea Analytica* (Lond. 1730), p. 1, in the form

$$x = \tfrac{1}{2}\sqrt[n]{\{l+\sqrt{(l^2-1)}\}} + \tfrac{1}{2}/\sqrt[n]{\{l+\sqrt{(l^2-1)}\}}, \text{ where } x=\cos\theta,\ l=\cos n\theta.$$

Since $\cos\theta - i\sin\theta = \cos(2\pi-\theta) + i\sin(2\pi-\theta)$,

we have, by (3) and (4),

$$\Pi(\cos\theta_1 - i\sin\theta_1) = \cos(\Sigma\theta_1) - i\sin(\Sigma\theta_1) \qquad (3');$$

and
$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta \qquad (4').$$

The theorem for a quotient corresponding to (1) may be obtained thus—

$$\frac{r(\cos\theta + i\sin\theta)}{r'(\cos\theta' + i\sin\theta')} = \frac{r(\cos\theta + i\sin\theta)(\cos\theta' - i\sin\theta')}{r'(\cos^2\theta' + \sin^2\theta')},$$

$$= \frac{r}{r'}\{(\cos\theta\cos\theta' + \sin\theta\sin\theta')$$

$$+ (\sin\theta\cos\theta' - \cos\theta\sin\theta')i\},$$

$$= \frac{r}{r'}\{\cos(\theta-\theta') + i\sin(\theta-\theta')\} \qquad (5).$$

Hence *the quotient of two complex numbers is a complex number whose modulus is the quotient of the moduli, and whose amplitude is to a multiple of 2π the difference of the amplitude of the two complex numbers.*

IRRATIONAL OPERATIONS WITH COMPLEX NUMBERS.

§ 16.] Since every irrational algebraical function involves only root extraction in addition to the four rational operations, and since we have shown that rational operations with complex numbers reproduce complex numbers and such only, if we could prove that the nth root of a complex number has for its value, or values, a complex number, or complex numbers and such only, then we should have established that all algebraical operations with complex numbers reproduce complex numbers and such only.

The chief means of arriving at this result is Demoivre's Theorem; but, before resorting to this powerful analytical engine, we shall show how to treat the particular case of the square root without its aid.

Let us suppose that

$$\sqrt{(x + yi)} = X + Yi \qquad (1).$$

Then $$x + yi = X^2 - Y^2 + 2XYi.$$
Hence, since X and Y are real, we must have, by § 3,
$$X^2 - Y^2 = x \qquad (2),$$
$$2XY = y \qquad (3).$$
Squaring both sides of (2) and (3), and adding, we deduce
$$(X^2 + Y^2)^2 = x^2 + y^2;$$
whence, since $X^2 + Y^2$ is necessarily positive, we deduce
$$X^2 + Y^2 = +\sqrt{(x^2 + y^2)} \qquad (4).$$
From (2) and (4), by addition, we derive
$$2X^2 = +\sqrt{(x^2 + y^2)} + x,$$
that is, $$X^2 = \frac{+\sqrt{(x^2 + y^2)} + x}{2}.$$
We therefore have $$X = \pm\sqrt{\left\{\frac{+\sqrt{(x^2 + y^2)} + x}{2}\right\}} \qquad (5).$$
In like manner we derive from (2) and (4), by subtraction, &c.,
$$Y = \pm\sqrt{\left\{\frac{+\sqrt{(x^2 + y^2)} - x}{2}\right\}} \qquad (6).$$

Since $x^2 + y^2$ is numerically greater than x^2, $+\sqrt{(x^2 + y^2)}$ is numerically greater than x. Hence the quantities under the sign of the square root in (5) and (6) are both real and positive. The values of X and Y assigned by these equations are therefore real.

Since $2XY = y$, like signs must be taken in (5) and (6), or unlike signs, according as y is positive or negative.

We thus have finally
$$\sqrt{(x + yi)} = \pm\left\{\sqrt{\left(\frac{\sqrt{(x^2 + y^2)} + x}{2}\right)} + i\sqrt{\left(\frac{\sqrt{(x^2 + y^2)} - x}{2}\right)}\right\} \qquad (7),$$
if y be positive;
$$= \pm\left\{\sqrt{\left(\frac{\sqrt{(x^2 + y^2)} + x}{2}\right)} - i\sqrt{\left(\frac{\sqrt{(x^2 + y^2)} - x}{2}\right)}\right\} \qquad (8),$$
if y be negative.

Example 1.

Express $\sqrt{(8 + 6i)}$ as a complex number.

Let $$\sqrt{(8 + 6i)} = x + yi.$$
Then $$x^2 - y^2 = 8, \quad 2xy = 6.$$
Hence $$(x^2 + y^2)^2 = 64 + 36 = 100$$
Hence $$x^2 + y^2 = 10;$$
and $$x^2 - y^2 = 8;$$

therefore $$2x^2=18, \quad 2y^2=2.$$
Hence $$x=\pm 3, \quad y=\pm 1.$$

Since $2xy=6$, we must have either $x=+3$ and $y=+1$, or $x=-3$ and $y=-1$.

Finally, therefore, we have

$$\sqrt{(8+6i)}=\pm(3+i);$$

the correctness of which can be immediately verified by squaring.

Example 2.

$$\sqrt{(3-7i)}=\pm\left\{\sqrt{\left(\frac{\sqrt{(58)}+3}{2}\right)}-i\sqrt{\left(\frac{\sqrt{(58)}-3}{2}\right)}\right\}.$$

Example 3.

Express $\sqrt{(+i)}$ and $\sqrt{(-i)}$ as complex numbers.

Let $$\sqrt{(+i)}=x+yi;$$
then $$i=x^2-y^2+2xyi.$$
Hence $$x^2-y^2=0 \quad (\alpha), \qquad 2xy=1 \quad (\beta).$$

From (α) we have $(x+y)(x-y)=0$; that is, either $y=-x$ or $y=x$. The former alternative is inconsistent with (β); hence the latter must be accepted. We then have, from (β), $2x^2=1$, whence $x^2=1/2$ and $x=\pm 1/\sqrt{2}$. Since $y=x$, we have, finally,

$$\sqrt{+i}=\pm\frac{1+i}{\sqrt{2}} \qquad (\gamma).$$

Similarly we show that

$$\sqrt{-i}=\pm\frac{1-i}{\sqrt{2}} \qquad (\delta).$$

Example 4.

To express the 4th roots of $+1$ and -1 as complex numbers.

$$\sqrt[4]{+1}=\sqrt{(\sqrt{+1})}=\sqrt{\pm 1}=\sqrt{+1} \text{ or } \sqrt{-1}=\pm 1 \text{ or } \pm i.$$

Hence we obtain four 4th roots of $+1$, namely, $+1$, -1, $+i$, $-i$.

Again $$\sqrt[4]{-1}=\sqrt{(\sqrt{-1})}=\sqrt{\pm i}.$$

Hence, by Example 3,

$$\sqrt[4]{-1}=\pm\frac{1+i}{\sqrt{2}}, \text{ or } \pm\frac{1-i}{\sqrt{2}}.$$

§ 17.] We now proceed to the general case of the nth root of any complex number, $r(\cos\theta+i\sin\theta)$.

Since r is a positive number, $\sqrt[n]{r}$ has (see chap. x., § 2) one real positive value, which we may denote by $r^{1/n}$.

Consider the n complex numbers—

$$r^{1/n}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right) \qquad (1),$$

$$r^{1/n}\left(\cos\frac{2\pi+\theta}{n}+i\sin\frac{2\pi+\theta}{n}\right) \qquad (2),$$

$$r^{1/n}\left(\cos\frac{4\pi+\theta}{n}+i\sin\frac{4\pi+\theta}{n}\right) \qquad (3),$$

. . . .

$$r^{1/n}\left(\cos\frac{2s\pi+\theta}{n}+i\sin\frac{2s\pi+\theta}{n}\right) \qquad (s+1),$$

. . . .

$$r^{1/n}\left(\cos\frac{2(n-1)\pi+\theta}{n}+i\sin\frac{2(n-1)\pi+\theta}{n}\right) \qquad (n).$$

No two of these are equal, since the amplitudes of any two differ by less than 2π. The nth power of any one of them is $r(\cos\theta+i\sin\theta)$; for take the $(s+1)$th, for example, and we have

$$\left\{r^{1/n}\left(\cos\frac{2s\pi+\theta}{n}+i\sin\frac{2s\pi+\theta}{n}\right)\right\}^n$$

$$=\left(r^{1/n}\right)^n\left(\cos\frac{2s\pi+\theta}{n}+i\sin\frac{2s\pi+\theta}{n}\right)^n,$$

$$=r\left(\cos n\frac{2s\pi+\theta}{n}+i\sin n\frac{2s\pi+\theta}{n}\right),$$

by Demoivre's Theorem,

$$=r(\cos(2s\pi+\theta)+i\sin(2s\pi+\theta)),$$

$$=r(\cos\theta+i\sin\theta).$$

Hence the complex numbers (1), (2), . . ., (n) are n different nth roots of $r(\cos\theta+i\sin\theta)$.

We cannot, by giving values to s exceeding $n-1$, obtain any new values of the nth root, for the values of the series (1), (2), . . ., (n) repeat, owing to the periodicity of the trigonometrical functions involved. We have, for example, $r^{1/n}(\cos.(2n\pi+\theta)/n+i\sin.(2n\pi+\theta)/n)=r^{1/n}(\cos.\theta/n+i\sin.\theta/n)$; and so on.

We can, in fact, prove that there cannot be more than n values of the nth root. Let us denote the complex number $r(\cos\theta+i\sin\theta)$ by a, for shortness; and let z stand for any nth root of a. Then must $z^n=a$, and therefore $z^n-a=0$. Hence every nth root of a, when substituted for z in z^n-a, causes this integral function of z to vanish. Hence, if z_1, z_2, . . .,

z_s be s nth roots of a, $z - z_1$, $z - z_2$, . . ., $z - z_s$ will all be factors of $z^n - a$. Now $z^n - a$ is of the nth degree in z, and cannot have more than n factors (see chap. v., § 16). Hence s cannot exceed n; that is to say, there cannot be more than n nth roots of a.

We conclude therefore that *every complex number has n nth roots and no more; and each of these nth roots can be expressed as a complex number.*

Cor. 1. Since every real number is merely a complex number whose imaginary part vanishes, it follows that *every real number, whether positive or negative, has n nth roots and no more, each of which is expressible as a complex number.*

Cor. 2. *The imaginary nth roots of any real number can be arranged in conjugate pairs.* For we have seen that, if $x + yi$ be any nth root of a, then $(x + yi)^n - a = 0$. Hence, if a be real (but not otherwise), it follows, by § 5, Cor. 4, that $(x - yi)^n - a = 0$; that is, $x - yi$ is also an nth root of a.

N.B.—This does not hold for the roots of a *complex* number generally.

§ 18.] Every real positive quantity can be written in the form

$$r(\cos 0 + i \sin 0) \qquad (A);$$

and every real negative quantity in the form

$$r(\cos \pi + i \sin \pi) \qquad (B);$$

where r is a real positive quantity. Hence, if we know the n nth roots of $\cos 0 + i \sin 0$, that is, of $+1$, and the n nth roots of $\cos \pi + i \sin \pi$, that is, of -1, the problem of finding the n nth roots of any real quantity, whether positive or negative, is reduced to finding the real positive value of the nth root of a real positive quantity r (see chap. xi., § 15).

By means of the nth roots of ± 1 we can, therefore, completely fill the lacuna left in chap. x., § 2. In addition to their use in this respect, the nth roots of ± 1 play an exceedingly im-

portant part in the theory of equations, and in higher algebra generally. We therefore give their fundamental properties here, leaving the student to extend his knowledge of this part of algebra as he finds need for it.

Putting $r=1$ and $\theta=0$ in 1, . . ., n of § 17, and remembering that $1^{1/n}=1$, we obtain for the n nth roots of $+1$,

$$\cos 0+i\sin 0,\quad \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},\quad \ldots\ldots,$$
$$\cos\frac{2(n-1)\pi}{n}+i\sin\frac{2(n-1)\pi}{n}.$$

Putting $r=1$, $\theta=\pi$, we obtain for the nth roots of -1,

$$\cos\frac{\pi}{n}+i\sin\frac{\pi}{n},\quad \cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n},\quad \ldots\ldots,$$
$$\cos\frac{(2n-1)\pi}{n}+i\sin\frac{(2n-1)\pi}{n}.$$

Cor. 1. Since $\cos.2(n-1)\pi/n=\cos.2\pi/n$, $\sin.2(n-1)\pi/n=-\sin.2\pi/n$; $\cos.2(n-2)\pi/n=\cos.4\pi/n$, $\sin.2(n-2)\pi/n=-\sin.4\pi/n$, and so on, *we can arrange the roots of* $+1$ *as follows*:—

nth roots of $+1$, n even, $=2m$ say,

$$+1,\ \cos\frac{2\pi}{n}\pm i\sin\frac{2\pi}{n},\quad \cos\frac{4\pi}{n}\pm i\sin\frac{4\pi}{n},\ \ldots,$$
$$\cos\frac{2(m-1)\pi}{n}\pm i\sin\frac{2(m-1)\pi}{n},\ -1 \qquad (C);$$

nth roots of $+1$, n odd, $=2m+1$ say,

$$+1,\ \cos\frac{2\pi}{n}\pm i\sin\frac{2\pi}{n},\quad \cos\frac{4\pi}{n}\pm i\sin\frac{4\pi}{n},\ \ldots,$$
$$\cos\frac{2m\pi}{n}\pm i\sin\frac{2m\pi}{n} \qquad (D).$$

Similarly *we can arrange the roots of* -1 *as follows* :—

nth roots of -1, n even, $= 2m$ say,

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\quad \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,$$
$$\cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n} \qquad \text{(E)};$$

nth roots of -1, n odd, $= 2m+1$ say,

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\quad \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,$$
$$\cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n},\ -1 \qquad \text{(F)}.$$

From (C), (D), (E), (F) we see, in accordance with chap. x., § 2, that the nth root of $+1$ has one real value if n be odd, and two real values if n be even; and that the nth root of -1 has one real value if n be odd, and no real value if n be even.

We have also a verification of the theorem of § 17, Cor. 2, that the imaginary roots of a real quantity consist of a set of pairs of conjugate complex numbers.

Cor. 2. The first of the imaginary roots of $+1$ in the series (1), . . ., (n), namely, $\cos.2\pi/n + i\sin.2\pi/n$, is called a *primitive** nth root of $+1$. Let us denote this root by ω.

Then since, by Demoivre's Theorem,

$$\omega^s = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^s = \cos\frac{2s\pi}{n} + i\sin\frac{2s\pi}{n},$$

and, in particular,

$$\omega^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos 2\pi + i\sin 2\pi,$$
$$= 1,$$

* By a primitive imaginary nth root of $+1$ in general is meant an nth root which is not also a root of lower order. For example, $\cos.2\pi/3 + i\sin.2\pi/3$ is a 6th root of $+1$, but it is also a cube root of $+1$, therefore $\cos.2\pi/3 + i\sin.2\pi/3$ is not a primitive 6th root of $+1$. It is obvious that $\cos.2\pi/n + i\sin.2\pi/n$ is a primitive nth root; but there are in general others, and it may be shown that any one of these has the property of Cor. 2.

we see that, *if ω be a primitive imaginary nth root of* $+1$, *then the n nth roots of* $+1$ *are*

$$\omega^1,\ \omega^2,\ \omega^3,\ \dots,\ \omega^n \qquad (G).$$

Similarly, *if* $\omega' = \cos.\pi/n + i\sin.\pi/n$, *which we may call a primitive imaginary nth root of* -1, *then the n nth roots of* -1 *are*

$$\omega'^1,\ \omega'^3,\ \omega'^5,\ \dots,\ \omega'^{2n-1} \qquad (H).$$

§ 19.] The results of last paragraph, taken in conjunction with the remainder theorem (see chap. v., § 15), show that

*Every binomial integral function, $x^n \pm A$, can be resolved into n factors of the 1st degree, whose coefficients may or may not be wholly real, or into at most two real factors of the 1st degree, and a number of real factors of the 2nd degree.**

Take, for example, $x^{2m} - a^{2m}$. This function vanishes whenever we substitute for x any $2m$th root of a^{2m}; that is, it vanishes whenever x has any of the values $a\omega$, $a\omega^2$, . . ., $a\omega^{2m}$ where ω stands for a primitive $2m$th root of $+1$.

Hence the resolution into linear factors is given by

$$x^{2m} - a^{2m} \equiv (x - a\omega)(x - a\omega^2)\dots(x - a\omega^{2m}).$$

To obtain the resolution into real factors, we observe that, corresponding to the roots $+a$ and $-a$, we have the factors $x-a$, $x+a$; and that, corresponding to the roots $a(\cos.s\pi/m \pm i\sin.s\pi/m)$, we have the factors

$$\left(x - a\cos\frac{s\pi}{m} - ai\sin\frac{s\pi}{m}\right)\left(x - a\cos\frac{s\pi}{m} + ai\sin\frac{s\pi}{m}\right),$$

$$\equiv \left(x - a\cos\frac{s\pi}{m}\right)^2 + a^2\sin^2\frac{s\pi}{m},$$

$$\equiv x^2 - 2ax\cos\frac{s\pi}{m} + a^2.$$

Hence the resolution into real factors is given by

$$x^{2m} - a^{2m} \equiv (x-a)(x+a)\left(x^2 - 2ax\cos\frac{\pi}{m} + a^2\right)\left(x^2 - 2ax\cos\frac{2\pi}{m} + a^2\right)\dots$$

We may treat $x^{2m} + a^{2m}$, $x^{2m+1} - a^{2m+1}$, and $x^{2m+1} + a^{2m+1}$ in a similar way.

Example 1.

To find the cube roots of $+1$ and -1. We have $+1 = 1\{\cos 0 + i\sin 0\}$. Hence the cube roots of $+1$ are

$$\cos 0 + i\sin 0, \quad \cos.2\pi/3 \pm i\sin.2\pi/3,$$

that is to say, $\qquad +1,\ -1/2 \pm i\sqrt{3}/2.$

---

* The solution of this problem was first found in a geometrical form by Cotes; it was published without demonstration in the *Harmonia Mensurarum* (1722), p. 113. Demoivre (*Misc. Anal.*, p. 17) gave a demonstration, and also found the real quadratic factors of the trinomial $1 + 2\cos\theta x^n + x^{2n}$.

Again $-1=1\{\cos\pi+i\sin\pi\}$. Hence the cube roots of -1 are

$$\cos.\pi/3\pm i\sin.\pi/3, \quad \cos\pi+i\sin\pi,$$

that is to say, $$1/2\pm i\sqrt{3}/2, \ -1.$$

Example 2.

To find the cube roots of $1+i$. We have $1+i=\sqrt{2}(1/\sqrt{2}+i1/\sqrt{2})$ $=\sqrt{2}(\cos 45^\circ+i\sin 45^\circ)$. Hence the cube roots of $1+i$ are

$$2^{\frac{1}{6}}(\cos 15^\circ+i\sin 15^\circ), \quad 2^{\frac{1}{6}}(\cos 135^\circ+i\sin 135^\circ), \quad 2^{\frac{1}{6}}(\cos 255^\circ+i\sin 255^\circ),$$

that is,

$$2^{\frac{1}{6}}(\cos 15^\circ+i\sin 15^\circ), \quad 2^{\frac{1}{6}}(-\cos 45^\circ+i\sin 45^\circ), \quad 2^{\frac{1}{6}}(-\cos 75^\circ-i\sin 75^\circ),$$

that is,

$$2^{\frac{1}{6}}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+i\frac{\sqrt{3}-1}{2\sqrt{2}}\right), \quad 2^{\frac{1}{6}}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right), \quad 2^{\frac{1}{6}}\left(-\frac{\sqrt{3}-1}{2\sqrt{2}}-i\frac{\sqrt{3}+1}{2\sqrt{2}}\right),$$

that is, $$\frac{(\sqrt{3}+1)+(\sqrt{3}-1)i}{2^{\frac{4}{3}}}, \quad \frac{-1+i}{2^{\frac{1}{3}}}, \quad -\frac{(\sqrt{3}-1)+(\sqrt{3}+1)i}{2^{\frac{4}{3}}}.$$

Here it will be observed that the roots are not arranged in conjugate pairs, as they would necessarily have been had the radicand been real.

Example 3.

To find approximately one of the imaginary 7th roots of $+1$. One of the imaginary roots is

$$\cos 51^\circ 25' 43''+i\sin 51^\circ 25' 43''.$$

By the table of natural sines and cosines, this gives

$$\cdot 6234893+\cdot 7818318i$$

as one approximate value for the 7th root of $+1$.

Example 4.

If ω be one of the imaginary cube roots of $+1$, to show that $1+\omega+\omega^2=0$, and that $(\omega x+\omega^2 y)(\omega^2 x+\omega y)$ is real.

We have $1+\omega+\omega^2=(1-\omega^3)/(1-\omega)=0$, since $\omega^3=1$ and $1-\omega\neq 0$.

Again,

$$(\omega x+\omega^2 y)(\omega^2 x+\omega y)=\omega^3 x^2+(\omega^4+\omega^2)xy+\omega^3 y^2.$$

Now $\omega^3=1$; and $\omega^4+\omega^2=\omega^3\omega+\omega^2=\omega+\omega^2=-1$, since $1+\omega+\omega^2=0$.

Hence

$$(\omega x+\omega^2 y)(\omega^2 x+\omega y)=x^2-xy+y^2.$$

FUNDAMENTAL PROPOSITION IN THE THEORY OF EQUATIONS.

§ 20.] *If* $f(z)\equiv A_0+A_1 z+A_2 z^2+\ldots+A_n z^n$ *be an integral function of* z *of the nth degree, whose coefficients* $A_0, A_1, \ldots, A_n$ *are given complex numbers, or, in particular, real numbers, where, of course,* $A_n\neq 0$, *then* $f(z)$ *can always be expressed as the product of* n *factors, each of the 1st degree in* z, *say* $z-z_1$, $z-z_2$, $z-z_3$, $\ldots$, $z-z_n$, $z_1, z_2, \ldots, z_n$ *being in general complex numbers.*

It is obvious that this proposition can be deduced from the following subsidiary theorem :—

One value of z, in general a complex number, can always be found which causes $f(z)$ to vanish.

For, let us suppose that $f(z_1)=0$, then, by the remainder theorem, $f(z)\equiv f_1(z)(z-z_1)$, where $f_1(z)$ is an integral function of z of the $(n-1)$th degree. Now, by our theorem, one value of z at least, say z_2, can be found for which $f_1(z)$ vanishes. We have, therefore, $f_1(z_2)=0$; and therefore $f_1(z)=f_2(z)(z-z_2)$, where $f_2(z)$ is now of the $(n-2)$th degree; and so on. Hence we prove finally that

$$f(z)\equiv A(z-z_1)(z-z_2)\ldots(z-z_n),$$

where A is a constant.

§ 21.] We shall now prove that there is always at least one finite value of z, say $z=a$, such that by taking z sufficiently near to a, that is by making $|z-a|$ small enough, we can make $|f(z)|$ as small as we please. So that in this sense every integral equation $f(z)=0$ has at least one finite root.

Let $|z|=R$. Then, since

$$|f(z)|=|A_n|R^n|1+A_{n-1}/A_nz+\quad\ldots\quad+A_0/A_nz^n|,$$

we have, by § 14,

$$|f(z)|>|A_n|R^n\{1-|A_{n-1}/A_nz+\quad\ldots+A_0/A_nz^n|\},$$

provided $|z|$, (*i.e.* R), be large enough; therefore

$$|f(z)|>|A_n|R^n\{1-C(1/R+\quad\ldots\quad+1/R^n)\},$$

where C is the greatest of $|A_{n-1}/A_n|,\ \ldots,\ |A_0/A_n|$. Therefore, taking provisionally $R>1$, we have

$$\begin{aligned}|f(z)|&>|A_n|R^n\{1-C(1-1/R^n)/R(1-1/R)\},\\&>|A_n|R^n\{1-C/(R-1)\}\end{aligned}\qquad(1),$$

provided $R>C+1$.

Hence, by taking $|z|$ sufficiently large, we can make $|f(z)|$ as large as we please; and we also see that there can be no root of $f(z)=0$ whose modulus exceeds $C+1$.

Let now w be the value of z at any finite point in the Argand Plane, so that $|f(w)|$ is finite. It follows from what has

just been proved that we can describe about the origin a circle S of finite radius, such that, at all points on and outside S, $|f(z)| > |f(w)|$. Then, since $|f(w)|$ is real and positive, if we consider all points within S, we see that there must be a finite lower limit L to the value of $|f(w)|$; that is to say, a quantity L which is not greater than any of the values of $|f(w)|$ within S, and such that by properly choosing u we can make $|f(w)| = \mathrm{L} + \epsilon$, where ϵ is a real positive quantity as small as we please.

We shall show that L must be zero. For, suppose $\mathrm{L} > 0$, and choose w so that $|f(w)| = \mathrm{L} + \epsilon$. Let h be a complex number, say $r(\cos\theta + i\sin\theta)$. Then

$$\begin{aligned} f(w+h) &= \mathrm{A}_0 + \mathrm{A}\,(w+h) + \quad . \; . \; . + \mathrm{A}_n(w+h)^n, \\ &= f(w) + \mathrm{B}_1 h + \mathrm{B}_2 h^2 + \quad . \; . \; . + \mathrm{A}_n h^n \qquad (2), \end{aligned}$$

where A_n is independent both of w and h, and by hypothesis cannot vanish, but $\mathrm{B}_1, \ldots, \mathrm{B}_{n-1}$ are functions of w, one or more of which may vanish. Suppose that B_m is the first of the B's that does not vanish, and let $b_m(\cos a_m + i\sin a_m)$, etc., be the normal forms of the complex numbers $\mathrm{B}_m/f(w)$, etc. Then, since $|f(w)|$ is not zero, b_m, etc., are all finite. Also we have, by Demoivre's Theorem,

$$f(w+h)/f(w) = 1 + b_m r^m \Theta_m + b_{m+1} r^{m+1} \Theta_{m+1} + \; . \; . \; . + b_n r^n \Theta_n \quad (3),$$

where $\Theta_m = \cos(m\theta + a_m) + i\sin(m\theta + a_m)$, etc.

We have h, and therefore both r and θ at our disposal. Let us first determine θ so that $\cos(m\theta + a_m) = -1$, $\sin(m\theta + a_m) = 0$; that is, give θ any one of the m values $(\pi - a_m)/m$, $(3\pi - a_m)/m, \ldots$, $(\overline{2m-1}\,.\,\pi - a_m)/m$, say the first. Then we have $\Theta_m = -1$; and Θ_{m+1}, etc., assume definite values, say, Θ'_{m+1}, etc. We now have

$$f(w+h)/f(w) = 1 - b_m r^m + b_{m+1} r^{m+1} \Theta'_{m+1} + \; . \; . \; . + b_n r^n \Theta'_n \quad (4).$$

Considering the right hand side of (4) as the sum of $1 - b_m r^m$ and $b_{m+1} r^{m+1} \Theta'_{m+1} + \; . \; . \; . + b_n r^n \Theta'_n$, we see, by § 14, that the modulus of $f(w+h)/f(w)$ lies between the difference and the sum of the moduli of these two. Also

$$|b_{m+1}r^{m+1}\Theta'_{m+1} + \ldots + b_n\Theta'_n r^n| \leqq b_{m+1}r^{m+1} + \ldots + b_n r^n,$$
$$\leqq b(r^{m+1} + \ldots + r^n),$$

where b is the greatest of $b_{m+1}, \ldots, b_n$,

$$\leqq (n-m)br^{m+1},$$

provided we take $r < 1$, so that $r^{m+1} > r^{m+2} > \ldots > r^n$.
Therefore we have

$$1 - b_m r^m - (n-m)br^{m+1} \leqq |f(w+h)/f(w)|$$
$$\leqq 1 - b_m r^m + (n-m)br^{m+1} \quad (5);$$

provided r be so chosen that $1 - b_m r^m$ and $1 - b_m r^m - (n-m)br^{m+1}$ are both positive. Let us further choose r so that $(n-m)br^{m+1} < b_m r^m$. All these conditions will obviously be satisfied if we give a finite value to r less than the least of the three,

$$1, \quad 1/(2b_m)^{1/m}, \quad b_m/(n-m)b. \quad (6).$$

When r is thus chosen, $|f(w+h)/f(w)|$ will lie between two positive proper fractions, so that $|f(w+h)/f(w)| = 1 - \mu$, where μ is a positive proper fraction; and we have

$$|f(w+h)| = (1-\mu)|f(w)| = (1-\mu)(\mathrm{L}+\epsilon) = \mathrm{L} + \epsilon - \mu(\mathrm{L}+\epsilon) \quad (7).$$

which, since ϵ may be as small as we please, is less than L by a finite amount. L is therefore not a finite lower limit as supposed; in other words, L must be zero. Our fundamental theorem is thus established.

By reasoning as above we can easily show that, if

$$f(z) \equiv A_0 + A_s z^s + \ldots + A_n z^n,$$

then

$$|\mathrm{A}_0|\{1 - (n-s+1)d|z|^s\} \leqq |f(z)|$$
$$\leqq |\mathrm{A}_0|\{1 + (n-s+1)d|z|^s\} \quad (8),$$

where d is the greatest of $|\mathrm{A}_s/\mathrm{A}_0|, \ldots, |\mathrm{A}_n/\mathrm{A}_0|$, provided $|z|$ is less than the lesser of the two quantities $1, 1/\{(n-s+1)d\}^{1/s}$.

Combining this result with one obtained incidentally above, we have the following useful theorem on the delimitation of the roots (real or imaginary) of an equation.

Cor. 1. *The equation* $\mathrm{A}_0 + \mathrm{A}_s z^s + \ldots + \mathrm{A}_n z^n = 0$ *can have no root whose modulus exceeds the greatest of the quantities* $1 + |\mathrm{A}_0/\mathrm{A}_n|$,

$1+|A_s/A_n|, \ldots, 1+|A_{n-1}/A_n|$, *or whose modulus is less than the least of* $1, 1/\{(n-s+1)|A_s/A_0|\}^{1/s}, \ldots, 1/\{(n-s+1)|A_n/A_0|\}^{1/s}$.

Cor. 2. *We can always assign a positive quantity* η, *such that, if* $|h|<\eta$, $|f(z+h)-f(z)|<\epsilon$, *where* ϵ *is a positive quantity as small as we please.*

This is expressed by saying that the integral function $f(z)$ is continuous for all complex values of its argument which have a finite modulus. The proof is obvious after what has already been done.

The above demonstration is merely a version of the proof given by Argand in his famous *Essai*,* amplified to meet some criticisms on the briefer statement in earlier editions of this work. The criticisms in question touch the formulation, but not the essential principle of Argand's proof, which is both ingenious and profound. As some of the critics appear to me to have missed the real point involved, perhaps the following remarks, which the student will appreciate more fully after reading chap. xv. §§ 17-19, and chap. xxix., may be useful.

Taking any value of $z=x+yi$, let $f(z)=u+vi$, where u and v are real functions of the real variables x and y. Plot u- v- and x- y- Argand diagrams. Then to each point (x, y) there corresponds one point (u, v), although it may happen that to one (u, v) point there correspond more than one (x, y) point. If we start with any given point (x, y), and consider $|f(z)|=\sqrt{(u^2+v^2)}$, it is obvious that the direction in which $|f(z)|$ varies most rapidly is obtained by causing (x, y) to move in the x- y- plane, so that (u, v) moves along the radius vector towards the origin in the u- v- plane; also that the rate of variation in the perpendicular direction is zero. If, therefore, we trace one of the curves $v/u=$constant in the x- y- plane, the tangent to this curve at every point z on it is the direction of most rapid variation. We may call these curves the *Gradient Curves* of $f(z)$.†

---

* See in particular his amplified demonstration given in a note in Gergonne's *Ann. de Math.* t. v. pp. 197-209 (1814-15).

† These curves, together with the curves $u^2+v^2=$constant, which we may call the *Equimodular Curves* of $f(z)$, possess a number of interesting properties. Since the Equimodulars and Gradients are orthomorphosed (see chap. xxix. § 36) from a series of concentic circles and their pencil of common radii, they form two mutually orthogonal systems. Through every given point (x', y'), which is not the affixe of a root of $f(z)=0$, there passes one equimodular $u^2+v^2=u'^2+v'^2$ and one gradient $u'v-v'u=0$. Every gradient $u'v-v'u=0$ passes through all the intersections of $u=0$, $v=0$, *i.e.* through the affixes of all the roots of $f(z)=0$. Near the root points the equimodulars take the form of small ovals enclosing these points.

It may readily be shown that the process by which we pass from w to $w+h$ ($=w_1$, say) in the above demonstration simply amounts to passing a certain distance along a tangent to the gradient curve through the point w. We may repeat this process, starting from w_1 and passing along the tangent to the gradient through w_1, and so on. We shall thus have $f(w)>f(w_1)>f(w_2)> \ldots$ This process we may call *Argand's Progression towards the root of an equation.*

Since $b_n < b$, $b_m r^m + (n-m)br^{m+1} < 1$, and $(n-m)br^{m+1} < b_m r^m$, it follows that

$$r < \{|f(w)|/2(n-m)|A_n|\}^{1/(m+1)} \text{ and } r < |B_m|/(n-m)|A_n|.$$

The first of these conditions shows that the longest admissible steps of Argand's Progression become smaller and smaller as we approach a root. This is expressed by saying that the Progression becomes asymptotic as we approach a root.

But the second condition shows that Progression also becomes asymptotic as we approach a point at which $B_1=0$; or $B_1=0$, $B_2=0$; or $B_1=0$, $B_2=0$, $B_3=0$, etc. Such points are stationary points for the variation of $|f(z)|$ on any path which passes through them. They are also multiple points on the gradient curves which pass through them; so that at them we have 2, 3, 4, etc., directions of most rapid variation of $|f(z)|$.

If the original progression leads towards one of these points (for which of course $|f(w)| \neq 0$), we must infer its existence from the asymptotic approach, and start afresh from that point along one of the tangents to the gradient that passes through it; or we may avoid the point altogether by starting afresh on a path which does not lead to it.

An interesting example is to take $z^2-z+1=0$, and start from a point on the real axis in the x- y- diagram. Argand's Progression will lead first to the minimum point (1/2, 0) on the x- axis; then along a line parallel to the y- axis to the points, $(1/2, \pm\sqrt{3}/2)$, which are the affixes of the two imaginary roots. The diagram of chap. XV. § 19 will also furnish a curious illustration by taking initial points on one or other of the two dotted lines.

It should be noted that the question as to whether $|f(z)|$ actually reaches its lower limit is not essential in Argand's proof, if we merely propose to show that a value of z can be found such that $|f(z)|$ is less than any assigned positive quantity, however small. Nor do we raise the question whether the root is rational or irrational, which would involve the subtle question of the ultimate logical definition of an irrational number (see vol. ii. (ed. 1900) chap. XXV. §§ 28-41).

§ 22.] We have now shown that in all cases

$$f(z) \equiv A(z-z_1)(z-z_2)\ldots(z-z_n),$$

where A is a constant.

$z_1, z_2, \ldots, z_n$ may be real, or they may be complex numbers of the general form $x+yi$. They may be all different, or two

or more of them may be identical, as may be easily seen by considering the above demonstration.

The general proposition thus established is equivalent to the following:—

If $f(z)$ be an integral function of z of the nth degree, there are n values of z for which $f(z)$ vanishes. These values may be real or complex numbers, and may or may not be all unequal.

We have already seen in chap. v., § 16, that there cannot be more than n values of z for which $f(z)$ vanishes, otherwise all its coefficients would vanish, that is, the function would vanish for all values of z. We have also seen that the constant A is equal to the coefficient A_n. We have therefore the unique resolution

$$f(z) \equiv A_n(z - z_1)(z - z_2) \ldots (z - z_n).$$

§ 23.] If the coefficients of $f(z)$ be all real, then we have seen that if $f(x + yi)$ vanish $f(x - yi)$ will also vanish. In this case the imaginary values among $z_1, z_2, \ldots, z_n$ will occur in conjugate pairs.

If $\alpha + \beta i$, $\alpha - \beta i$ be such a conjugate pair, then, corresponding to them, we have the factor

$$(z - \alpha - \beta i)(z - \alpha + \beta i) = (z - \alpha)^2 + \beta^2,$$

that is to say, a real factor of the 2nd degree.

It may of course happen that the conjugate pair $\alpha \pm \beta i$ is repeated, say s times, among the values $z_1, z_2, \ldots, z_n$. In that case we should have the factor $(z - \alpha)^2 + \beta^2$ repeated s times; so that there would be a factor $\{(z - \alpha)^2 + \beta^2\}^s$ in the function $f(z)$.

Hence, *every integral function of z, whose coefficients are all real, can be resolved into a product of real factors, each of which is either a positive integral power of a real integral function of the 1st degree, or a positive integral power of a real integral function of the 2nd degree.*

This is the general proposition of which the theorem of § 19 is a particular case.

Exercises XVI.

Express as complex numbers—

(1.) $(a+bi)^8+(a-bi)^8$.

(2.) $\dfrac{1+i}{1+2i}+\dfrac{1-i}{1-2i}$.

(3.) $\dfrac{2+36i}{6+8i}+\dfrac{7-26i}{3-4i}$.

(4.) $\left(\dfrac{p+qi}{p-qi}\right)^2-\left(\dfrac{p-qi}{p+qi}\right)^2$.

(5.) $\dfrac{69-7\sqrt{(15)}+(\sqrt{3}-6\sqrt{5})i}{3-(\sqrt{3}-3\sqrt{5})i}$.

(6.) Show that

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n=-1,$$

if n be any integer which is not a multiple of 3.

(7.) Expand and arrange according to the powers of x

$$(x-1-i\sqrt{2})\,(x-1+i\sqrt{2})\,(x-2+i\sqrt{3})\,(x-2-i\sqrt{3}).$$

(8.) Show that

$$\{(2a-b-c)+i(b-c)\sqrt{3}\}^3=\{(2b-c-a)+i(c-a)\sqrt{3}\}^3.$$

(9.) Show that

$$\{(\sqrt{3}+1)+(\sqrt{3}-1)i\}^3=16(1+i).$$

(10.) If $\xi+\eta i$ be a value of x for which $ax^2+bx+c=0$, a, b, c being all real, then $2a\xi\eta+b\eta=0$, $a\eta^2=a\xi^2+b\xi+c$.

(11.) If $\sqrt[3]{(x+yi)}=X+Yi$, show that $4(X^2-Y^2)=x/X+y/Y$.

(12.) If n be a multiple of 4, show that

$$1+2i+3i^2+\ .\ .\ .+(n+1)i^n=\tfrac{1}{2}(n+2-ni).$$

(13.) Show that $|a_0+a_1z+\ .\ .\ .+a_nz^n|\geqslant|a_n|\ |z|^n(1-nc/|z|)$, provided $|z|$ exceeds the greater of 1 and nc, and c is the greatest of $|a_0/a_n|$, . . ., $|a_{n-1}/a_n|$.

(14.) Find the modulus of

$$\frac{(2-3i)\,(3+4i)}{(6+4i)\,(15-8i)}.$$

(15.) Find the modulus of

$$\{x+\sqrt{(x^2+y^2i)}\}^2.$$

(16.) Find the modulus of

$$bc(b-ci)+ca(c-ai)+ab(a-bi).$$

(17.) Show that

$$|1+ix+i^2x^2+i^3x^3+\ .\ .\ .\ \text{ad}\ \infty|=1/\sqrt{(1+x^2)},\ \text{where}\ x<1.$$

(18.) Find the moduli of $(x+yi)^n$ and $(x+yi)^n/(x-yi)^n$.

Express the following as complex numbers :—

(19.) $\sqrt{-7+24i}$. (20.) $\sqrt{6+i\sqrt{13}}$.

(21.) $\sqrt{-7/36+2i/3}$. (22.) $\sqrt{4ab+2(a^2-b^2)i}$.

(23.) $\sqrt{1+2x\sqrt{(x^2-1)}i}$. (24.) $\sqrt{1+i\sqrt{(x^4-1)}}$.

(25.) Find the 4th roots of $-119+120i$.

(26.) Resolve $x^6 - a^6$ into factors of the 1st degree.

(27.) Resolve $x^5 + 1$ into real factors of the 1st or of the 2nd degree.

(28.) Resolve $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ into real factors of the 2nd degree.

(29.) Resolve $x^{2m} - 2\cos\theta a^m x^m + a^{2m}$ into real factors of the 1st or of the 2nd degree.

(30.) If ω be an imaginary nth root of $+1$, show that $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$.

(31.) Show that, if ω be an imaginary cube root of $+1$, then

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),$$

and

$$(x + \omega y + \omega^2 z)^3 + (x + \omega^2 y + \omega z)^3 = (2x - y - z)(2y - z - x)(2z - x - y).$$

(32.) Show that $(x + y)^m - x^m - y^m$ is divisible by $x^2 + xy + y^2$ for every odd value of m which is greater than 3 and not a multiple of 3.

(33.) Show that

$$\left\{\left(X\cos\frac{2r\pi}{n} - Y\sin\frac{2r\pi}{n}\right) + \left(X\sin\frac{2r\pi}{n} + Y\cos\frac{2r\pi}{n}\right)i\right\}^n = (X + Yi)^n.$$

(34.) Simplify

$$\frac{(\cos 2\theta - i\sin 2\theta)(\cos\phi + i\sin\phi)^2}{\cos(\theta + \phi) + i\sin(\theta + \phi)} + \frac{(\cos 2\theta + i\sin 2\theta)(\cos\phi - i\sin\phi)^2}{\cos(\theta + \phi) - i\sin(\theta + \phi)}.$$

(35.) If $\sqrt{(a + bi)} + \sqrt{(c + di)} = \sqrt{(x + yi)}$, show that

$$(x - a - c)^2 + (y - b - d)^2 = 4\sqrt{\{(a^2 + b^2)(c^2 + d^2)\}}.$$

(36.) Prove that one of the values of $\sqrt[4]{(a + bi)} + \sqrt[4]{(a - bi)}$ is

$$\sqrt{[\sqrt{\{2a + 2\sqrt{(a^2 + b^2)}\}} + 2\sqrt[4]{(a^2 + b^2)}]}.$$

(37.) If $w = \cos\pi/7 + i\sin\pi/7$, prove that $(x - w)(x + w^2)(x - w^3)(x + w^4)(x - w^5)(x + w^6) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$.

(38.) Find the value of $w^r_1 + w^r_2 + \ldots + w^r_n$; $w_1, w_2 \ldots w_n$ being the nth roots of 1, and r a positive integer. What modification of the result is necessary if r is a negative integer?

Prove that $1/(1 + w_1 x) + 1/(1 + w_2 x) + \ldots + 1/(1 + w_n x) = n/(1 - x^n)$.

(39.) Decompose $1/(1 + x + x^2)$ into partial fractions of the form $a/(bx + c)$. Hence show that

$$x/(1 + x + x^2) = x - x^2 + x^4 - x^5 + x^7 - x^8 + \ldots + x^{3n+1} - x^{3n+2} + R,$$

where x^3, x^6, etc., are wanting; and find R.

(40.) Find the equation of least degree, having real rational coefficients, one of whose roots is $\sqrt{2} + i$.

One root of $x^4 + 3x^3 - 30x^2 + 366x - 310 = 0$ is $3 + 5i$, find the other three roots.

(41.) If a be a given complex number, and z a complex number whose affixe lies on a given straight line, find the locus of the affixe of $a + z$.

(42.) Show that the area of the triangle whose vertices are the affixes of z_1, z_2, z_3 is $\Sigma\{(z_2 - z_3)\mid z_1\mid^2/4iz_1\}$.

(43.) If $z = (a + \gamma\cos\theta) + i(\beta + \gamma\sin\theta)$, where α, β, γ are constant and θ variable, find the maximum and minimum values of $|z|$; and of amp z when such values exist.

(44.) If the affixe of $x+yi$ move on the line $3x+4y+5=0$, prove that the minimum value of $|x+yi|$ is 1.

(45.) If u and v are two complex numbers such that $u=v+1/v$, show that, if the affixe of v describes a circle about the origin in Argand's diagram, then the affixe of u describes an ellipse $(x^2/a^2+y^2/b^2=1)$; and, if the affixe of u describes a circle about the origin, then the affixe of v describes a quartic curve, which, in the particular case where the radius of the circle described by the affixe of u is 2, breaks up into two circles whose centres are on the i-axis.

(46.) If x and y be real, and $x+y=1$, show that the affixe of xz_1+yz_2 lies on the line joining the affixes of z_1 and z_2. Hence show that the affixe of xz_1+yz_2 lies on a fixed straight line provided $lx+my=1$, l and m being constants.

(47.) If $\xi+\eta i$ be an imaginary root of $x^3+2x+1=0$, prove that (ξ, η) is one of the intersections of the graphs of $\eta^2=3\xi^2+2$ and $\eta^2=1/2+3/8\xi$. Draw the graphs: and mark the intersections which correspond to the roots of the equation.

If a be the real root of this cubic, show that the imaginary roots are $\frac{1}{2}\{-a\pm i\sqrt{(2-3/a)}\}$.

(48.) If $\xi\pm\eta i$ be a pair of imaginary roots of $x^3-px+q=0$, show that (ξ, η) are co-ordinates of the real intersections of $3\xi^2-\eta^2-p=0$, $8\xi\eta^2+2p\xi-3q=0$. Hence prove that the roots of the cubic are all real, or one real and two imaginary, according as $4p^3<>27q^2$. What happens if $4p^3=27q^2$?

(49.) If $x^3+qx+r=0$ has imaginary roots, the real part of each is positive or negative according as r is positive or negative.

(50.) The cubic $x^3-9x^2+33x-65=0$ has an imaginary root whose modulus is $\sqrt{13}$; find all its roots.

(51.) Find the real quadratic factors of $x^{2n}+x^{2n-1}+\ .\ .\ .\ +1$; and hence prove that

$$2^n \sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\ .\ .\ .\ \sin\frac{n\pi}{2n+1}=\sqrt{(2n+1)}.$$

(52.) Find in rational integral form the equation which results by eliminating θ from the equations $x=a\cos\theta+b\cos 3\theta$, $y=a\sin\theta+b\sin 3\theta$. (Use Demoivre's Theorem.) Give a geometrical interpretation of your analysis.

Historical Note.—Imaginary quantities appear for the first time in the works of the Italian mathematicians of the 16th century. Cardano, in his *Artis Magnæ sive de Regulis Algebraicis Liber Unus* (1545), points out (cap. xxxvii., p. 66) that, if we solve in the usual way the problem to divide 10 into two parts whose product shall be 40, we arrive at two formulæ which, in modern notation, may be written $5+\sqrt{-15}$, $5-\sqrt{-15}$. He leaves his reader to imagine the meaning of these "sophistic" numbers, but shows that, if we add and multiply them in formal accordance with the ordinary algebraic rules, their sum and product do come out as required in the evidently impossible problem; and he adds "hucusque progreditur Arithmetica subtilitas, cujus hoc extremum ut dixi adeo est subtile, ut sit inutile." Bombelli in his *Algebra* (1522), following Cardano, devoted considerable attention to the theory of complex numbers, more especially in connection with the solution of cubic equations.

There is clear indication in the fragment *De Arte Logistica* (see above, p. 201) that Napier was in possession to some extent at least of the theory. He was fully cognisant of the independent existence of negative quantity ("quantitates defectivæ minores nihilo"), and draws a clear distinction between the roots of positive and of negative numbers. He points out (Napier's Ed., p. 85) that roots of even order have no real value, either positive or negative, when the radicand is negative. Such roots he calls "nugacia"; and expressly warns against the error of supposing that $\sqrt{-9} = -\sqrt{9}$. In this passage there occurs the curious sentence, "Hujus arcani magni algebraici fundamentum superius, Lib. i. cap. 6, jecimus: quod (quamvis a nemine quod sciam revelatum sit) quantum tamen emolumenti adferat huic arti, et cæteris mathematicis postea patebit." There is nothing farther in the fragment *De Arte Logistica* to show how deeply he had penetrated the secret which was to be hidden from mathematicians for 200 years.

The theory of imaginaries received little notice until attention was drawn to it by the brilliant results to which the use of them led Euler (1707-1783) and his contemporaries and followers. Notwithstanding the use made by Euler and others of complex numbers in many important investigations, the fundamental principles of their logic were little attended to, if not entirely misunderstood. To Argand belongs the honour of first clearing up the matter in his *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (1806). He there gives geometrical constructions for the sum and product of two complex numbers, and deduces a variety of conclusions therefrom. He also was one of the first to thoroughly understand and answer the question of § 21 regarding the existence of a root of every integral function. Argand was anticipated to a considerable extent by a Danish mathematician, Caspar Wessel, who in 1797 presented to the Royal Academy of Denmark a remarkable memoir *Om Directionens analytiske Betegning, et Forsög, anvendt fornemmelig til plane og sphaeriske Polygoners Oplösning*, which was published by the Academy in 1799, but lay absolutely unknown to mathematicians, till it was republished by the same body in 1897. See an interesting address by Beman to Section A of the American Association for the Promotion of Science (1897). Even Argand's results appear to have been at first little noticed; and, as a matter of history, it was Gauss who first initiated mathematicians into the true theory of the imaginaries of ordinary algebra. He first used the phrase *complex number*, and introduced the use of the symbol i for the imaginary unit. He illustrated the twofold nature of a complex number by means of a diagram, as Argand had done; gave a masterly discussion of the fundamental principles of the subject in his memoir on Biquadratic Residues (1831) (see his Works, vol. ii., pp. 101 and 171); and furnished three distinct proofs (the first published in 1799) of the proposition that every equation has a root.

From the researches of Cauchy (1789-1857) and Riemann (1826-1866) on complex numbers has sprung a great branch of modern pure mathematics, called on the Continent function-theory. The student who wishes to attain a full comprehension of the generality of even the more elementary theorems of algebraic analysis will find a knowledge of the theory of complex quantity indispensable; and without it he will find entrance into many parts of the higher mathematics impossible.

For further information we may refer the reader to Peacock's *Algebra*, vol. ii. (1845); to De Morgan's *Trigonometry and Double Algebra* (1849), where a list of most of the English writings on the subject is given; and to Hankel's *Vorlesungen über die complexen Zahlen* (1867), where a full historical account of Continental researches will be found. It may not be amiss to add that the theory of complex numbers is closely allied to Hamilton's theory of Quaternions, Grassmann's Ausdehnungslehre, and their modern developments.

CHAPTER XIII.

Ratio and Proportion.

RATIO AND PROPORTION OF ABSTRACT QUANTITIES.

§ 1.] *The ratio of the abstract quantity a to the abstract quantity b is simply the quotient of a by b.*

When the quotient $a \div b$, or a/b, or $\frac{a}{b}$ is spoken of as a ratio, it is often written $a:b$; a is called the *antecedent* and b the *consequent* of the ratio.

There is a certain convenience in introducing this new name, and even the new fourth notation, for a quotient. So far, however, as mere abstract quantity is concerned, the propositions which we proceed to develop are simply results in the theory of algebraical quotients, arising from certain conditions to which we subject the quantities considered.

If $a > b$, that is, if $a - b$ be positive, $a:b$ is said to be a *ratio of greater inequality*.

If $a < b$, that is, if $a - b$ be negative, $a:b$ is said to be a *ratio of less inequality*.

When two ratios are multiplied together, they are said to be *compounded*. Thus, the ratio $aa' : bb'$ is said to be compounded of the ratios $a:b$ and $a':b'$.

The compound of two equal ratios, $a:b$ and $a:b$, namely, $a^2 : b^2$, is called the *duplicate* of the ratio $a:b$.

Similarly, $a^3 : b^3$ is the *triplicate* of the ratio $a:b$.*

* Formerly $a^2 : b^2$ was spoken of as the double of the ratio $a:b$. Similarly $\sqrt{a} : \sqrt{b}$ was called the half or subduplicate of $a:b$, and $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ the sesquiplicate of $a:b$.

§ 2.] *Four abstract numbers, a, b, c, d, are said to be proportional when the ratio $a:b$ is equal to the ratio $c:d$.*

We then write

$$a:b=c:d.*$$

a and d are called the *extremes*, and b and c the *means*, of the proportion. a and c are said to be homologues, and b and d to be homologues.

If a, b, c, d, e, f, &c., be such that $a:b=b:c=c:d=d:e=e:f$ = &c., a, b, c, d, e, f, &c., are said to be in *continued proportion.*

If a, b, c be in continued proportion, b is said to be a *mean proportional* between a and c.

If a, b, c, d be in continued proportion, b and c are said to be *two mean proportionals* between a and d; and so on.

§ 3.] *If b be positive, and $a>b$, the ratio $a:b$ is diminished by adding the same positive quantity to both antecedent and consequent; and increased by subtracting the same positive quantity ($<b$) from both antecedent and consequent.*

If $a<b$, the words "increased" and "diminished" must be interchanged in the above statement.

For,
$$\frac{a+x}{b+x}-\frac{a}{b}=\frac{b(a+x)-a(b+x)}{b(b+x)}$$
$$=\frac{x(b-a)}{b(b+x)}.$$

Now, if $a>b$, $b-a$ is negative; and x, b, $b+x$ are all positive by the conditions imposed; hence $x(b-a)/b(b+x)$ is negative.

Hence
$$\frac{a+x}{b+x}-\frac{a}{b}\text{ is negative,}$$

that is,
$$\frac{a+x}{b+x}<\frac{a}{b}.$$

Again,
$$\frac{a-x}{b-x}-\frac{a}{b}=\frac{x(a-b)}{b(b-x)}.$$

But, since $a>b$, $a-b$ is positive, and x and b are positive, and, since $x<b$, $b-x$ is positive. Hence $x(a-b)/b(b-x)$ is positive.

* Formerly in writing proportions the sign :: (originally introduced by Oughtred) was used instead of the ordinary sign of equality.

Hence $$\frac{a-x}{b-x} > \frac{a}{b}.$$

The rest of the proposition may be established in like manner.

The reader will obtain an instructive view of this proposition by comparing it with Exercise 7, p. 267.

§ 4.] *Permutations of a Proportion.*

If $$a : b = c : d \qquad (1),$$
then $$b : a = d : c \qquad (2),$$
$$a : c = b : d \qquad (3),$$
and $$c : a = d : b \qquad (4).$$

For, from (1), we have $$\frac{a}{b} = \frac{c}{d}.$$

Hence $$1/\frac{a}{b} = 1/\frac{c}{d},$$

that is, $$\frac{b}{a} = \frac{d}{c};$$

that is, $$b : a = d : c,$$
which establishes (2).

Again, from (1), $$\frac{a}{b} = \frac{c}{d},$$

multiplying both sides by $\frac{b}{c}$, we have

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c},$$

that is, $$\frac{a}{c} = \frac{b}{d};$$

that is, $$a : c = b : d,$$
which proves (3).

(4) follows from (3) in the same way as (2) from (1).

§ 5.] *The product of the extremes of a proportion is equal to the product of the means; and, conversely, if the product of two quantities be equal to the product of two others, the four form a proportion, the extremes being the constituents of one of the products, the means the constituents of the other.*

For, if $$a : b = c : d,$$

that is, $$\frac{a}{b} = \frac{c}{d},$$

then $$\frac{a}{b} \times bd = \frac{c}{d} \times bd,$$

whence $$ad = bc.$$

Again, if $$ad = bc,$$

then $$ad/bd = bc/bd,$$

whence $$\frac{a}{b} = \frac{c}{d}.$$

Cor. *If three of the terms of a proportion be given, the remaining one is uniquely determined.*

For, when three of the quantities a, b, c, d are given, the equation

$$ad = bc,$$

which results by the above from their being in proportion, becomes an equation of the 1st degree (see chap. xvi.) to determine the remaining one.

Suppose, for example, that the 1st, 3rd, and 4th terms of the proportion are $\frac{1}{2}$, $\frac{4}{3}$, and $\frac{6}{8}$; and let x denote the unknown 2nd term.

Then $$\tfrac{1}{2} : x = \tfrac{4}{3} : \tfrac{6}{8};$$

whence $$\tfrac{4}{3} \times x = \tfrac{1}{2} \times \tfrac{6}{8}.$$

Multiplying by $\frac{3}{4}$, we have $$x = \tfrac{1}{2} \times \tfrac{6}{8} \times \tfrac{3}{4},$$
$$= \tfrac{9}{32}.$$

§ 6.] *Relations connecting quantities in continued proportion.*

If three quantities, a, b, c, be in continued proportion, then

$$a : c = a^2 : b^2 = b^2 : c^2;$$

and $$b = \sqrt{(ac)}.$$

If four quantities, a, b, c, d, be in continued proportion, then

$$a : d = a^3 : b^3 = b^3 : c^3 = c^3 : d^3,$$

and $$b = \sqrt[3]{(a^2 d)}, \quad c = \sqrt[3]{(ad^2)}.$$

For the general proposition, see Exercise 12, p. 267.

For, if $$a : b = b : c,$$

then $$\frac{a}{b} = \frac{b}{c}.$$

Therefore $$\frac{a}{b} \times \frac{b}{c} = \frac{b}{c} \times \frac{b}{c},$$

whence $$\frac{a}{c} = \frac{b^2}{c^2} = \frac{a^2}{b^2} \qquad (1).$$

Also $$ac = b^2,$$

whence $$b = \sqrt{(ac)} \qquad (2).$$

Equations (1) and (2) establish the first of the two propositions above stated.

Again, if $$a : b = b : c = c : d,$$

then $$\frac{a}{b} = \frac{b}{c}, \quad \frac{a}{b} = \frac{c}{d}.$$

Also $$\frac{a}{b} = \frac{a}{b},$$

hence $$\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d},$$

that is, $$\frac{a^3}{b^3} = \frac{a}{d},$$

therefore $$\frac{a}{d} = \frac{a^3}{b^3} = \frac{b^3}{c^3} = \frac{c^3}{d^3} \qquad (3).$$

Further, since $$\frac{a}{d} = \frac{a^3}{b^3},$$

$$b^3 = a^2 d;$$

whence $$b = \sqrt[3]{(a^2 d)}. \qquad (4).$$

Also, since $$\frac{a}{d} = \frac{c^3}{d^3},$$

$$c^3 = ad^2;$$

whence $$c = \sqrt[3]{(ad^2)} \qquad (5).$$

It should be noticed that the result (2) shows that the finding of a mean proportional between two given quantities a and c depends on the extraction of a square root. For example, the mean proportional between 1 and 2 is

$$\sqrt{(1 \times 2)} = \sqrt{2} = 1 \cdot 4142 \ . \ . \ .$$

Again, (4) and (5) enable us to insert two mean proportionals between two given quantities by extracting certain cube roots. For example, the two mean proportionals between 1 and 2 are

$$\sqrt[3]{(1\times 2)} = \sqrt[3]{2} = 1{\cdot}2599 \ldots$$

and

$$\sqrt[3]{(1\times 2^2)} = \frac{2}{\sqrt[3]{2}} = 1{\cdot}5874 \ldots$$

Conversely, of course, the finding of the cube root of 2, which again corresponds to the famous Delian problem of antiquity, the duplication of the cube, could be made to depend on the finding of two mean proportionals, a result well known to the Greek geometers of Plato's time.

§ 7.] After what has been done, the student will have no difficulty in showing that

if $$a : b = c : d,$$

then $$ma : mb = nc : nd \qquad (1),$$

and $$ma : nb = mc : nd \qquad (2).$$

§ 8.] Also that

if $$a_1 : b_1 = c_1 : d_1,$$
$$a_2 : b_2 = c_2 : d_2,$$
$$\vdots$$
$$a_n : b_n = c_n : d_n,$$

then $$a_1a_2 \ldots a_n : b_1b_2 \ldots b_n = c_1c_2 \ldots c_n : d_1d_2 \ldots d_n \qquad (1).$$

Cor.

If $$a : b = c : d,$$

then $$a^n : b^n = c^n : d^n.$$

(Here n, see chap. x., may be positive or negative, integral or fractional, provided a^n, &c., be real, and of the same sign as a, &c.)

§ 9.] *If* $$a : b = c : d,$$

then $$a \pm b : b = c \pm d : d \qquad (1),$$

$$a + b : a - b = c + d : c - d \qquad (2),$$

$$la + mb : pa + qb = lc + md : pc + qd \qquad (3),$$

$$la^r + mb^r : pa^r + qb^r = lc^r + md^r : pc^r + qd^r \qquad (4),$$

where l, m, p, q, r are any quantities, positive or negative.

Also, if $a_1 : b_1 = a_2 : b_2 = a_3 : b_3 = \ldots = a_n : b_n$,
then each of these ratios is equal to

$$a_1 + a_2 + \ldots + a_n : b_1 + b_2 + \ldots + b_n \qquad (5);$$

and also to

$$\sqrt[r]{(l_1 a_1^r + l_2 a_2^r + \ldots + l_n a_n^r)} : \sqrt[r]{(l_1 b_1^r + l_2 b_2^r + \ldots + l_n b_n^r)} \quad (6).$$

Though outwardly somewhat different in appearance, these six results are in reality very much allied. Two different methods of proof are usually given.

FIRST METHOD.

Let us take, for example, (1) and (2).

Since $$\frac{a}{b} = \frac{c}{d},$$

therefore $$\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1;$$

whence $$\frac{a \pm b}{b} = \frac{c \pm d}{d};$$

this establishes the two results in (1).

Writing these separately we have

$$\frac{a + b}{b} = \frac{c + d}{d},$$

$$\frac{a - b}{b} = \frac{c - d}{d};$$

whence $$\frac{(a + b)}{b} \Big/ \frac{(a - b)}{b} = \frac{(c + d)}{d} \Big/ \frac{(c - d)}{d},$$

that is, $$\frac{a + b}{a - b} = \frac{c + d}{c - d},$$

which establishes (2).

Similar treatment may be applied to the rest of the six results.

SECOND METHOD.

Let us take, for example, (2).

Since $a/b = c/d$, we may denote each of these ratios by the same symbol, ρ, say. We then have

$$\frac{a}{b} = \rho, \qquad \frac{c}{d} = \rho\,;$$

whence $$a = \rho b, \qquad c = \rho d \qquad (\alpha).$$

Now, using (α), we have

$$\begin{aligned}\frac{a+b}{a-b} &= \frac{\rho b + b}{\rho b - b},\\ &= \frac{b(\rho+1)}{b(\rho-1)},\\ &= \frac{\rho+1}{\rho-1}.\end{aligned}$$

In exactly the same way, we have

$$\begin{aligned}\frac{c+d}{c-d} &= \frac{\rho d + d}{\rho d - d},\\ &= \frac{\rho+1}{\rho-1}.\end{aligned}$$

Hence $$\frac{a+b}{a-b} = \frac{\rho+1}{\rho-1} = \frac{c+d}{c-d}.$$

Again, let us take (5).

We have $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ . \ . \ . = \frac{a_n}{b_n}, \text{ each } = \rho, \text{ say,}$$

hence $$a_1 = \rho b_1, \quad a_2 = \rho b_2, \quad . \ . \ ., \quad a_n = \rho b_n\,;$$

therefore $$\begin{aligned}\frac{a_1 + a_2 + \ . \ . \ . + a_n}{b_1 + b_2 + \ . \ . \ . + b_n} &= \frac{\rho b_1 + \rho b_2 + \ . \ . \ . + \rho b_n}{b_1 + b_2 + \ . \ . \ . + b_n},\\ &= \frac{\rho(b_1 + b_2 + \ . \ . \ . + b_n)}{b_1 + b_2 + \ . \ . \ . + b_n},\\ &= \rho,\end{aligned}$$

hence

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \&\text{c.} = \rho = \frac{a_1 + a_2 + \ . \ . \ . + a_n}{b_1 + b_2 + \ . \ . \ . + b_n}.$$

Finally, let us take (6).

Since
$$a_1^r = (\rho b_1)^r = \rho^r b_1^r,$$
$$a_2^r = (\rho b_2)^r = \rho^r b_2^r, \ \&c.$$
we have
$$\sqrt[r]{(l_1 a_1^r + l_2 a_2^r + \ . \ . \ . \ + l_n a_n^r)} = \sqrt[r]{(\rho^r(l_1 b_1^r + l_2 b_2^r + \ . \ . \ . \ + l_n b_n^r))},$$
$$= \rho \sqrt[r]{(l_1 b_1^r + l_2 b_2^r + \ . \ . \ . \ + l_n b_n^r)},$$
(see chap. x., § 4). It follows that
$$\frac{\sqrt[r]{(l_1 a_1^r + l_2 a_2^r + \ . \ . \ . \ + l_n a_n^r)}}{\sqrt[r]{(l_1 b_1^r + l_2 b_2^r + \ . \ . \ . \ + l_n b_n^r)}} = \rho = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c.$$

Of the two methods there can be no doubt that the second is the clearer and more effective. The secret of its power lies in the following principle:—

In establishing an equation between conditioned quantities, if we first express all the quantities involved in the equation in terms of the fewest quantities possible under the conditions, then the verification of the equation involves merely the establishment of an algebraical identity. In establishing (2), for instance, we expressed all the quantities involved in terms of the three b, d, ρ, so many being necessary, by § 5, to determine a proportion.

A good deal of the art of algebraical manipulation consists in adroitly taking advantage of this principle, without at the same time destroying the symmetry of the functions involved.

§ 10.] The following general theorem contains, directly or indirectly, all the results of last article as particular cases; and will be found to be a compendium of a very large class of favourite exercises on the present subject, some of which will be found at the end of the present paragraph.

If $\phi(x_1, x_2, \ . \ . \ ., x_n)$ be any homogeneous integral function of the variables $x_1, x_2, \ . \ . \ ., x_n$ of the rth degree, or a homogeneous function of degree r, according to the extended notion of homogeneity and degree given at the foot of p. 73, and if
$$a_1 : b_1 = a_2 : b_2 = \ . \ . \ . \ = a_n : b_n,$$
then each of these ratios is equal to
$$\sqrt[r]{\phi(a_1, a_2, \ . \ . \ ., a_n)} : \sqrt[r]{\phi(b_1, b_2, \ . \ . \ ., b_n)}.$$

This theorem is an immediate consequence of the property of homogeneous functions given in chap. iv., p. 73.

Example 1.

Which is the greater ratio, $x^2+y^2:x+y$, or $x^2-y^2:x-y$, x and y being each positive?

$$\frac{x^2+y^2}{x+y}-\frac{x^2-y^2}{x-y}=\frac{(x^2+y^2)(x-y)-(x^2-y^2)(x+y)}{(x+y)(x-y)},$$
$$=\frac{2xy^2-2x^2y}{(x+y)(x-y)},$$
$$=-\frac{2xy(x-y)}{(x+y)(x-y)},$$
$$=-\frac{2xy}{x+y}.$$

Now, if x and y be each positive, $-2xy/(x+y)$ is essentially negative. Hence

$$x^2+y^2:x+y<x^2-y^2:x-y.$$

Example 2.

If $a:b=c:d$, and $A:B=C:D$, then $a\sqrt{A}-b\sqrt{B}:c\sqrt{C}-d\sqrt{D}=a\sqrt{A}+b\sqrt{B}:c\sqrt{C}+d\sqrt{D}$.

Let each of the ratios $a:b$ and $c:d=\rho$, and each of the two $A:B$ and $C:D=\sigma$, then $a=\rho b$, $c=\rho d$; $A=\sigma B$, $C=\sigma D$. We then have

$$\frac{a\sqrt{A}-b\sqrt{B}}{c\sqrt{C}-d\sqrt{D}}=\frac{\rho b\sqrt{(\sigma B)}-b\sqrt{B}}{\rho d\sqrt{(\sigma D)}-d\sqrt{D}},$$
$$=\frac{(\rho\sqrt{\sigma}-1)b\sqrt{B}}{(\rho\sqrt{\sigma}-1)d\sqrt{D}}=\frac{b\sqrt{B}}{d\sqrt{D}} \qquad (\alpha).$$

In the same way we get

$$\frac{a\sqrt{A}+b\sqrt{B}}{c\sqrt{C}+d\sqrt{D}}=\frac{(\rho\sqrt{\sigma}+1)b\sqrt{B}}{(\rho\sqrt{\sigma}+1)d\sqrt{D}}=\frac{b\sqrt{B}}{d\sqrt{D}} \qquad (\beta).$$

From (α) and (β) the required result follows.

Example 3.

If b be a mean proportional between a and c, show that

$$(a+b+c)(a-b+c)=a^2+b^2+c^2 \qquad (\alpha),$$

and
$$(a+b+c)^2+a^2+b^2+c^2=2(a+b+c)(a+c) \qquad (\beta).$$

Taking (α) we have

$$(a+b+c)(a-b+c)=(\overline{a+c}+b)(\overline{a+c}-b),$$
$$=(a+c)^2-b^2.$$

Now, by data, $a/b=b/c$, and therefore $b^2=ac$; hence

$$(a+c)^2-b^2=(a+c)^2-ac,$$
$$=a^2+ac+c^2,$$
$$=a^2+b^2+c^2,$$

since $b^2=ac$. Hence (α) is proved.

Taking now (β), and, for variety, adopting the second method of § 9, let us put

$$\frac{a}{b}=\frac{b}{c}=\rho.$$

Hence $a=\rho b$, $b=\rho c$; so that $a=\rho(\rho c)=\rho^2 c$.

We have to verify the identity

$$(\rho^2c+\rho c+c)^2+(\rho^2c)^2+(\rho c)^2+c^2=2(\rho^2c+\rho c+c)(\rho^2c+c);$$

that is to say,

$$\{(\rho^2+\rho+1)^2+(\rho^4+\rho^2+1)\}c^2=2(\rho^2+\rho+1)(\rho^2+1)c^2 \qquad (\gamma).$$

Now

$$\begin{aligned}\{(\rho^2+\rho+1)^2+(\rho^4+\rho^2+1)\}c^2&=(\rho^2+\rho+1)\{(\rho^2+\rho+1)+(\rho^2-\rho+1)\}c^2,\\&=2(\rho^2+\rho+1)(\rho^2+1)c^2,\end{aligned}$$

which proves the truth of (γ), and therefore establishes (β).

Example 4.*

If $x/(b+c-a)=y/(c+a-b)=z/(a+b-c)$, then $(b-c)x+(c-a)y+(a-b)z=0$.

Let us put

$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\rho,$$

then

$$\begin{aligned}x&=(b+c-a)\rho,\\y&=(c+a-b)\rho,\\z&=(a+b-c)\rho.\end{aligned}$$

Now, from the last three equations, we have—

$$\begin{aligned}(b-c)x+(c-a)y+(a-b)z&\\&=(b-c)(b+c-a)\rho+(c-a)(c+a-b)\rho+(a-b)(a+b-c)\rho,\\&=\{(b^2-c^2+c^2-a^2+a^2-b^2)-(a(b-c)+b(c-a)+c(a-b))\}\rho,\\&=\{0-0\}\rho,\\&=0.\end{aligned}$$

Example 5.

If

$$\frac{bz+cy}{b-c}=\frac{cx+az}{c-a}=\frac{ay+bx}{a-b} \qquad (1),$$

then

$$(a+b+c)(x+y+z)=ax+by+cz \qquad (2).$$

Let each of the ratios in (1) be equal to ρ, then

$$\begin{aligned}bz+cy&=\rho(b-c) &(3),\\cx+az&=\rho(c-a) &(4),\\ay+bx&=\rho(a-b) &(5).\end{aligned}$$

From (3), (4), (5), by addition,

$$\begin{aligned}(b+c)x+(c+a)y+(a+b)z&=\rho\{(b-c)+(c-a)+(a-b)\},\\&=\rho 0,\\&=0 \qquad (6).\end{aligned}$$

If now we add $ax+by+cz$ to both sides of (6) we obtain equation (2).

---

* Examples 4, 5, and 6 illustrate a species of algebraical transformation which is very common in geometrical applications. In reality they are examples of a process which is considered more fully in chap. xiv.

Example 6.

If
$$\frac{cy+bz}{qb+rc-pa}=\frac{az+cx}{rc+pa-qb}=\frac{bx+ay}{pa+qb-rc} \qquad (1),$$
show that
$$\frac{x}{a\{pa(a+b+c)-qb(a+b-c)-rc(a-b+c)\}}$$
$$=\frac{y}{b\{qb(a+b+c)-pa(a+b-c)-rc(-a+b+c)\}},$$
$$=\frac{z}{c\{rc(a+b+c)-qb(-a+b+c)-pa(a-b+c)\}} \qquad (2).$$

Let each of the fractions of (1) be $=\rho$; and observe that the three equations,
$$\left.\begin{aligned} cy+bz&=(qb+rc-pa)\rho \quad (\alpha)\\ az+cx&=(rc+pa-qb)\rho \quad (\beta)\\ bx+ay&=(pa+qb-rc)\rho \quad (\gamma)\end{aligned}\right\} \qquad (3),$$
which thus arise are symmetrical in the triple set $\left\{\begin{matrix} xyz\\ abc\\ pqr\end{matrix}\right\}$, so that the simultaneous interchange of the letters in two of the vertical columns simply changes each of the equations (3) into another of the same set. It follows, then, that a similar interchange made in any equation derived from (3) will derive therefrom another equation also derivable from (3).

Now, if we multiply both sides of (β) by b, and both sides of (γ) by c, we obtain, by addition from the two equations thus derived,
$$2bcx+a\{cy+bz\}=\rho\{b(rc+pa-qb)+c(pa+qb-rc)\} \qquad (4).$$

Now, using the value of $cy+bz$ given by (α), we have
$$2bcx+pa(qb+rc-pa)=\rho\{pa(b+c)-qb(b-c)-rc(-b+c)\} \qquad (5).$$

Subtracting $pa(qb+rc-pa)$ from both sides of (5), we have
$$2bcx=\rho\{pa(a+b+c)-qb(a+b-c)-rc(a-b+c)\} \qquad (6).$$

From (6), we have
$$\frac{x}{a\{pa(a+b+c)-qb(a+b-c)-rc(a-b+c)\}}=\frac{\rho}{2abc} \qquad (7).$$

We may in (7) make the interchange $\begin{pmatrix} xap\\ \text{into}\\ ybq\end{pmatrix}$, or $\begin{pmatrix} xap\\ \text{into}\\ zcr\end{pmatrix}$, and we shall obtain two other equations derivable from (3) by a process like that used to derive (7) itself. These interchanges leave the right-hand side of (7) unaltered, but change the left-hand side into the second and third members of (2) respectively. Hence the three members of (2) are all equal, each being in fact equal to $\rho/2abc$.

This is a good example of the use of the principle of symmetry in complicated algebraical calculations.

Exercises XVII.

(1.) Which is the greater ratio, $5:7$ or $151:208$?

(2.) If the ratio $3:4$ be duplicated by subtracting x from both antecedent and consequent, show that $x=1\frac{5}{7}$.

(3.) What quantity x added to the antecedent and to the consequent of $a:b$ will convert this ratio into $c:d$?

(4.) Find the fourth proportional to $3\frac{1}{2}$, $5\frac{3}{4}$, $6\frac{4}{8}$; also the third proportional to $1+\sqrt{2}$ and $3+2\sqrt{2}$.

(5.) Insert a mean proportional between 11 and 19; and also two mean proportionals between the same two numbers.

(6.) Find a simple surd number which shall be a mean proportional between $\sqrt{7}-\sqrt{5}$ and $11\sqrt{7}+13\sqrt{5}$.

(7.) If x and y be such that when they are added to the antecedent and consequent respectively of the ratio $a:b$ its value is unaltered, show that $x:y=a:b$.

(8.) If x and y be such that when they are added respectively to the antecedent and consequent and to the consequent and antecedent of $a:b$ the two resulting ratios are equal, show that either $x=y$ or $x+y=-a-b$.

(9.) Find a quantity x such that when it is added to the four given quantities a, b, c, d the result is four quantities in proportion. Exemplify with 3, 4, 9, 13; and with 3, 4, $1\frac{1}{2}$, 2.

(10.) If four quantities be proportional, the sum of the greatest and least is always greater than the sum of the other two.

(11.) If the ratio of the difference of the antecedents of two ratios to the sum of their consequents is equal to the difference of the two ratios, then the antecedents are in the duplicate ratio of the consequents.

(12.) If the n quantities $a_1, a_2, \ldots, a_n$ be in continued proportion, then $a_1:a_n=a_1^{n-1}:a_2^{n-1}=a_2^{n-1}:a_3^{n-1}=$&c.; and

$$a_2=\sqrt[n-1]{(a_1^{n-2}a_n)},\quad a_3=\sqrt[n-1]{(a_1^{n-3}a_n^2)},\quad \ldots,\quad a_r=\sqrt[n-1]{(a_1^{n-r}a_n^{r-1})}.$$

(13.) If
$$(pa+qb+rc+sd)(pa-qb-rc+sd)$$
$$=(pa-qb+rc-sd)(pa+qb-rc-sd),$$
then
$$bc:ad=ps:qr;$$
and, if either of the two sets a, b, c, d or p, q, r, s form a proportion, the other will also.

(14.) If $a:b=c:d=e:f$,

then
$$a^3+3a^2b+b^3:c^3+3c^2d+d^3=a^3+b^3:c^3+d^3 \qquad (a);$$

$$\sqrt{\left(\frac{a^2c^2}{e^2}+\frac{a^2e^2}{c^2}+\frac{c^2e^2}{a^2}\right)}:\sqrt{\left(\frac{b^2d^2}{f^2}+\frac{b^2f^2}{d^2}+\frac{d^2f^2}{b^2}\right)}$$
$$=a^3df+c^3bf+e^3bd:b^3ce+d^3ae+f^3ac \qquad (\beta);$$

$$pa-qc+re:pb-qd+rf=\sqrt[3]{ace}:\sqrt[3]{bdf}$$
$$=\sqrt{(a^2-c^2+e^2+2ae)}:\sqrt{(b^2-d^2+f^2+2bd)} \qquad (\gamma).$$

(15.) If $a : a' = b : b'$,

then
$$a^{m+n} + a^m b^n + b^{m+n} : a'^{m+n} + a'^m b'^n + b'^{m+n}$$
$$= (a + b)^{m+n} : (a' + b')^{m+n}.$$

(16.) If $a : b = c : d$, and $\alpha : \beta = \gamma : \delta$,

then
$$a^3\alpha^2 + (a^2b + ab^2)\alpha\beta + b^3\beta^2 : (a^3 + b^3)(\alpha^2 + \beta^2)$$
$$= c^3\gamma^2 + (c^2d + cd^2)\gamma\delta + d^3\delta^2 : (c^3 + d^3)(\gamma^2 + \delta^2).$$

(17.) If $a : b = b : c = c : d$, then
$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2 \qquad (\alpha);$$
$$(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2 \qquad (\beta);$$
$$ab + cd + ad = (a + b + c)(b - c + d) \qquad (\gamma);$$
$$a + b - c - d = (a + b)(b - d)/b \qquad (\delta);$$
$$(a + b + c + d)(a - b - c + d) = 2(ab - cd)(ac - bd)/(ad + bc) \qquad (\epsilon).$$

(18.) If a, b, c be in continued proportion,

then
$$a^2 + ab + b^2 : b^2 + bc + c^2 = a : c \qquad (\alpha);$$
$$a^2(a - b + c)(a + b + c) = a^4 + a^2b^2 + b^4 \qquad (\beta);$$
$$(b + c)^2/(b - c) + (c + a)^2/(c - a) + (a + b)^2/(a - b) = 4b(a + b + c)/(a - c) \qquad (\gamma).$$

(19.) If a, b, c, d be in continued proportion,

then
$$(a - c)(b - d) - (a - d)(b - c) = (b - c)^2 \qquad (\alpha);$$
$$\sqrt{(ab)} + \sqrt{(bc)} + \sqrt{(cd)} = \sqrt{\{(a + b + c)(b + c + d)\}} \qquad (\beta).$$

(20.) If $ab = cd = ef$, then
$$(ac + ce + ea)/dbf(d + b + f) = (a^2 + c^2 + e^2)/(b^2d^2 + d^2f^2 + f^2b^2).$$

(21.) If $(a - b)/(d - c) = (b - c)/(c - f)$, then each of them
$$= \{b(f - d) + (cd - af)\}/e(f - d).$$

(22.) If $\xi/yx = \eta/x^2 = \zeta/yz$, then $x/\xi\eta = y/\xi^2 = z/\eta\zeta$.

(23.) If $\quad 2x + 3y : 3y + 4z : 4z + 5x = 4a - 5b : 3b - a : 2b - 3a$,

then
$$7x + 6y + 8z = 0.$$

(24.) If $ax + cy : by + dz = ay + cz : bz + dx = az + cx : bx + dy$, and if $x + y + z \neq 0$, $ab - cd \neq 0$, $ad - bc \neq 0$, then each of these ratios $= a + c : b + d$; and $x^2 + y^2 + z^2 = yz + zx + xy$.

(25.) If $(a - ny + mz)/l' = (b - lz + nx)/m' = (c - mx + ly)/n'$, then
$$\left(x - \frac{m'c - n'b}{ll' + mm' + nn'}\right)/l = \left(y - \frac{n'a - l'c}{ll' + mm' + nn'}\right)/m = \left(z - \frac{l'b - m'a}{ll' + mm' + nn'}\right)/n.*$$

RATIO AND PROPORTION OF CONCRETE QUANTITIES.

§ 11.] We have now to consider how the theorems we have established regarding the ratio and proportion of abstract numbers are to be applied to concrete quantities. We shall base

* Important in the theory of the central axis of a system of forces, &c.

this application on the theory of units. This, for practical purposes, is the most convenient course, but the student is not to suppose that it is the only one open to us. It may be well to recall once more that any theory may be expressed in algebraical symbols, provided the fundamental principles of its logic are in agreement with the fundamental laws of algebraical operation.

§ 12.] *If* A *and* B *be two concrete quantities of the same kind, which are expressible in terms of one and the same unit by the commensurable numbers a and b respectively, then the ratio of* A *to* B *is defined to be the ratio or quotient of these abstract numbers, namely,* $a:b$, *or* a/b.

It should be observed that, by properly choosing the unit, the ratio of two concrete quantities which are each commensurable with any finite unit at all can always be expressed as the ratio of two integral numbers. For example, if the quantities be lengths of $3\frac{1}{4}$ feet and $4\frac{3}{8}$ feet respectively, then, by taking for unit $\frac{1}{8}$th of a foot, the quantities are expressible by 26 and 35 respectively; and the ratio is $26:35$. This follows also from the algebraical theorem that $(3+\frac{1}{4})/(4+\frac{3}{8})=26/35$.

If A, B *be two concrete quantities of the same kind, whose ratio is* $a:b$, *and* C, D *two other concrete quantities of the same kind (but not necessarily of the same kind as* A *and* B), *whose ratio is* $c:d$, *then* A, B, C, D *are said to be proportional when the ratio of* A *to* B *is equal to the ratio of* C *to* D, *that is, when*

$$a:b=c:d.$$

We may speak of the ratio A : B, of the concrete magnitudes themselves, and of the proportion A : B = C : D, without alluding explicitly to the abstract numbers which measure the ratios; but all conclusions regarding these ratios will, in our present manner of treating them, be interpretations of algebraical results such as we have been developing in the earlier part of this chapter, obtained by operating with a, b, c, d. The theory of the ratio and proportion of concrete quantity is thus brought under the theory of the ratio and proportion of abstract quantities.

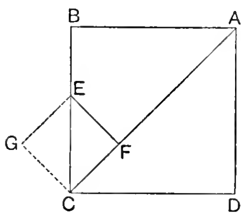
There are, however, several points which require a nearer examination.

§ 13.] In the first place, it must be noticed that in a concrete

ratio the antecedent and the consequent must be quantities of the same kind; and in a concrete proportion the two first terms must be alike in kind, and the two last alike in kind. Thus, from the present point of view at least, there is no sense in speaking of the ratio of an area to a line, or of a ton of coals to a sum of money. Accordingly, some of the propositions proved above—those regarding the permutations of a proportion, for instance—could not be immediately cited as true regarding a proportion among four concrete magnitudes, unless all the four were of the same kind.

This, however, is a mere matter of the interpretation of algebraical formulæ—a matter, in short, regarding the putting of a problem into, and the removing of it from, the algebraical machine.

§ 14.] A more important question arises from the consideration that, if we take two concrete magnitudes of the same kind at random, there is no reason to expect that there exists any unit in terms of which each is exactly expressible by means of commensurable numbers.



Let us consider, for example, the historically famous case of the side AB and diagonal AC of a square ABCD. On the diagonal AC lay off AF = AB, and draw FE perpendicular to AC. It may be readily shown that

$$BE = EF = FC.$$

Hence
$$CF = AC - AB \qquad (1),$$
$$CE = CB - CF \qquad (2).$$

Now, if AB and AC were each commensurably expressible in terms of any finite unit, each would, by the remark in § 12, be an integral multiple of a certain finite unit. But from (1) it follows that if this were so, CF would be an integral multiple of the same unit; and, again, from (2), that CE would be an integral multiple of the same unit. Now CF and CE are the side and diagonal of a square, CFEG, whose side is less than half the side

of ABCD; and from CFEG could in turn be derived a still smaller square whose side and diagonal would be integral multiples of our supposed unit; and so on, until we had a square as small as we please, whose side and diagonal are integral multiples of a finite unit; which is absurd. Hence the side and diagonal of a square are not magnitudes such as A and B are supposed to be in our definition of concrete ratio.

§ 15.] The difficulty which thus arises in the theory of concrete ratio is surmounted as follows :—

We assume, as axiomatic regarding concrete ratio, that if A′ and A″ be two quantities respectively less and greater than A, then the ratio A : B is greater than A′ : B and less than A″ : B; and we show that A′ and A″ can be found such that, while each is commensurable with B, they differ from each other, and therefore each differs from A by as little as we please.

Suppose, in fact, that we take for our unit the nth part of B, then there will be two consecutive integral multiples of B/n, say mB/n and $(m+1)B/n$, between which A will lie. Take these for our values of A′ and A″; then

$$\begin{aligned} A'' - A' &= (m+1)B/n - mB/n, \\ &= B/n. \end{aligned}$$

Hence $A'' - A'$ can, by sufficiently increasing n, be made as small as we please.

We thus obtain, in accordance with the definition of § 12, two ratios, m/n and $(m+1)/n$, between which the ratio A : B lies, each of which may be made to differ from A : B by as little as we please.

Practically speaking, then, we can find for the ratio of two incommensurables an expression which shall be as accurate as we please. Regarding this matter, see vol. ii., chap. xxv., §§ 26-41.

Example.

If B be the side and A the diagonal of a square, to find a rational value of A : B which shall be correct to 1/1000th.

If we take for unit the 1/1000th part of B, then $B=1000$, and $A^2=2{,}000{,}000$. Now $1414^2=1999396$, and $1415^2=2002225$. Hence $1414/1000 < A/B < 1415/1000$. But $1415/1000 - 1414/1000 = 1/1000$. Hence we have $A/B = 1{\cdot}414$, the error being $< 1/1000$.

§ 16.] The theory of proportion given in Euclid's *Elements* gets over the difficulty of incommensurables in a very ingenious although indirect manner. No working definition of a ratio is attempted, but the proportionality of four magnitudes is defined substantially as follows :—

If there be four magnitudes A, B, C, D, such that, always,

$$mA >,\ =,\ \text{or} < nB,$$

according as

$$mC >,\ =,\ \text{or} < nD,$$

m and n being any integral numbers whatsoever, then A, B, C, D are said to be proportional.

Here no use is made of the notion of a unit, so that the difficulty of incommensurability is not raised. On the other hand, there is substituted a somewhat indirect and complicated method for testing the subsistence or non-subsistence of proportionality.

It is easy to see that, if A, B, C, D be proportional according to the algebraical definition, they have the property of Euclid's definition. For, if $a:b$ and $c:d$ be the numerical measures of the ratios A : B and C : D, we have

$$\frac{a}{b}=\frac{c}{d};$$

hence

$$\frac{ma}{nb}=\frac{mc}{nd},$$

from which it follows that $ma >,\ =,$ or $< nb$, according as $mc >,\ =,$ or $< nd$.

The converse, namely, that, if A, B, C, D be proportional according to Euclid's definition, then

$$\frac{a}{b}=\frac{c}{d},$$

can be proved by means of the following lemma.

Given any commensurable quantity a/b, another commensurable quantity can be found which shall exceed or fall short of a/b by as little as we please.

Let n be an integral number, and let mb be the least multiple of b which exceeds na, so that

$$na=mb-r,$$

where $r<b$.

Dividing both sides of this equation by nb, we have

$$\frac{a}{b}=\frac{m}{n}-\frac{r}{nb};$$

whence

$$\frac{m}{n}-\frac{a}{b}=\frac{r}{nb},$$

so that m/n exceeds a/b by r/nb. Now, since r never exceeds the given quantity b, by making n sufficiently great, we can make r/nb as small as we please ; that is to say, we can make m/n exceed a/b by as little as we please.

Similarly we may show that another commensurable quantity may be found falling short of a/b by as little as we please.

From this it follows that, *if two commensurable quantities differ by ever so little, we can always find another commensurable quantity which lies between*

them; for we can find another commensurable quantity which exceeds the less of the two by less than the difference between it and the greater.

Suppose now that

$$ma >, =, \text{ or } < nb,$$

according as

$$mc >, =, \text{ or } < nd,$$

m and n being any integers whatever, then we must have

$$\frac{a}{b}=\frac{c}{d}.$$

For, if these fractions (which we may suppose to be commensurable by virtue of § 15) differ by ever so little, it will be possible to find another fraction, n/m say, where n and m are integers, which lies between them. Hence, if a/b be the less of the two, we must have

$$\frac{a}{b}<\frac{n}{m}, \text{ that is, } ma<nb;$$

$$\frac{c}{d}>\frac{n}{m}, \text{ that is, } mc>nd.$$

In other words we have found two integers, m and n, such that we have at once

$$ma<nb$$

and

$$mc>nd.$$

But, by hypothesis, when $ma<nb$, we must have $mc<nd$. Hence the fractions a/b and c/d cannot be unequal.

"VARIATION."

§ 17.] There are an infinite number of ways in which we may conceive one quantity y to depend upon, be calculable from, or, in technical mathematical language, be a function of, another quantity x. Thus we may have, for example,

$$y = 3x,$$
$$y = 17x^2,$$
$$y = ax + b,$$
$$y = ax^2 + bx + c,$$
$$y = 2\sqrt{x},$$

and so on.

For convenience x is called the *independent variable*, and y the *dependent variable*; because we imagine that any value we please is given to x, and the corresponding value of y derived from it by means of the functional relation. All the other symbols of quantity that occur in the above equations, such as 3, 17, a, b, c,

2, &c., are supposed to remain fixed, and are therefore called *constants*.

Here we attach meanings to the words *variable* and *constant* more in accordance with their use in popular language than those given above (chap. ii., § 6).

The justification of the double usage, if not already apparent, will be more fully understood when we come to discuss the theory of equations, and to consider more fully the variations of functions of various kinds (see chaps. xv.-xviii.)

§ 18.] In the meantime, we propose to discuss very briefly the simplest of all cases of the functional dependence of one quantity upon another, that, namely, which is characterised by the following property.

Let the following scheme

Values of the Independent Variable.	Corresponding Values of the Dependent Variable.
x	y
x'	y'

denote any two corresponding pairs whatever of values of the independent and dependent variables, then the dependence is to be such that always

$$y : y' = x : x' \qquad (1).$$

It is obvious that this property completely determines the nature of the dependence of y upon x, as soon as any single corresponding pair of values are given. Suppose, in fact, that, when x has the value x_0, y has the value y_0, then, by (1),

$$\frac{y}{y_0} = \frac{x}{x_0};$$

whence

$$y = \left(\frac{y_0}{x_0}\right)x.$$

Now we may keep x_0 and y_0 as a fixed standard pair, for reference as it were; their ratio y_0/x_0 is therefore a given con-

stant quantity, which we may denote by a, say. We therefore have

$$y = ax \qquad (2),$$

that is to say, y is a given constant multiple of x; or, in the language of chap. iv., § 17, a homogeneous integral function of x of the 1st degree.

Example. Let us suppose that we have for any two corresponding pairs y, x and y', x' the relation $y:x=y':x'$; and that when $x=3$, $y=6$. Then since 6 and 3 are corresponding pairs $y:x=6:3$. Hence $y/x=6/3=2$. Hence $y=2x$.

Conversely, of course, the property (2) leads to the property (1). For, from (2),

$$y = ax;$$

hence, if x' and y' be other two corresponding values,

$$y' = ax'.$$

Hence
$$\frac{y}{y'} = \frac{ax}{ax'} = \frac{x}{x'}.$$

When y depends on x in the manner just explained it is said *to vary directly as x*, or, more shortly, *to vary as x*.

A better * phrase, which is also in use, is "*y is proportional to x*."

This particular connection between y and x is sometimes expressed by writing

$$y \propto x.$$

§ 19.] In place of x, we might write in equation (2) x^2, $1/x$, $1/x^2$, $x+b$, and so on; we should then have

$$y = ax^2 \qquad (\alpha),$$
$$y = a/x \qquad (\beta),$$
$$y = a/x^2 \qquad (\gamma),$$
$$y = a(x+b) \qquad (\delta).$$

* The use of the word "Variation" in the present connection is unfortunate, because the qualifying particle "as" is all that indicates that we are here concerned not with variation in general, as explained in § 17, but merely with the simplest of all the possible kinds of it. There is a tendency in uneducated minds to suppose that this simplest of all kinds of functionality is the only one; and this tendency is encouraged by the retention of the above piece of antiquated nomenclature.

The corresponding forms of equation (1) would then be

$$y : y' = x^2 : x'^2 \qquad (\alpha'),$$
$$y : y' = 1/x : 1/x' \qquad (\beta'),$$
$$y : y' = 1/x^2 : 1/x'^2 \qquad (\gamma'),$$
$$y : y' = x + b : x' + b \qquad (\delta').$$

y is then said to vary as, or be proportioned to, x^2, $1/x$, $1/x^2$, $x + b$. In cases (β) and (γ) y is sometimes said to vary inversely as x, and inversely as the square of x respectively.

Still more generally, instead of supposing the dependent variable to depend on one independent variable, we may suppose the dependent variable u to depend on two or more independent variables, x, y, z, &c.

For example, we may have, corresponding to (2),

$$u = axy \qquad (\epsilon),$$
$$u = axyz \qquad (\zeta),$$
$$u = a(x + y) \qquad (\eta),$$
$$u = ax/y \qquad (\theta);$$

and, corresponding to (1),

$$u : u' = xy : x'y' \qquad (\epsilon'),$$
$$u : u' = xyz : x'y'z' \qquad (\zeta'),$$
$$u : u' = x + y : x' + y' \qquad (\eta'),$$
$$u : u' = x/y : x'/y' \qquad (\theta').$$

In case (ϵ) u is sometimes said to vary as x and y jointly; in case (θ) directly as x and inversely as y.

§ 20.] The whole matter we are now discussing is to a large extent an affair of nomenclature and notation, and a little attention to these points is all that the student will require to prove the following propositions. We give the demonstrations in one or two specimen cases.

(1.) *If $z \propto y$ and $y \propto x$, then $z \propto x$.*

Proof.—By data $z = ay$, $y = bx$, where a and b are constants; therefore $z = abx$. Hence $z \propto x$, since ab is constant.

(2.) *If $y_1 \propto x_1$ and $y_2 \propto x_2$, then $y_1y_2 \propto x_1x_2$.*

Proof.—By data $y_1 = a_1x_1$, $y_2 = a_2x_2$, where a_1 and a_2 are con-

stants. Hence $y_1y_2 = a_1a_2x_1x_2$, which proves the proposition, since a_1a_2 is constant.

In general if $y_1 \propto x_1$, $y_2 \propto x_2$, . . ., $y_n \propto x_n$, *then* $y_1y_2 \ldots y_n \propto x_1x_2 \ldots x_n$. *And, in particular, if* $y \propto x$, *then* $y^n \propto x^n$.

(3.) *If* $y \propto x$, *then* $zy \propto zx$, *whether* z *be variable or constant.*

(4.) *If* $z \propto xy$, *then* $x \propto z/y$, *and* $y \propto z/x$.

(5.) *If* z *depend on* x *and* y, *and on these alone, and if* $z \propto x$ *when* y *is constant, and* $z \propto y$ *when* x *is constant, then* $z \propto xy$ *when both* x *and* y *vary.*

Proof.—Consider the following system of corresponding values of the variables involved.

Dependent Variable.	Independent Variables.
z	x, y.
z_1	x', y.
z'	x', y'.

Then, since y has the same value for both z and z_1, we have, by data,

$$\frac{z}{z_1} = \frac{x}{x'}.$$

Again, since x' is the same for both z_1 and z', we have, by data,

$$\frac{z_1}{z'} = \frac{y}{y'}.$$

From these two equations we have

$$\frac{z}{z_1} \times \frac{z_1}{z'} = \frac{x}{x'} \times \frac{y}{y'},$$

that is,

$$\frac{z}{z'} = \frac{xy}{x'y'},$$

which proves that $z \propto xy$.

A good example of this case is the dependence of the area of a triangle upon its base and altitude.

We have

$$\text{Area} \propto \text{base (altitude constant)};$$
$$\text{Area} \propto \text{altitude (base constant)}.$$

Hence area $\propto$ base $\times$ altitude, when both vary.

(6.) In a similar manner we may prove that *if z depend on $x_1, x_2, \ldots, x_n$, and on these alone, and vary as any one of these when the rest remain constant, then $z \propto x_1x_2 \ldots x_n$ when all vary.*

(7.) *If $z \propto x$ (y constant) and $z \propto 1/y$ (x constant), then $z \propto x/y$ when both vary.*

For example, if V, P, T denote the volume, pressure, and absolute temperature of a given mass of a perfect gas, then

$$V \propto 1/P \text{ (T constant)}, \; V \propto T \text{ (P constant)}.$$

Hence in general $V \propto T/P$.

Example 1.

If $s \propto t^2$ when f is constant, and $s \propto f$ when t is constant, and $2s = f$ when $t = 1$, find the relation connecting s, f, t.

It follows by a slight extension of § 20 (5) that, when f and t both vary, $s \propto ft^2$. Hence $s = aft^2$, where a is a constant, which we have to determine.

Now, when $t = 1$, $s = \frac{1}{2}f$, hence $\frac{1}{2}f = af1^2$, that is, $\frac{1}{2}f = af$; in other words, we must have $a = \frac{1}{2}$. The relation required is, therefore, $s = \frac{1}{2}ft^2$.

Example 2.

The thickness of a grindstone is unaltered in the using, but its radius gradually diminishes. By how much must its radius diminish before the half of its mass is worn away? Given that the mass varies directly as the square of the radius when the thickness remains unaltered.

Let m denote the mass, r the radius, then by data, $m = ar^2$, where a is constant.

Let now r become r', and, in consequence, m become $\frac{1}{2}m$, then $\frac{1}{2}m = ar'^2$, hence

$$\frac{ar'^2}{ar^2} = \frac{\frac{1}{2}m}{m},$$

that is,

$$\frac{r'^2}{r^2} = \frac{1}{2};$$

whence

$$\frac{r'}{r} = \frac{1}{\sqrt{2}}.$$

It follows, therefore, that the radius of the stone must be diminished in the ratio $1 : \sqrt{2}$.

Example 3.

A and B are partners in a business in which their interests are in the ratio $a : b$. They admit C to the partnership, without altering the whole amount of capital, in such a way that the interests of the three partners in the business are then equal. C contributes £c to the capital of the firm.

How is the sum £c which is withdrawn from the capital to be divided between A and B? and what capital had each in the business originally?

Solution.—Since what C pays in is his share of the capital, they each have finally £c in the business; let now £x be A's share of C's payment, so that £$(c-x)$ is B's share of the same. In effect, A takes £x and B £$(c-x)$ out of the business. Hence they had originally £$(c+x)$ and £$(c+c-x)$ in the business. By data, then, we must have

$$\frac{c+x}{2c-x}=\frac{a}{b},$$

hence $$b(c+x)=a(2c-x);$$

we have, therefore, $$bc+bx=2ac-ax.$$

From this last equation we derive, by adding $ax-bc$ to both sides,

$$(a+b)x=(2a-b)c.$$

Hence, dividing by $a+b$, we have

$$x=\frac{(2a-b)c}{a+b} \qquad (1).$$

Hence $$c-x=c-\frac{(2a-b)c}{a+b},$$

$$=\frac{(2b-a)c}{a+b} \qquad (2).$$

It appears, then, that A and B take £$(2a-b)c/(a+b)$ and £$(2b-a)c/(a+b)$ respectively out of the business. C's payment must be divided between them in the ratio of these sums, that is, in the ratio $2a-b:2b-a$. They had in the business originally £$3ac/(a+b)$ and £$3bc/(a+b)$ respectively.

EXERCISES XVIII.

(1.) If $y \propto x$, and if $y=3\frac{5}{8}$ when $x=6\frac{1}{2}$, find the value of y when $x=\frac{5}{8}$.

(2.) y varies inversely as x^2; and z varies directly as x^2. When $x=2$, $y+z=340$; when $x=1$, $y-z=1275$. For what value of x is $y=z$?

(3.) $z \propto u-v$; $u \propto x$; $v \propto x^2$. When $x=2$, $z=48$; when $x=5$, $z=30$. For what values of x is $z=0$?

(4.) If $xy \propto x^2+y^2$, and $x=3$ when $y=4$, find the equation connecting y and x.

(5.) If $x+y \propto x-y$, then $x^2+y^2 \propto xy$ and $x^3+y^3 \propto xy(x\pm y)$.

(6.) If $(x+y+z)(x+y-z)(x-y+z)(-x+y+z) \propto x^2y^2$, then either $x^2+y^2 \propto z^2$ or $x^2+y^2-z^2 \propto xy$.

(7.) If $x \propto y$, then $x^2+y^2 \propto xy$.

(8.) If $x^3+\dfrac{1}{y^3} \propto x^3-\dfrac{1}{y^3}$, then $y \propto 1/x$.

(9.) If $x \propto y^2$, $y^3 \propto z^4$, $z^5 \propto u^6$, $u^7 \propto v^4$, then $(x/v)(y/v)(z/v)(u/v)$ is constant.

(10.) Two trains take 3 seconds to clear each other when passing in opposite directions, and 35 seconds when passing in the same direction: find the ratio of their velocities.

(11.) A watch loses $2\frac{1}{2}$ minutes per day. It is set right on the 15th March at 1 P.M.: what will the proper time be when it indicates 9 A.M. on the 20th April?

(12.) A small disc is placed between two infinitely small sources of radiant heat of equal intensity, at a point on the line joining them equidistant from the two. It is then moved parallel to itself through a distance $a/2\sqrt{3}$ towards one of the two sources, a being the distance between them: show that the whole radiation falling on the disc is trebled.

(The radiation falling on the disc varies inversely as the square of the distance from the source, when the disc is moved parallel to itself towards or from the source.)

(13.) The radius of a cylinder is r, and its height h. It is found that by increasing either its radius or its height by x its volume is increased by the same amount. Show that $x=r(r-2h)/h$. What condition is there upon r and h in order that the problem may be possible?

(Given that the volume of a cylinder varies directly as its height when its radius is constant, and directly as the square of its radius when its height is constant.)

(14.) A solid spherical mass of glass, 1 inch in diameter, is blown into a shell bounded by two concentric spheres, the diameter of the outer one being 3 inches. Calculate the thickness of the shell. (The volume of a sphere varies directly as the cube of its diameter.)

(15.) Find the radius of a sphere whose volume is the sum of the volumes of two spheres whose radii are $3\frac{1}{2}$ feet and 6 feet respectively.

(16.) Two equal vessels contain spirits and water, the ratios of the amount of spirit to the amount of water being $\rho:1$ and $\rho':1$ respectively. The contents of the two are mixed: show that the ratio of the amount of spirit to the amount of water in the mixture is $\rho+\rho'+2\rho\rho':2+\rho+\rho'$.

CHAPTER XIV.

On Conditional Equations in General.

DEFINITIONS AND GENERAL NOTIONS.

§ 1.] It will be useful for the student at this stage to attempt to form a wider conception than we have hitherto presupposed of what is meant by an *analytical function* in general. Dividing the subjects of operation into variables (x, y, z, . . .) and constants (a, b, c, . . .), we have already seen what is meant by a rational integral algebraical function of the variables x, y, z, . . .; and we have also had occasion to consider rational fractional algebraical functions of x, y, z, . . . We saw that in distinguishing the nature of such functions attention was paid to the way in which the *variables alone* were involved in the function. We have already been led to consider functions like $\sqrt{(x + \sqrt{y})}$, or $\sqrt[3]{(x + \sqrt[5]{y})}$, or $ax^{\frac{3}{2}} + bx^{\frac{1}{2}} + c$, where the variables are involved by way of root extraction. Such functions as these are called irrational algebraical functions. These varieties exhaust the category of what are usually called *Ordinary* Algebraical Functions*. In short, *any intelligible concatenation of operations, in which the operands selected for notice and called the variables are involved in no other ways than by addition, subtraction, multiplication, division, and root extraction, is called an Ordinary Algebraical Function of these variables.*

Although we have thus exhausted the category of ordinary algebraical functions, we have by no means exhausted the possi-

* The adjective "Ordinary" is introduced to distinguish the class of functions here defined from algebraical functions as more generally defined in chap. xxx., § 10. The word "Synthetic" is often used for "Ordinary" in the present connection.

bilities of analytical expression. Consider for example a^x, where, as usual, x denotes a variable and a a constant. Here x is not involved in any of the ways recognised in the definition of an algebraical function, but appears as an index or exponent. a^x is therefore called an *exponential function* of x. It should be carefully noted that the discrimination turns solely on the way in which the *variable* enters. Thus, while a^x is an exponential function of x, x^a is an algebraical function of x. There are other functions in ordinary use,—for example, $\sin x$, $\log x$,—and an infinity besides that might be imagined, which do not come under the category of *algebraical*; all such, for the present, we class under the general title of *transcendental functions*, so that transcendental simply means non-algebraical. We use the term *analytical function*, or simply *function*, to include all functions, whether algebraical or transcendental, and we denote a function of the variables $x, y, z, \ldots$, in which the constants $a, b, c, \ldots$ are also involved, by

$$\phi(x, y, z, \ldots a, b, c, \ldots);$$

or, if explicit mention of the constants is unnecessary, by

$$\phi(x, y, z, \ldots).$$

§ 2.] Consider any two functions whatever, say $\phi(x, y, z, \ldots a, b, c, \ldots)$, and $\psi(x, y, z, \ldots a, b, c, \ldots)$, of the variables $x, y, z, \ldots$, involving the constants $a, b, c, \ldots$

If the equation

$$\phi(x, y, z, \ldots a, b, c, \ldots) = \psi(x, y, z, \ldots a, b, c, \ldots) \quad (1)$$

be such that the left-hand side can, *for all values of the variables* $x, y, z, \ldots$, be transformed into the right by merely applying the fundamental laws of algebra, it is called an *identity*. With equations of this kind the student is already very familiar.

If, on the other hand, the left-hand side of the equation (1) can be transformed into the right *only when* $x, y, z, \ldots$ *have certain values*, or *are conditioned in some way*, then it is said to be a *Conditional Equation*, or an *Equation of Condition*.* Examples

* When it is necessary to distinguish between an equation of identity and an equation of condition, the sign $\equiv$ is used for the former, and the sign $=$ for the latter. Thus, we should write $(x+1)(x-1) \equiv x^2 - 1$; but $2x + 2 = 2$.

of such equations have already occurred, more especially in chap. xiii. One of the earliest may be seen in chap. iv., § 24, where, *inter alia*, it was required to determine B so that we should have $2B + 2 = 2$; in other words, to find a value of x to satisfy the equation

$$2x + 2 = 2 \qquad (2).$$

Here $2x + 2$ can be transformed into 2 when (and, as we shall hereafter see, only when) $x = 0$.

Every determinate problem, wherein it is required to determine certain unknown quantities in terms of certain other given or known quantities by means of certain given conditions, leads, when expressed in analytical language, to one or more equations of condition; to as many equations, in fact, as there are conditions. The quantities involved are therefore divided into two classes, *known* and *unknown*. The known quantities are denoted by the so-called constant letters; the unknown by the variable letters. Hence, in the present chapter, *constant* and *known* are convertible terms; and so are *variable* and *unknown*. The constants may be actual numerical quantities, real and positive or negative (-4, $-\frac{1}{2}$, 0, $+1$, $+\frac{3}{2}$, &c.), or imaginary or complex numbers ($-i$, $1 + 2i$, &c.); or they may be letters standing for any such quantities in general.

§ 3.] Equations are classified according to their form, and according to the number of variables that occur in them.

If transcendental functions appear, as, for example, in $2^x = 3^x + 2$, the equation is said to be *transcendental*. With such for the present we shall have little to do.

If only the ordinary algebraical functions appear, as, for example, in $\surd(x + y) + \surd(x - y) = 1$, the equation is called an *algebraical equation*. Such an equation may, of course, be rational or irrational, and, if rational, either fractional or integral, according to circumstances.

It will be shown presently that every algebraical equation can be connected with, or made to depend upon, an equation of the form

$$\phi(x, y, z, \ldots) = 0,$$

where ϕ is a rational integral function. Such equations are therefore of great analytical importance; and it is to them that the "Theory of Equations," as ordinarily developed, mainly applies. An integral equation of this kind is described by assigning its *degree* and the number of its variables. The degree of the equation is simply the degree of the function ϕ. Thus, $x^2 + 2xy + y^2 - 2 = 0$ is said to be an equation of the 2nd degree in two variables.

§ 4.] Equations of condition may occur in sets of one or of more than one. In the latter case we speak of the set as a set or system of *simultaneous equations.*

The main problem which arises in connection with every system of equations of condition is to find a set or sets of values of the variables which shall render every equation of the system an identity literal or numerical.

Such a set of values of the variables is said to *satisfy* the system, and is called a *solution* of the system of equations. If there be only one equation, and only one variable, a value of that variable which satisfies the equation is called a *root.* We also say that a solution of a system of equations *satisfies the system,* meaning that it renders each equation of the system an identity.

It is important to distinguish between two very different kinds of solution. When the values of the variables which constitute the solution are closed expressions, that is, functions of known form of the constants in the given equations, we have what may be called a *formal solution* of the system of equations. In particular, if these values be ordinary algebraical functions of the constants, we have an *algebraical solution.* Such solutions cannot in general be found. In the case of integral algebraical equations of one variable, for example, if the degree exceed the 4th, it has been shown by Abel and others that algebraical solutions do not exist except in special cases, so that the formal solution, if it could be found, would involve transcendental functions.

When the values of the variables which constitute the solution are given *approximately* as numbers, real or complex, the solution is said to be an *approximate numerical* solution. In this case the

words "render the equation a numerical identity" are understood to mean "reduce the two sides of the equation to values which shall differ by less than some quantity which is assigned." For example, if *real* values of the two sides, say P and P′, are in question, then these must be made to differ by less than some given small quantity, say 1/100,000; if complex values are in question, say $P+Qi$ and $P'+Q'i$, then these must be so reduced that the modulus of their difference, namely, $\sqrt{\{(P-P')^2+(Q-Q')^2\}}$, shall be less than some given small quantity, say 1/100,000. (Cf. chap. xii., § 21.)

As a matter of fact, numerical solutions can often be obtained where formal solutions are out of the question. Integral algebraical equations, for example, can always be solved numerically to any desired approximation, no matter what their degree.

Example 1.

$$2x+2=2.$$

$x=0$ is a solution, for this value of x reduces the equation to

$$2\times 0+2=2,$$

which is a numerical identity. Strictly speaking, this is a case of algebraical solution.

Example 2.

$$ax-b^2=0.$$

$x=b^2/a$ reduces the equation to

$$a\frac{b^2}{a}-b^2=0,$$

which is a literal identity; hence $x=b^2/a$ is an algebraical solution.

Example 3.

$$x^2-2=0.$$

Here $x=+\sqrt{2}$ and $x=-\sqrt{2}$ each reduce the equation to the identity

$$2-2=0;$$

these therefore are two algebraical solutions.

On the other hand, $x=+1{\cdot}4142$ and $x=-1{\cdot}4142$ are approximate numerical solutions, for each of them reduces x^2-2 to $-{\cdot}00003836$, which differs from 0 by less than ·00004.

Example 4.

$$(x-1)^2+2=0.$$

$x=1+\sqrt{2}i$ and $x=1-\sqrt{2}i$ are algebraical solutions, as the student will easily verify.

$x=1{\cdot}0001+1{\cdot}4142i$ and $x=1{\cdot}0001-1{\cdot}4142i$ are approximate numerical solutions, for they reduce $(x-1)^2+2$ to ${\cdot}00003837+{\cdot}00028284i$ and ${\cdot}00003837-{\cdot}00028284i$ respectively, complex numbers whose moduli are each less than ·0003.

Example 5.

$$x-y=1.$$

Here $x=1$, $y=0$, is a solution; so is $x=1\cdot 5$, $y=\cdot 5$; so is $x=2$, $y=1$; and, in fact, so is $x=a+1$, $y=a$, where a is any quantity whatsoever.

Here, then, there are an infinite number of solutions.

Example 6. Consider the following system of two equations:—

$$x-y=1, \quad 2x+y=5.$$

Here $x=2$, $y=1$ is a solution; and, as we shall show in chap. xvi., there is no other.

The definition of the solution of a conditional equation suggests two remarks of some importance.

1*st*. *Every conditional equation is a hypothetical identity. In all operations with the equation we suppose the variables to have such values as will render it an identity.*

2*nd*. *The ultimate test of every solution is that the values which it assigns to the variables shall satisfy the equations when substituted therein.*

No matter how elaborate or ingenious the process by which the solution has been obtained, if it do not stand this test, it is no solution; and, on the other hand, no matter how simply obtained, provided it do stand this test, it is a solution.* In fact, as good a way of solving equations as any other is to guess a solution and test its accuracy by substitution.†

§ 5.] The consideration of particular cases, such as Examples 1-6 of § 4, teaches us that the number of solutions of a system of one or more equations may be finite or infinite. If the number be finite, we say that the solution is *determinate* (singly determinate, or multiply determinate according as there are one or more solutions); if there be a continuous infinity of solutions, we say that the solution is *indeterminate*.

The question thus arises, Under what circumstances is the solution of a system of equations determinate? Part at least of the answer is given by the following fundamental propositions.

Proposition I. *The solution of a system of equations is in general determinate (singly, or multiply according to circumstances) when the number of the equations is equal to the number of the variables.*

* A little attention to these self-evident truths would save the beginner from many a needless blunder.

† This is called solving by "inspection."

Rightly considered, this is an ultimate logical principle which may be discussed, but not in any strictly general sense proved. Let us illustrate by a concrete example. The reader is aware that a rectilinear triangle is determinable in a variety of ways by means of three elements, and that consequently three conditions will in general determine the figure. To translate this into analytical language, let us take for the three determining elements the three sides, whose lengths, at present unknown, we denote by x, y, z respectively. Any three conditions upon the triangle may be translated into three equations connecting x, y, z with certain given or constant quantities; and these three equations will in general be sufficient to determine the three variables, x, y, z. The general principle * common to this and like cases is simply Proposition I. The truth is that this proposition stands less in need of proof than of limitation. What is wanted is an indication of the circumstances under which it is liable to exception. To return to our particular case: What would happen, for example, if one of the conditions imposed upon our triangle were that the sum of two of the sides should fall short of the third by a given positive quantity? This condition could be expressed quite well by an equation (namely, $x + y = z - q$, say), but it is fulfilled by no real triangle.† Again, it might chance that the last of the three given conditions was merely a consequence of the two first. We should then have in reality only two conditions—that is to say, analytically speaking, it might chance that the last of the three equations was merely one derivable from the two first, and then there would be an infinite number of solutions of the system of three variables. Such a system is

$$\begin{aligned} x + y + z &= 6, \\ 3x + 2y + z &= 10, \\ 2x + y &= 4, \end{aligned}$$

for example, for, as the reader may easily verify, it is satisfied by $x = a - 2$, $y = 8 - 2a$, $z = a$, where a is any quantity whatsoever.

* A name seems to be required for this all-pervading logical principle: the *Law of Determinate Manifoldness* might be suggested.

† See below, chap. xix.

It will be seen in following chapters how these difficulties are met in particular cases. Meantime, let us observe that, if we admit Proposition I., two others follow very readily.

Proposition II. *If the number of equations be less than the number of variables, the solution is in general indeterminate.*

Proposition III. *If the number of independent equations be greater than the number of variables, there is in general no solution, and the system of equations is said to be inconsistent.*

For, let the number of variables be n, and the number of equations m, say, where $m < n$. Let us assign to the first $n-m$ variables any set of values we please, and regard these as constant. This we may do in an infinity of ways. If we substitute any such set of values in the m equations, we have now a set of m equations to determine the last m variables; and this, by Proposition I., they will do determinately. In other words, for every set of values we like to give to the first $n-m$ variables, the m equations give us a determinate set of values for the last m. We thus get an infinite number of solutions; that is, the solution is indeterminate.

Next, let m be $> n$. If we take the first n equations, these will in general, by Proposition I., give a determinate set, or a finite number of determinate sets of values for all the n variables. If we now take one of these sets of values, and substitute it in one of the remaining $m-n$ equations, that equation will not in general be satisfied; for, if we take an equation at random, and a solution at random, the latter will not in general fit the former. The system of m equations will therefore in general be inconsistent.

It may, of course, happen, *in exceptional cases*, that this proposition does not hold; witness the following system of three equations in two variables:—

$$x - y = 1, \quad 2x + y = 5, \quad 3x + 2y = 8,$$

which has the common solution $x = 2$, $y = 1$.

§ 6.] We have also the further question, When the system is determinate, how many solutions are there? The answer to this, in the case of integral equations, is furnished by the two following propositions:—

Proposition I. *An integral equation of the nth degree in one variable has n roots and no more, which may be real or complex, and all unequal or not all unequal, according to circumstances.*

Proposition II. *A determinate system of m integral equations with m variables, whose degrees in these variables are p, q, r, . . . respectively, has, at most, pqr . . . solutions, and has, in general, just that number.*

Proposition I. was proved in the chapter on complex numbers, where it was shown that for any given integral function of x of the nth degree there are just n values of x and no more that reduce that function to zero, these values being real or complex, and all unequal or not as the case may be.

Proposition II. will not be proved in this work, except in particular cases which occur in chapters to follow. General proofs will be found in special treatises on the theory of equations. We set it down here because it is a useful guide to the learner in teaching him how many solutions he is to expect. It will also enable him, occasionally, to detect when a system is indeterminate, for, if a number of solutions be found exceeding that indicated by Proposition II., then the system is certainly indeterminate, that is to say, has an infinite number of solutions.

Example. The system $x^2+y^2=1$, $x-y=1$ has, by Proposition II., $2\times1=2$ solutions. As a matter of fact, these solutions are $x=0$, $y=-1$, and $x=1$, $y=0$.

EQUIVALENCE OF SYSTEMS OF EQUATIONS.

§ 7.] *Two systems of equations,* A *and* B *(each of which may consist of one or more equations), are said to be equivalent when every solution of* A *is a solution of* B, *and every solution of* B *a solution of* A.

From any given system, A, of equations, we may in an infinity of ways deduce another system, B; but it will not necessarily be the case that the two systems are equivalent. In other words, we may find in an infinity of ways a system, B, of equations which will be satisfied by all values of the

variables for which A is satisfied; but it will not follow conversely that A will be satisfied for all values for which B is satisfied. To take a very simple example, $x-1=0$ is satisfied by the value $x=1$, and by no other; $x(x-1)=0$ is satisfied by $x=1$, that is to say, $x(x-1)=0$ is satisfied when $x-1=0$ is satisfied. On the other hand, $x(x-1)=0$ is satisfied either by $x=0$ or by $x=1$, therefore $x-1=0$ is not always satisfied when $x(x-1)=0$ is so; for $x=0$ reduces $x-1$ to -1, and not to 0. Briefly, $x(x-1)=0$ may be derived from $x-1=0$, but is not equivalent to $x-1=0$.

$x(x-1)=0$ is, in fact, more than equivalent to $x-1=0$, for it involves $x-1=0$ and $x=0$ as alternatives. It will be convenient in such cases to say that $x(x-1)=0$ is equivalent to

$$\left\{\begin{array}{r} x=0 \\ x-1=0 \end{array}\right\}.$$

When by any step we derive from one system another which is exactly equivalent, we may call that step a *reversible* derivation, because we can make it backwards without fallacy. If the derived system is not equivalent, we may call the step *irreversible*, meaning thereby that the backward step requires examination.

There are few parts of algebra more important than the logic of the derivation of equations, and few, unhappily, that are treated in more slovenly fashion in elementary teaching. No mere blind adherence to set rules will avail in this matter; while a little attention to a few simple principles will readily remove all difficulty.

It must be borne in mind that in operating with conditional equations we always *suppose* the variables to have such values as will render the equations identities, although we may not at the moment actually substitute such values, or even know them. *We are therefore at every step, hypothetically at least, applying the fundamental laws of algebraical transformation just as in chap. i.*

The following general principle, already laid down for real quantities, and carefully discussed in chap. xii., § 12, for complex quantities, may be taken as the root of the whole matter.

Let P *and* Q *be two functions of the variables* x, y, z, . . ., *which do not become infinite* * *for any values of those variables that we have to consider. If* $P \times Q = 0$ *and* $Q \neq 0$, *then will* $P = 0$, *and if* $P \times Q = 0$ *and* $P \neq 0$, *then will* $Q = 0$.

Otherwise, *the only values of the variables which make* $P \times Q = 0$ *are such as make either* $P = 0$, *or* $Q = 0$, *or both* $P = 0$ *and* $Q = 0$.

§ 8.] It follows by the fundamental laws of algebra *that if*

$$P = Q \qquad (1),$$

then

$$P \pm R = Q \pm R \qquad (2),$$

where R *is either constant or any function of the variables.*

We shall show that *this derivation is reversible.*

For, if
$$P \pm R = Q \pm R,$$
then
$$P \pm R \mp R = Q \pm R \mp R,$$
that is,
$$P = Q;$$
in other words, if (2) holds so does (1).

Cor. 1. *If we transfer any term in an equation from the one side to the other, at the same time reversing its sign of addition or subtraction, or if we reverse all the signs on both sides of an equation, we deduce in each case an equivalent equation.*

For, if
$$P + Q = R + S, \text{ say},$$
then
$$P + Q - S = R + S - S,$$
that is,
$$P + Q - S = R.$$

Again, if
$$P + Q = R + S,$$
then
$$P + Q - P - Q - R - S = R + S - P - Q - R - S,$$
that is,
$$-R - S = -P - Q,$$
or
$$-P - Q = -R - S.$$

Cor. 2. *Every equation can be reduced to an equivalent equation of the form—*

$$R = 0.$$

For, if the equation be $P = Q$,

* In all that follows all functions of the variables that appear are supposed not to become infinite for any values of the variables contemplated. Cases where this understanding is violated must be considered separately.

we have $$P - Q = Q - Q,$$
that is $$P - Q = 0,$$
which is of the form $$R = 0.$$

Example.
$$-3x^3 + 2x^2 + 3x = x^2 - x - 3.$$
Subtracting $x^2 - x - 3$ from both sides, we have the equivalent equation
$$-3x^3 + x^2 + 4x + 3 = 0.$$
Changing all the signs, we have
$$3x^3 - x^2 - 4x - 3 = 0.$$
In this way an integral equation can always be arranged with all its terms on one side, so that the coefficient of the highest term is positive.

§ 9.] It follows from the fundamental laws of algebra that

if $$P = Q \qquad (1),$$
then $$PR = QR \qquad (2),$$
the step being reversible if R *is a constant differing from* 0, *but not if* R *be a function of the variables.**

For, if $$PR = QR,$$
an equivalent equation is, by § 8,
$$PR - QR = 0 \qquad (3),$$
that is, $$(P - Q)R = 0 \qquad (4).$$
Now, if R be a constant $\neq 0$, it will follow from (4), by the general principle of § 7, that
$$P - Q = 0,$$
which is equivalent to $$P = Q.$$
But, if R be a function of the variables, (4) may also be satisfied by values of the variables that satisfy
$$R = 0 \qquad (5);$$
and such values will not in general satisfy (1).

In fact, (2) is equivalent, not to (1), but to (1) and (5) as alternatives.

---

* This is spoken of as "multiplying the equation by R." Similarly the process of § 8 is spoken of as "adding or subtracting R to or from the equation." This language is not strictly correct, but is so convenient that we shall use it where no confusion is to be feared.

Cor. 1. From the above it follows that *dividing both sides of an equation by any function other than a constant not equal to zero is not a legitimate process of derivation, since we may thereby lose solutions.*

Thus $PR = QR$ is equivalent to $\left\{\begin{array}{ll} P - Q = 0 \\ R \qquad = 0 \end{array}\right\}$;

whereas $$PR/R = QR/R^*$$

gives $$P = Q,$$

which is equivalent merely to

$$P - Q = 0.$$

Example. If we divide both sides of the equation

$$(x-1)x^2 = 4(x-1) \qquad (\alpha)$$

by $x-1$, we reduce it to $$x^2 = 4 \qquad (\beta),$$

which is equivalent to $(x-2)(x+2)=0$.

(α), on the other hand, is equivalent to

$$(x-1)(x-2)(x+2)=0.$$

Hence (α) has the three solutions $x=1$, $x=2$, $x=-2$; while (β) has only the two $x=2$, $x=-2$.

Cor. 2. To multiply or divide both sides of an equation by any constant quantity differing from zero is a reversible process of derivation. Hence, *if the coefficients of an integral equation be fractional either in the algebraical or in the arithmetical sense, we can always find an equivalent equation in which the coefficients are all integral, and have no common measure.*

Also, *we can always so arrange an integral equation that the coefficient of any term we please, say the highest, shall be* $+1$.

Example 1.

$$\frac{3x+2}{4} + \frac{6x+3}{5} = \frac{2x+4}{8}$$

gives, on multiplying both sides by 40,

$$10(3x+2)+8(6x+3)=5(2x+4),$$

that is, $$30x+20+48x+24=10x+20,$$

whence, after subtracting $10x+20$ from both sides,

$$68x+24=0;$$

---

* As we are here merely establishing a negative proposition, the reader may, to fix his ideas, assume that all the letters stand for integral functions of a single variable.

whence again, after division of both sides by 68,

$$x+\frac{6}{17}=0.$$

Example 2.

$$\left(\frac{p+q}{q}x+\frac{p}{p-q}y\right)\left(\frac{p-q}{p}x+\frac{q}{p+q}y\right)=2xy.$$

If we multiply both sides by $pq(p-q)(p+q)$, that is, by $pq(p^2-q^2)$, we derive the equivalent equation

$$\{(p^2-q^2)x+pqy\}\{(p^2-q^2)x+pqy\}=2pq(p^2-q^2)xy,$$

that is, $$(p^2-q^2)^2x^2+2pq(p^2-q^2)xy+p^2q^2y^2=2pq(p^2-q^2)xy,$$

which is equivalent to $$(p^2-q^2)^2x^2+p^2q^2y^2=0.$$

Cor. 3. *From every rational algebraical equation an integral equation can be deduced; but it is possible that extraneous solutions may be introduced in the process.*

Suppose we have $$\mathrm{P}=\mathrm{Q} \qquad (\alpha),$$
where P and Q are rational, but not integral. Let L be the L.C.M. of the denominators of all the fractions that occur either in P or in Q, then LP and LQ are both integral. Hence, if we multiply both sides of (α) by L, we deduce the integral equation

$$\mathrm{LP}=\mathrm{LQ} \qquad (\beta).$$

Since, however, the multiplier L contains the variables, it is possible that some of the solutions of $\mathrm{L}=0$ may satisfy (β), and such solutions would in general be extraneous to (α). We say *possible*; in general, however, this will not happen, because P and Q contain fractions whose denominators are factors in L. Hence the solutions of $\mathrm{L}=0$ will in general make either P or Q infinite, and therefore $(\mathrm{P}-\mathrm{Q})\mathrm{L}$ not necessarily zero. The point at issue will be best understood by studying the two following examples :—

Example 1.

$$2x-3+\frac{x^2-6x+8}{x-2}=\frac{x-2}{x-3} \qquad (\alpha).$$

If we multiply both sides by $(x-2)(x-3)$, we deduce the equation

$$(2x-3)(x-2)(x-3)+(x^2-6x+8)(x-3)=(x-2)^2 \qquad (\beta),$$

which is integral, and is satisfied by any solution of (α). We must, however, examine whether any of the solutions of $(x-2)(x-3)=0$ satisfy (β). These solutions are $x=2$ and $x=3$. The second of these obviously does not satisfy (β), and need not be considered; but $x=2$ does satisfy (β), and we must examine (α) to see whether it satisfies that equation also.

Now, since $x^2-6x+8\equiv(x-2)(x-4)$, (α) may be written in the equivalent form

$$2x-3+x-4=\frac{x-2}{x-3},$$

which is obviously not satisfied by $x=2$.

It appears, therefore, that in the process of integralisation we have introduced the extraneous solution $x=2$.

Example 2.

$$2x-3+\frac{2x^2-6x+8}{x-2}=\frac{x-2}{x-3} \qquad (\alpha').$$

Proceeding as before, we deduce

$$(2x-3)(x-2)(x-3)+(2x^2-6x+8)(x-3)=(x-2)^2 \qquad (\beta').$$

It will be found that neither of the values $x=2$, $x=3$ satisfies (β'). Hence no extraneous solutions have been introduced in this case.

N.B.—The reason why $x=2$ satisfies (β) in Example 1 is that the numerator x^2-6x+8 of the fraction on the left contains the factor $x-2$ which occurs in the denominator.

Cor. 4. *Raising both sides of an equation to the same integral power is a legitimate, but not a reversible, process of derivation.*

The equation $$P=Q \qquad (1)$$
is equivalent to $$P-Q=0 \qquad (2).$$

If we multiply by $P^{n-1}+P^{n-2}Q+P^{n-3}Q^2+\ldots+Q^{n-1}$, we deduce from (2)

$$P^n-Q^n=0 \qquad (3),$$

which is satisfied by any solution of (1); (3), however, is not equivalent to (1), but to

$$\left\{\begin{array}{r} P=Q \\ P^{n-1}+P^{n-2}Q+\ldots+Q^{n-1}=0 \end{array}\right\}.$$

It will be observed that, if we start with an equation in the standard form $P-Q=0$, transfer the part Q to the right-hand side, and then raise both sides to the nth power, the result is the same as if we had multiplied both sides of the equation in its original form by a certain factor. To make the introduction of extraneous factors more evident we chose the latter process; but in practice the former may happen to be the more convenient.*

If the reader will reflect on the nature of the process described in chap. x. for rationalising an algebraical function by means of a rationalising factor, he will see that by repeated operations of this kind every algebraical equation can be reduced to a rational

* See below, § 12, Example 3.

form; but at each step extraneous solutions may be introduced. Hence

Cor. 5. *From every algebraical equation we can derive a rational integral equation, which will be satisfied by any solution of the given equation; but it does not follow that every solution, or even that any solution, of the derived equation will satisfy the original one.*

Example 1. Consider the equation

$$\sqrt{(x+1)}+\sqrt{(x-1)}=1 \qquad (\alpha),$$

where the radicands are supposed to be real and the square root to have the positive sign.*

(α) is equivalent to $\quad\sqrt{(x+1)}=1-\sqrt{(x-1)},$

whence we derive, by squaring,

$$x+1=1+x-1-2\sqrt{(x-1)},$$

which is equivalent to $\quad 1=-2\sqrt{(x-1)}.$

From this last again, by squaring, we derive

$$1=4(x-1),$$

which is equivalent to the integral equation

$$4x-5=0 \qquad (\beta),$$

the only solution of which, as we shall see hereafter, is $x=\frac{5}{4}$.

It happens here that $x=\frac{5}{4}$ is not a solution of (α), for $\sqrt{(\frac{5}{4}+1)}+\sqrt{(\frac{5}{4}-1)}=\frac{3}{2}+\frac{1}{2}=2$.

Example 2.

$$\sqrt{(x+1)}-\sqrt{(x-1)}=1 \qquad (\alpha').$$

Proceeding exactly as before we have

$$x+1=1+x-1+2\sqrt{(x-1)},$$
$$1=+2\sqrt{(x-1)},$$
$$1=4(x-1),$$
$$4x-5=0 \qquad (\beta'),$$

Here (β′) gives $x=\frac{5}{4}$, which happens this time to be a solution of the original equation.

We conclude this discussion by giving two propositions applicable to systems of equations containing more than one equation. These by no means exhaust the subject; but, as our object here is merely to awaken the intelligence of the student, the rest may be left to himself in the meantime.

§ 10.] *From the system*

$$P_1=0, \quad P_2=0, \quad \ldots, \quad P_n=0 \qquad (A)$$

we derive

$$L_1P_1+L_2P_2+\ldots+L_nP_n=0, \quad P_2=0, \quad \ldots, \quad P_n=0 \quad (B),$$

and the two will be equivalent if L_1 *be a constant differing from* 0.

---

* When $\sqrt{x}$ is imaginary, its "principal value" (see chap. xxix.) ought to be taken, unless it is otherwise indicated.

Any solution of the system (A) reduces $P_1, P_2, \ldots, P_n$ all to 0, and therefore reduces $L_1P_1 + L_2P_2 + \ldots + L_nP_n$ to 0, and hence satisfies (B).

Again, any solution of (B) reduces $P_2, P_3, \ldots, P_n$ all to 0, and therefore reduces $L_1P_1 + L_2P_2 + \ldots + L_nP_n = 0$ to $L_1P_1 = 0$, that is to say, if L_1 be a constant $\neq 0$, to $P_1 = 0$. Hence, in this case, any solution of (B) satisfies (A).

If L_1 contain the variables, then (B) is equivalent, not to (A) simply, but to

$$\begin{Bmatrix} P_1 = 0, & P_2 = 0, & \ldots, & P_n = 0 \\ L_1 = 0, & P_2 = 0, & \ldots, & P_n = 0 \end{Bmatrix}.$$

As a particular case of the above, we have that the two systems

$$P = Q, \qquad R = S;$$

and

$$P + R = Q + S, \; R = S$$

are equivalent. For these may be written

$$\overline{P - Q} = 0, \; \overline{R - S} = 0;$$

$$\overline{P - Q} + \overline{R - S} = 0, \; \overline{R - S} = 0.$$

If l, l', m, m' be constants, any one of which may be zero, but which are such that $lm' - l'm \neq 0$, then the two systems

$$U = 0, \; U' = 0,$$

and

$$lU + l'U' = 0, \; mU + m'U' = 0$$

are equivalent.

The proof is left to the reader. A special case is used and demonstrated in chap. xvi., § 4.

§ 11.] *Any solution of the system*

$$P = Q, \quad R = S \tag{A}$$

is a solution of the system

$$PR = QS, \; R = S \tag{B};$$

but the two systems are not equivalent.

From $P = Q$, we derive

$$PR = QR,$$

which, since $R = S$, is equivalent to

$$PR = QS.$$

It follows therefore that any solution of (A) satisfies (B).

Starting now with (B), we have

$$\mathrm{PR} = \mathrm{QS} \qquad (1),$$

$$\mathrm{R} = \mathrm{S} \qquad (2).$$

Since $\mathrm{R} = \mathrm{S}$, (1) becomes

$$\mathrm{PR} = \mathrm{QR},$$

which is equivalent to

$$(\mathrm{P} - \mathrm{Q})\mathrm{R} = 0,$$

that is, equivalent to

$$\left\{\begin{array}{r} \mathrm{P} - \mathrm{Q} = 0 \\ \mathrm{R} = 0 \end{array}\right\}.$$

Hence the system (B) is equivalent to

$$\left\{\begin{array}{l} \mathrm{P} = \mathrm{Q},\ \mathrm{R} = \mathrm{S} \\ \mathrm{R} = 0,\ \mathrm{R} = \mathrm{S} \end{array}\right\}.$$

that is to say, to

$$\left\{\begin{array}{l} \mathrm{P} = \mathrm{Q},\ \mathrm{R} = \mathrm{S} \\ \mathrm{R} = 0,\ \mathrm{S} = 0 \end{array}\right\}.$$

In other words, (B) involves, besides (A), the alternative system,

$$\mathrm{R} = 0, \quad \mathrm{S} = 0.$$

Example. From $x - 2 = 1 - y, \ x = 1 + y,$
a system which has the single solution $x = 2, \ y = 1$, we derive the system

$$x(x-2) = 1 - y^2, \ x = 1 + y,$$

which, in addition to the solution $x = 2, \ y = 1$, has also the solution $x = 0, \ y = -1$ belonging to the system

$$x = 0, \ 1 + y = 0.$$

§ 12.] In the process of solving systems of equations, one of the most commonly-occurring requirements is to deduce from two or more of the equations another that shall not contain certain assigned variables. This is called "*eliminating* the variables in question *between* the equations used for the purpose." In performing the elimination we may, of course, use any legitimate process of derivation, but strict attention must always be paid to the question of equivalence.

Example. Given the system

$$x^2 + y^2 = 1 \qquad (1),$$

$$x + y = 1 \qquad (2),$$

it is required to eliminate y, that is, to deduce from (1) and (2) an equation involving x alone.

(2) is equivalent to

$$y=1-x.$$

Hence (1) is equivalent to

$$x^2+(1-x)^2=1,$$

that is to say, to

$$2x^2-2x=0,$$

or, if we please, to

$$x^2-x=0\ ;$$

and thus we have eliminated y, and obtained an equation in x alone.

The method we have employed (simply substitution) is, of course, only one among many that might have been selected.

Observe that, as a result of our reasoning, we have that the system (1) and (2) is equivalent to the system

$$x^2-x=0 \qquad (3),$$
$$x+y=1 \qquad (4),$$

from which the reader will have no difficulty in deducing the solution of the given system.

§ 13.] Although, as we have said, the solution of a system of equations is the main problem, yet the reader will learn, especially when he comes to apply algebra to geometry, that much information—very often indeed all the information that is required—may be derived from a system without solving it, but merely by throwing it into various equivalent forms. The derivation of equivalent systems, elimination, and other general operations with equations of condition have therefore an importance quite apart from their bearing on ultimate solution.

We have appended to this chapter a large number of exercises in this branch of algebra, keeping exercises on actual solution for later chapters, which deal more particularly with that part of the subject. The student should work a sufficient number of the following sets to impress upon his memory the general principles of the foregoing chapter, and reserve such as he finds difficult for occasional future practice.

The following are worked out as specimens of various artifices for saving labour in calculations of the present kind :—

Example 1. Reduce the following equation to an integral form :—

$$\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax+b}{px+q} \qquad (\alpha).$$

We may write (α) in the form

$$\frac{x(ax+b)+c}{x(px+q)+r}=\frac{ax+b}{px+q} \qquad (\beta).$$

Multiplying (β) by $(px+q)\{x(px+q)+r\}$, we obtain

$$x(ax+b)(px+q)+c(px+q)=x(ax+b)(px+q)+r(ax+b) \quad (\gamma).$$

Now, (γ) is equivalent to

$$c(px+q)=r(ax+b) \quad (\delta),$$

which again is equivalent to

$$(cp-ra)x+(cq-rb)=0 \quad (\epsilon).$$

The only possibly irreversible step here is that from (β) to (γ).

Observe the use of the brackets in (β) and (γ) to save useless detail.

Example 2.

Integralise

$$\frac{(a-x)(x+m)}{x+n}=\frac{(a+x)(x-m)}{x-n} \quad (\alpha).$$

Since $x+m=(x+n)+(m-n)$, $x-m=(x-n)-(m-n)$, (α) may be written in the equivalent form,

$$(a-x)\left(1+\frac{m-n}{x+n}\right)=(a+x)\left(1-\frac{m-n}{x-n}\right) \quad (\beta),$$

whence the equivalent form

$$(a-x)-(a+x)+(m-n)\left(\frac{a-x}{x+n}+\frac{a+x}{x-n}\right)=0,$$

that is,

$$-2x+\frac{2(m-n)(n+a)x}{x^2-n^2}=0 \quad (\gamma).$$

Multiplying by $-\frac{1}{2}(x^2-n^2)$, we deduce from (γ) the integral equation

$$x\{x^2-n^2-(m-n)(n+a)\}=0 \quad (\delta).$$

In this case the only extraneous solutions that could be introduced are those of $x^2-n^2=0$.

Note the preliminary transformation in (β); and observe that the order in which the operations of collecting and distributing and of using any legitimate processes of derivation that may be necessary is quite unrestricted, and should be determined by considerations of analytical simplicity. Note also that, although we can remove the numerical factor 2 in (γ), it is not legitimate to remove the factor x; $x=0$ is, in fact, as the student will see by inspection, one of the solutions of (α).

Example 3.

X, Y, Z, U denoting rational functions, it is required to rationalise the equation

$$\sqrt{X}\pm\sqrt{Y}\pm\sqrt{Z}\pm\sqrt{U}=0 \quad (\alpha).$$

We shall take + signs throughout; but the reader will see, on looking through the work, that the final result would be the same whatever arrangement of signs be taken.

From (α),

$$\sqrt{X}+\sqrt{Y}=-\sqrt{Z}-\sqrt{U},$$

whence, by squaring,

$$X+Y+2\sqrt{(XY)}=Z+U+2\sqrt{(ZU)} \quad (\beta).$$

From (β),

$$X+Y-Z-U=-2\sqrt{(XY)}+2\sqrt{(ZU)},$$

whence, by squaring,

$$(X+Y-Z-U)^2=4XY+4ZU-8\sqrt{(XYZU)} \qquad (\gamma).$$

We get from (γ),

$$X^2+Y^2+Z^2+U^2-2XY-2XZ-2XU-2YZ-2YU-2ZU=-8\sqrt{(XYZU)},$$

whence, by squaring,

$$\{X^2+Y^2+Z^2+U^2-2XY-2XZ-2XU-2YZ-2YU-2ZU\}^2=64XYZU \quad (\delta).$$

Since X, Y, Z, U are, by hypothesis, all rational, (δ) is the required result.

As a particular instance, consider the equation

$$\sqrt{(2x+3)}+\sqrt{(3x+2)}-\sqrt{(2x+5)}-\sqrt{(3x)}=0 \qquad (\alpha').$$

Here $X=2x+3$, $Y=3x+2$, $Z=2x+5$, $U=3x$; and the student will find, from (δ) above, as the rationalised equation,

$$(48x^2+112x+24)^2=64(2x+3)(3x+2)(2x+5)3x \qquad (\delta').$$

After some reduction (δ') will be found to be equivalent to

$$(x-3)^2=0 \qquad (\epsilon').$$

It may be verified that $x=3$ is a common solution of (α') and (ϵ').

Although, for the sake of the theoretical insight it gives, we have worked out the general formula (δ), and although, as a matter of fact, it contains as particular cases very many of the elementary examples usually given, yet it is by no means advisable that the student should work particular cases by merely substituting in (δ); for, apart from the disciplinary advantage, it often happens that direct treatment is less laborious, owing to intervening simplifications. Witness the following treatment of the particular case (α') above given.

From (α'), by transposition,

$$\sqrt{(2x+3)}+\sqrt{(3x+2)}=\sqrt{(2x+5)}+\sqrt{(3x)},$$

whence, by squaring,

$$5x+5+2\sqrt{(6x^2+13x+6)}=5x+5+2\sqrt{(6x^2+15x)},$$

which reduces to the equivalent equation

$$\sqrt{(6x^2+13x+6)}=\sqrt{(6x^2+15x)} \qquad (\beta').$$

From (β'), by squaring,

$$6x^2+13x+6=6x^2+15x,$$

which is equivalent to

$$x-3=0 \qquad (\delta'').$$

Thus, not only is the labour less than that involved in reducing (δ), but (δ'') is itself somewhat simpler than (δ').

Example 4. If

$$x+y+z=0 \qquad (\alpha),$$

show that

$$\Sigma(y^2+yz+z^2)^3=3\Pi(y^2+yz+z^2) \qquad (\beta).$$

We have

$$\begin{aligned} y^2+yz+z^2 &= y^2+z(y+z), \\ &= (-z-x)^2+z(-x), \text{ by } (\alpha), \\ &= z^2+zx+x^2, \\ &= x^2+xy+y^2, \text{ by symmetry.} \end{aligned}$$

It follows then that

$$\Sigma(y^2+yz+z^2)^3 = 3(y^2+yz+z^2)^3 \qquad (\gamma),$$

and

$$3\Pi(y^2+yz+z^2) = 3(y^2+yz+z^2)^3 \qquad (\delta).$$

From (γ) and (δ), (β) follows at once.

Example 5. If

$$x+y+z=0 \qquad (\alpha),$$

show that

$$\Sigma\frac{(y^2+z^2)(z^2+x^2)}{(y+z)(z+x)}+5\Sigma yz=0 \qquad (\beta).$$

From (α),

$$y+z=-x \qquad (\gamma),$$

whence, squaring and then transposing, we have

$$y^2+z^2=x^2-2yz \qquad (\delta).$$

Similarly

$$z+x=-y \qquad (\gamma'),$$

$$z^2+x^2=y^2-2zx \qquad (\delta').$$

From the last four equations we have

$$\begin{aligned} \Sigma\frac{(y^2+z^2)(z^2+x^2)}{(y+z)(z+x)} &= \Sigma\frac{(x^2-2yz)(y^2-2zx)}{xy}, \\ &= \Sigma\frac{x^2y^2-2x^3z-2y^3z+4xyz^2}{xy}, \\ &= \Sigma\left(xy-2\frac{(x^3+y^3)z^2}{xyz}+4z^2\right), \\ &= \Sigma xy+4\Sigma x^2-\frac{2}{xyz}\Sigma x^2(y^3+z^3) \qquad (\epsilon). \end{aligned}$$

Now, from (α), by squaring and transposing,

$$\Sigma x^2=-2\Sigma xy \qquad (\zeta).$$

Also

$$\begin{aligned} \Sigma x^2(y^3+z^3) &\equiv \Sigma x^2y^2(x+y), \\ &= -\Sigma x^2y^2z, \text{ by } (\alpha), \\ &= -xyz\Sigma xy \qquad (\eta). \end{aligned}$$

If we use (ζ) and (η), (ϵ) reduces to

$$\Sigma\frac{(y^2+z^2)(z^2+x^2)}{(y+z)(z+x)}=-5\Sigma xy,$$

which is equivalent to (β).

The use of the principles of symmetry in conjunction with the Σ notation in shortening the calculations in this example cannot fail to strike the reader.

Example 6. If

$$\frac{yz-x^2}{y+z}=\frac{zx-y^2}{z+x} \qquad (\alpha),$$

and if x, y, z be all unequal, show that each of these expressions is equal to $(xy-z^2)/(x+y)$, and also to $x+y+z$.

Denote each of the sides of (α) by U. Then we have

$$\frac{yz-x^2}{y+z}=\mathrm{U} \qquad (\beta),$$

$$\frac{zx-y^2}{z+x}=\mathrm{U} \qquad (\gamma).$$

Since $y+z=0$ and $z+x=0$ would render the two sides of (α) infinite, we may assume that values of x, y, z fulfilling these conditions are not in question, and multiply (β) and (γ) by $y+z$ and $z+x$ respectively. We then deduce

$$yz-x^2-(y+z)\mathrm{U}=0 \qquad (\delta),$$
$$zx-y^2-(z+x)\mathrm{U}=0 \qquad (\epsilon).$$

From (δ) and (ϵ), by subtraction, we have

$$z(x-y)+(x^2-y^2)-(x-y)\mathrm{U}=0,$$

that is,
$$(x+y+z-\mathrm{U})(x-y)=0 \qquad (\zeta).$$

Now $x-y=0$ is excluded by our data; hence, by (ζ), we must have

$$x+y+z-\mathrm{U}=0, \qquad (\eta),$$

that is,
$$\mathrm{U}=x+y+z \qquad (\theta).$$

We have thus established one of the desired conclusions. To obtain the other it is sufficient to observe that (η) is symmetrical in x, y, z. For, if we start with (η) and multiply by $x-z$ (which, by hypothesis, $\neq 0$), we obtain

$$y(x-z)+(x^2-z^2)-(x-z)\mathrm{U}=0\;;$$

and, combining this by addition with (δ),

$$xy-z^2-(x+y)\mathrm{U}=0\;;$$

which gives (since $x+y\neq 0$)

$$\mathrm{U}=\frac{xy-z^2}{x+y}.$$

The reader should notice here the convenient artifice of introducing an auxiliary variable U. He should also study closely the logic of the process, and be sure that he sees clearly the necessity for the restrictions $x-y\neq 0$, $x+y\neq 0$.

Example 7. To eliminate x, y, z between the equations

$$y^2+z^2=ayz \qquad (\alpha),$$
$$z^2+x^2=bzx \qquad (\beta),$$
$$x^2+y^2=cxy \qquad (\gamma),$$

where $x\neq 0$, $y\neq 0$, $z\neq 0$.

In the first place, we observe that, although there are three variables, yet, since the equations are homogeneous, we are only concerned with the ratios of the three. We might, for example, divide each of the equations by x^2; we should then have to do merely with y/x and z/x, each of which might be regarded as a single variable. There are therefore enough equations for the purpose of the elimination.

From (α) and (β) we deduce, by subtraction,

$$x^2 - y^2 = (bx - ay)z \qquad (\delta).$$

We remark that it follows from this equation that $bx - ay \neq 0$; for $bx - ay = 0$ would give $x^2 = y^2$, and hence, by (γ), $x = 0$ (at least if we suppose $c \neq \pm 2$). This being so, we may multiply (β) by $(bx - ay)^2$. We thus obtain

$$z^2(bx - ay)^2 + x^2(bx - ay)^2 = bxz(bx - ay)^2,$$

whence, using (δ), we have

$$(x^2 - y^2)^2 + x^2(bx - ay)^2 = bx(bx - ay)(x^2 - y^2),$$

which reduces, after transposition, to

$$(x^2 - y^2)^2 = xy(ax - by)(bx - ay),$$

that is to say,
$$(x^2 + y^2)^2 - 4x^2y^2 = xy(ax - by)(bx - ay) \qquad (\epsilon).$$

Using (γ), we deduce from (ϵ)

$$(c^2 - 4)x^2y^2 = xy(ax - by)(bx - ay),$$

whence, bearing in mind that $xy \neq 0$, we get

$$(c^2 - 4)xy = ab(x^2 + y^2) - (a^2 + b^2)xy,$$

which is equivalent to

$$(a^2 + b^2 + c^2 - 4)xy = ab(x^2 + y^2) \qquad (\zeta).$$

Using (γ) once more, and transposing, we reach finally

$$(a^2 + b^2 + c^2 - 4 - abc)xy = 0,$$

whence, since $xy \neq 0$, we conclude that

$$a^2 + b^2 + c^2 - 4 - abc = 0 \qquad (\eta),$$

so that (η) is the required result of eliminating x, y, z between the equations (α), (β), (γ). Such an equation as (η) is often called the eliminant (or resultant) of the given system of equations.

Example 8. Show that, if the two first of the following three equations be given, the third can be deduced, it being supposed that $x \neq y \neq z \neq 0$.

$$a^2(y^2 + yz + z^2) - ayz(y + z) + y^2z^2 = 0 \qquad (\alpha),$$
$$a^2(z^2 + zx + x^2) - azx(z + x) + z^2x^2 = 0 \qquad (\beta),$$
$$a^2(x^2 + xy + y^2) - axy(x + y) + x^2y^2 = 0 \qquad (\gamma).$$

This is equivalent to showing that, if we eliminate z between (α) and (β), the result is (γ).

Arranging (α) and (β) according to powers of z, we have

$$a^2y^2 - a(-ay + y^2)z + (a^2 - ay + y^2)z^2 = 0 \qquad (\delta),$$
$$a^2x^2 - a(-ax + x^2)z + (a^2 - ax + x^2)z^2 = 0 \qquad (\epsilon).$$

Multiplying (δ) and (ϵ) by x^2 and y^2 respectively, and subtracting, we get

$$a^2xy(x - y)z + \{a^2(x + y) - axy\}(x - y)z^2 = 0,$$

whence, rejecting the factor $a(x - y)z$, which is permissible since $x \neq y$, $z \neq 0$,

$$axy + \{a(x + y) - xy\}z = 0 \qquad (\zeta).$$

Again, multiplying (δ) and (ϵ) by $a^2 - ax + x^2$ and $a^2 - ay + y^2$ respectively, and subtracting, we get, after rejecting the factor a^2,

$$a(x + y) - xy + \{a - (x + y)\}z = 0 \qquad (\eta).$$

Finally, multiplying (ζ) and (η) by $a(x+y)-xy$ and axy respectively, and subtracting, we get, since $z \neq 0$,

$$\{a(x+y)-xy\}^2 - axy\{a-(x+y)\} = 0,$$

which gives $$a^2(x^2+xy+y^2) - axy(x+y) + x^2y^2 = 0,$$

the required result.

Exercises XIX.

(On the Reduction of Equations to an Integral Form.)

Solve by inspection the following systems of equations:—

(1.) $$x^2 - 4 - 3(x+2) = 0.$$

(2.) $$\frac{a+b}{x-b} = \frac{2b}{x-a}.$$

(3.) $$(a-b)x - a^2 + b^2 = 0.$$

(4.) $$x(b-c) + y(c-a) + (a-b) = 0,$$
$$ax(b-c) + by(c-a) + c(a-b) = 0.$$

(5.) $$x+y+z = a+b+c,$$
$$ax+by+cz = a^2+b^2+c^2,$$
$$bx+cy+az = bc+ca+ab.$$

(6.) For what values of a and b does the equation

$$(x-a)(3x-2) = 3x^2+bx+10$$

become an identity?

Integralise the following equations; and discuss in each case the equivalence of the final equation to the given one.

(7.) $$\frac{x+2}{x-4} + \frac{2x+11}{x-2} = 14.$$

(8.) $$\frac{1-3/(x+1)}{x+1+1/(x+1)} = \frac{1+3/(x-1)}{x-1+1/(x-1)}.$$

(9.) $$\frac{1}{x+a} + \frac{1}{x-c} = \frac{1}{x-a} + \frac{1}{x+c}.$$

(10.) $$\frac{a^2}{x-a} + \frac{b^2}{x-b} = \frac{x^2}{x-a} + \frac{a^2+b^2}{x-b}.$$

(11.) $$\frac{(3-x)(x+10)}{x+11} = \frac{(3+x)(x-10)}{x-11}.$$

(12.) $$\frac{x^2+px+q}{x^2+rx+2q} = \frac{x^2+px+t}{x^2+rx+2t}.$$

(13.) $$\frac{(x-a)^3}{(c-a)(a-b)} + \frac{(x-b)^3}{(a-b)(b-c)} + \frac{(x-c)^3}{(b-c)(c-a)} = 0.$$

(14.) $$\frac{x+\mathrm{T}+\mathrm{U}}{x^2+(2-t)x+s(2-s-t)} + \frac{x+\mathrm{T}-\mathrm{U}}{x^2+(2-s)x+t(2-s-t)} = 1,$$

when $$2\mathrm{T} = s+t-s^2-st-t^2.$$

(15.) $$\frac{x^2+ax+b}{x+a}+\frac{x^2+cx+d}{x+c}=\frac{x^2+ax+b'}{x+a}+\frac{x^2+cx+d'}{x+c}.$$

Rationalise the following equations and reduce the resulting equation to as simple a form as possible :—

(16.) $\sqrt{X}+\sqrt{Y}+\sqrt{Z}=0$, where X, Y, Z are rational functions of the variables.

(17.) $$\sqrt{(x+a)}+\sqrt{(x+b)}+\sqrt{(x+c)}=0.$$

(18.) $$\sqrt{(1+x)}+\sqrt{(4+x)}-\sqrt{(9+x)}=0.$$

(19.) $$[x-c+\{(x-c)^2+y^2\}^{\frac{1}{2}}]/[x+c+\{(x-c)^2+y^2\}^{\frac{1}{2}}]=m.$$

(20.) $$x-a=\sqrt{\{a^2-\sqrt{(a^2x^2-x^4)}\}}.$$

(21.) $$\sqrt{x}+\sqrt{(x-7)}=21/\sqrt{(x-7)}.$$

(22.) $$\frac{\sqrt{x}+8}{\sqrt{x}+3}=\frac{\sqrt{x}+48}{\sqrt{x}+29}.$$

(23.) $$\frac{\sqrt{(2+x)}}{\sqrt{2}+\sqrt{(2+x)}}=\frac{\sqrt{(2-x)}}{\sqrt{2}-\sqrt{(2-x)}}.$$

(24.) $$\sqrt{(x+a)}+\sqrt{(x-a)}+\sqrt{(b+x)}+\sqrt{(b-x)}=0.$$

(25.) $$\frac{\sqrt{(1+x+x^2)}+\sqrt{(1-x+x^2)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}=1.$$

(26.) $$(y-z)(ax+b)^{\frac{1}{2}}+(z-x)(ay+b)^{\frac{1}{2}}+(x-y)(az+b)^{\frac{1}{2}}=0.$$

(27.) $\Sigma\sqrt{(y-z)}=0$; and show that $\Sigma x=\sqrt{(3\Sigma yz)}$ (three variables x, y, z).

(28.) $$\frac{1}{\sqrt{\{x+\sqrt{(x^2-1)}\}}}+\frac{1}{\sqrt{\{x-\sqrt{(x^2-1)}\}}}=\sqrt{\{2(x^3+1)\}}.$$

(29.) $$x^{\frac{1}{4}}+5x^{\frac{1}{2}}-22=0.$$

(30.) $$\frac{\sqrt[n]{(a+x)}}{x}+\frac{\sqrt[n]{(a+x)}}{a}=\frac{\sqrt[n]{x}}{c}.$$

(31.) $$\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}=\sqrt[3]{b}.$$

(32.) $$x^{\frac{1}{5}}+y^{\frac{1}{5}}+z^{\frac{1}{5}}=0,$$

where $$x+y+z=0.$$

Exercises XX.

(On the Transformation of Systems of Equations.)

[In working this set the student should examine carefully the logic of every step he takes, and satisfy himself that it is consistent with his data. He should also make clear to himself whether each step is or is not reversible.]

(1.) If $$\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}+2=0,\ x+y+z\neq0,$$

then $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x+y+z}.$$

(2.) If
$$x^3+y^3=z^3,$$
then
$$\{(x^3+z^3)y\}^3+\{(x^3-y^3)z\}^3=\{(y^3+z^3)x\}^3 \qquad (Tait).$$

(3.) If x, y, z be real, and if $x^4(y-z)+y^4(z-x)+z^4(x-y)=0$, then two at least of the three must be equal.

(4.) If
$$(x+y+z)^3=x^3+y^3+z^3,$$
then
$$(x+y+z)^{2n+1}=x^{2n+1}+y^{2n+1}+z^{2n+1}.$$

(5.) If
$$(p^2x+2pry+r^2z)(q^2x+2qsy+s^2z)=\{pqx+(ps+qr)y+rsz\}^2,$$
then either $y^2-zx=0$ or $ps-qr=0$.

(6.) If
$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$
where
$$r^2=x^2+y^2+z^2,$$
then
$$\frac{x^2}{b^2c^2-\rho^2}+\frac{y^2}{c^2a^2-\rho^2}+\frac{z^2}{a^2b^2-\rho^2}=0,$$
where
$$\rho^2=a^2x^2+b^2y^2+c^2z^2.$$
(Important in the theory of the wave surface.—*Tait.*

(7.) If
$$\frac{y+z}{b-c}=\frac{z+x}{c-a}=\frac{x+y}{a-b}, \text{ and } x+y+z=0,$$
show that each of them is equal to $\sqrt{\{\Sigma x^2/2(\Sigma a^2-\Sigma bc)\}}$.

(8.) If
$$a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz),$$
and if
$$a+b+c=0,$$
then
$$x+y+z=0.$$

(9.) If
$$\frac{x+2y}{2a+b}=\frac{y+2z}{2b+c}=\frac{z+2x}{2c+a},$$
then
$$\left(\frac{\Sigma x}{\Sigma a}\right)^2=\frac{\Sigma xy}{\Sigma ab}=\frac{\Sigma x^2}{\Sigma a^2}.$$

(10.) If
$$x=\frac{2ab+b^2}{a^2+ab+b^2}, \quad y=\frac{a^2-b^2}{a^2+ab+b^2},$$
then
$$x^3+y=y^3+x.$$

(11.) If
$$x=a+b+\frac{(a-b)^2}{4(a+b)}, \quad y=\frac{a+b}{4}+\frac{ab}{a+b},$$
then
$$(x-a)^2-(y-b)^2=b^2.$$

(12.) If
$$\alpha=ax+by+cz+dw,$$
$$\beta=bx+ay+dz+cw,$$
$$\gamma=cx+dy+az+bw,$$
$$\delta=dx+cy+bz+aw,$$
and if
$$f(\alpha,\beta,\gamma,\delta)=(\alpha+\beta+\gamma+\delta)(\alpha-\beta+\gamma-\delta)(\alpha-\beta-\gamma+\delta)(\alpha+\beta-\gamma-\delta),$$
then
$$f(\alpha,\beta,\gamma,\delta)=f(a,b,c,d)f(x,y,z,w).$$

(13.) If $x+y+z=0$, then $\Sigma 1/x^2=(\Sigma 1/x)^2$.

(14.) If $$\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m+\left(\frac{z}{c}\right)^m=1=\left(\frac{a}{d}\right)^m+\left(\frac{b}{d}\right)^m+\left(\frac{c}{d}\right)^m,$$

and $$\frac{x^m}{a^{m+n}}=\frac{y^m}{b^{m+n}}=\frac{z^m}{c^{m+n}},$$

show that $$(\Sigma x^{mn/(m+n)})(\Sigma x^{m^2/(m+n)})=d^m.$$

(15.) If $$a-\frac{yz}{x}=b-\frac{zx}{y}=c-\frac{xy}{z},\ x\neq 0,\ y\neq 0,\ z\neq 0,$$

then $$a+\frac{y^2-z^2}{b-c}=b+\frac{z^2-x^2}{c-a}=c+\frac{x^2-y^2}{a-b}.$$

(16.) If $x+(yz-x^2)/(x^2+y^2+z^2)$ be unaltered by interchanging x and y, it will be unaltered by interchanging x and z, provided x, y, z be all unequal; and it will vanish if $x+y+z=1$.

(17.) If $zxy/(y+z)-x^2=xyz/(z+x)-y^2$, and $x\neq y$, then each of these is equal to $xyz/(x+y)-z^2$ and also to $yz+zx+xy$.

(18.) Of the three equations

$$\frac{x}{x^2-w^2}=\frac{y+z}{(m+1)w^2-(n+1)yz},$$
$$\frac{y}{y^2-w^2}=\frac{z+x}{(m+1)w^2-(n+1)zx},$$
$$\frac{z}{z^2-w^2}=\frac{x+y}{(m+1)w^2-(n+1)xy},$$

where $x\neq y\neq z$, any two imply the third (*Cayley*).

(19.) Given

$$\frac{1}{1+x+xz}+\frac{y}{1+y+xy}+\frac{yz}{1+z+yz}=1,$$
$$\frac{x}{1+x+xz}+\frac{xy}{1+y+xy}+\frac{1}{1+z+yz}=1,$$

none of the denominators being zero, then $x=y=z$.

(20.) Given $\Sigma(y+z)^2/x=3\Sigma x$, $\Sigma x\neq 0$, prove $\Sigma(y+z-x)^3+\Pi(y+z-x)=0$.

(21.) Given $\Sigma x=0$, prove $\Sigma(x^3+y^3)/(x+y)+5xyz\Sigma(1/x)=0$.

(22.) Given $\Sigma x=0$, prove that $\Sigma x^3\Sigma x^2/\Sigma x^5$ is independent of x, y, z.

(23.) If $\Sigma x^5=-5xyz\Sigma xy$, then $\Sigma x=0$, or $\Sigma x^4-\Sigma x^3y+\Sigma x^2y^2+2\Sigma x^2yz=0$.

(24.) If $\Pi(x^2+1)=a^2+1$, $\Pi(x^2-1)=a^2-1$, and $\Sigma xy=0$, then $x+y+z=0$ or $=\pm a$.

(25.) If $x+y+z+u=0$, then $4\Sigma x^3+3\Sigma(y+z)(u+y)(u+z)=0$, where the Σ refers to the four variables x, y, z, u.

Exercises XXI.

(*On Elimination.*)

(1.) Eliminate x between the equations

$$x+1/x=y,\quad x^5+1/x^5=z.$$

(2.) If $z=\sqrt{(ay^2-a^2/y)}$, $y=\sqrt{(ax^2-a^2/x)}$, express $\sqrt{(az^2-a^2/z)}$ in terms of x.

(3.) If
$$\phi(x)=(a^x-a^{-x})/(a^x+a^{-x}),$$
$$F(x)=2/(a^x+a^{-x}),$$
then
$$\phi(x+y)=(\phi(x)+\phi(y))/(1+\phi(x)\phi(y)),$$
$$F(x+y)=F(x)F(y)/(1+\phi(x)\phi(y)).$$

(4.) Given
$$\frac{x(y+z-x)}{a}=\frac{y(z+x-y)}{b}=\frac{z(x+y-z)}{c},$$
prove that
$$\frac{a(b+c-a)}{x}=\frac{b(c+a-b)}{y}=\frac{c(a+b-c)}{z}.$$

(5.) Given $bz+cy=cx+az=ay+bx$, $x^2+y^2+z^2=2yz+2zx+2xy$, prove that one of the functions $a\pm b\pm c=0$.

(6.) Show that the result of eliminating x and y between the equations
$$\frac{x}{a}+\frac{y}{b}=1,\quad \frac{x^2}{c^2}+\frac{y^2}{d^2}=1,\quad xy=p^2,$$
is $(b^2c^2+a^2d^2)^2p^4+2abc^2d^2(b^2c^2+a^2d^2-2a^2b^2)p^2+a^2b^2c^2d^2(a^2-c^2)(b^2-d^2)=0.$

(7.) Eliminate x, y, x', y' from
$$ax+by=c^2,\quad x^2+y^2=c^2,$$
$$a'x'+b'y'=c'^2,\quad x'^2+y'^2=c'^2,\quad xy'+x'y=0.$$

(8.) If $1/(x+a)+1/(y+a)+1/(z+a)=1/a$, with two similar equations in which b and c take the place of a, show that $\Sigma(1/a)=0$, provided a, b, c be all different.

(9.) Show that any two of the following equations can be deduced from the other three:—
$$ax+bc=zu,\quad by+ca=uv,\quad cz+db=vx,\quad du+ec=xy,\quad ev+ad=yz.$$

(10.) Eliminate x, y, z from the three equations
$$(z+x-y)(x+y-z)=ayz,$$
$$(x+y-z)(y+z-x)=bzx,\quad (y+z-x)(z+x-y)=cxy\ ;$$
and show that the result is $abc=(a+b+c-4)^2$.

CHAPTER XV.

Variation of Functions.

§ 1.] The view which we took of the theory of conditional equations in last chapter led us to the problem of finding a set of values of the variables which should render a given conditional equation an identity. There is another order of ideas of at least equal analytical importance, and of wider practical utility, which we now proceed to explain. Instead of looking merely at the values of the variables x, y, z, . . . which satisfy the equation

$$f(x, y, z, \ldots) = 0,$$

that is, which render the function $f(x, y, z, \ldots)$ zero, we consider all possible values of the variables, and all possible corresponding values of the function; or, at least, we consider a number of such values sufficient to give us a clear idea of the whole; then, among the rest, we discover those values of the variables which render the function zero. The two methods might be illustrated by the two possible ways of finding a particular man in a line of soldiers. We might either go straight to some part of the ranks where a preconceived theory would indicate his presence; or we might walk along from one end of the line to the other looking till we found him. In this new way of looking at analytical functions, the graphical method, as it is called, is of great importance. This consists in representing the properties of the function in some way by means of a geometrical figure, so that we can with the bodily eye take a comprehensive view of the peculiarities of any individual case.

Now, since $x^2-6x+8\equiv(x-2)(x-4)$, (α) may be written in the equivalent form

$$2x-3+x-4=\frac{x-2}{x-3},$$

which is obviously not satisfied by $x=2$.

It appears, therefore, that in the process of integralisation we have introduced the extraneous solution $x=2$.

Example 2.

$$2x-3+\frac{2x^2-6x+8}{x-2}=\frac{x-2}{x-3} \qquad (\alpha').$$

Proceeding as before, we deduce

$$(2x-3)(x-2)(x-3)+(2x^2-6x+8)(x-3)=(x-2)^2 \qquad (\beta').$$

It will be found that neither of the values $x=2$, $x=3$ satisfies (β'). Hence no extraneous solutions have been introduced in this case.

N.B.—The reason why $x=2$ satisfies (β) in Example 1 is that the numerator x^2-6x+8 of the fraction on the left contains the factor $x-2$ which occurs in the denominator.

Cor. 4. *Raising both sides of an equation to the same integral power is a legitimate, but not a reversible, process of derivation.*

The equation $$\mathrm{P}=\mathrm{Q} \qquad (1)$$
is equivalent to $$\mathrm{P}-\mathrm{Q}=0 \qquad (2).$$

If we multiply by $\mathrm{P}^{n-1}+\mathrm{P}^{n-2}\mathrm{Q}+\mathrm{P}^{n-3}\mathrm{Q}^2+\ldots+\mathrm{Q}^{n-1}$, we deduce from (2)

$$\mathrm{P}^n-\mathrm{Q}^n=0 \qquad (3),$$

which is satisfied by any solution of (1); (3), however, is not equivalent to (1), but to

$$\left\{\begin{array}{r}\mathrm{P}=\mathrm{Q} \\ \mathrm{P}^{n-1}+\mathrm{P}^{n-2}\mathrm{Q}+\ldots+\mathrm{Q}^{n-1}=0\end{array}\right\}.$$

It will be observed that, if we start with an equation in the standard form $\mathrm{P}-\mathrm{Q}=0$, transfer the part Q to the right-hand side, and then raise both sides to the nth power, the result is the same as if we had multiplied both sides of the equation in its original form by a certain factor. To make the introduction of extraneous factors more evident we chose the latter process; but in practice the former may happen to be the more convenient.*

If the reader will reflect on the nature of the process described in chap. x. for rationalising an algebraical function by means of a rationalising factor, he will see that by repeated operations of this kind every algebraical equation can be reduced to a rational

* See below, § 12, Example 3.

form ; but at each step extraneous solutions may be introduced. Hence

Cor. 5. *From every algebraical equation we can derive a rational integral equation, which will be satisfied by any solution of the given equation ; but it does not follow that every solution, or even that any solution, of the derived equation will satisfy the original one.*

Example 1. Consider the equation

$$\sqrt{(x+1)}+\sqrt{(x-1)}=1 \qquad (\alpha),$$

where the radicands are supposed to be real and the square root to have the positive sign.*

(α) is equivalent to $\qquad \sqrt{(x+1)}=1-\sqrt{(x-1)},$

whence we derive, by squaring,

$$x+1=1+x-1-2\sqrt{(x-1)},$$

which is equivalent to $\qquad 1=-2\sqrt{(x-1)}.$

From this last again, by squaring, we derive

$$1=4(x-1),$$

which is equivalent to the integral equation

$$4x-5=0 \qquad (\beta),$$

the only solution of which, as we shall see hereafter, is $x=\frac{5}{4}$.

It happens here that $x=\frac{5}{4}$ is not a solution of (α), for $\sqrt{(\frac{5}{4}+1)}+\sqrt{(\frac{5}{4}-1)}$ $=\frac{3}{2}+\frac{1}{2}=2$.

Example 2.

$$\sqrt{(x+1)}-\sqrt{(x-1)}=1 \qquad (\alpha').$$

Proceeding exactly as before we have

$$x+1=1+x-1+2\sqrt{(x-1)},$$
$$1=+2\sqrt{(x-1)},$$
$$1=4(x-1),$$
$$4x-5=0 \qquad (\beta'),$$

Here (β') gives $x=\frac{5}{4}$, which happens this time to be a solution of the original equation.

We conclude this discussion by giving two propositions applicable to systems of equations containing more than one equation. These by no means exhaust the subject ; but, as our object here is merely to awaken the intelligence of the student, the rest may be left to himself in the meantime.

§ 10.] *From the system*

$$P_1=0, \quad P_2=0, \quad . \; . \; ., \quad P_n=0 \qquad (A)$$

we derive

$$L_1P_1+L_2P_2+\;.\;.\;.+L_nP_n=0, \quad P_2=0, \quad . \; . \; ., \quad P_n=0 \quad (B),$$

and the two will be equivalent if L_1 *be a constant differing from* 0.

---

* When $\sqrt{x}$ is imaginary, its "principal value" (see chap. xxix.) ought to be taken, unless it is otherwise indicated.

Any solution of the system (A) reduces $P_1, P_2, \ldots, P_n$ all to 0, and therefore reduces $L_1P_1 + L_2P_2 + \ldots + L_nP_n$ to 0, and hence satisfies (B).

Again, any solution of (B) reduces $P_2, P_3, \ldots, P_n$ all to 0, and therefore reduces $L_1P_1 + L_2P_2 + \ldots + L_nP_n = 0$ to $L_1P_1 = 0$, that is to say, if L_1 be a constant $\neq 0$, to $P_1 = 0$. Hence, in this case, any solution of (B) satisfies (A).

If L_1 contain the variables, then (B) is equivalent, not to (A) simply, but to

$$\left\{\begin{array}{llll} P_1 = 0, & P_2 = 0, & \ldots, & P_n = 0 \\ L_1 = 0, & P_2 = 0, & \ldots, & P_n = 0 \end{array}\right\}.$$

As a particular case of the above, we have that the two systems

$$P = Q, \qquad R = S;$$

and

$$P + R = Q + S, \; R = S$$

are equivalent. For these may be written

$$\overline{P - Q} = 0, \; \overline{R - S} = 0;$$

$$\overline{P - Q} + \overline{R - S} = 0, \; \overline{R - S} = 0.$$

If l, l', m, m' be constants, any one of which may be zero, but which are such that $lm' - l'm \neq 0$, then the two systems

$$U = 0, \; U' = 0,$$

and

$$lU + l'U' = 0, \; mU + m'U' = 0$$

are equivalent.

The proof is left to the reader. A special case is used and demonstrated in chap. xvi., § 4.

§ 11.] *Any solution of the system*

$$P = Q, \quad R = S \qquad \text{(A)}$$

is a solution of the system

$$PR = QS, \; R = S \qquad \text{(B)};$$

but the two systems are not equivalent.

From $P = Q$, we derive

$$PR = QR,$$

which, since $R = S$, is equivalent to

$$PR = QS.$$

It follows therefore that any solution of (A) satisfies (B).

Starting now with (B), we have

$$PR = QS \qquad (1),$$
$$R = S \qquad (2).$$

Since $R = S$, (1) becomes

$$PR = QR,$$

which is equivalent to

$$(P - Q)R = 0,$$

that is, equivalent to

$$\left\{\begin{array}{r} P - Q = 0 \\ R = 0 \end{array}\right\}.$$

Hence the system (B) is equivalent to

$$\left\{\begin{array}{l} P = Q,\ R = S \\ R = 0,\ R = S \end{array}\right\}.$$

that is to say, to

$$\left\{\begin{array}{l} P = Q,\ R = S \\ R = 0,\ \ S = 0 \end{array}\right\}.$$

In other words, (B) involves, besides (A), the alternative system,

$$R = 0, \quad S = 0.$$

Example. From $x - 2 = 1 - y,\ x = 1 + y,$ a system which has the single solution $x = 2,\ y = 1$, we derive the system

$$x(x - 2) = 1 - y^2,\ x = 1 + y,$$

which, in addition to the solution $x = 2,\ y = 1$, has also the solution $x = 0,\ y = -1$ belonging to the system

$$x = 0,\ 1 + y = 0.$$

§ 12.] In the process of solving systems of equations, one of the most commonly-occurring requirements is to deduce from two or more of the equations another that shall not contain certain assigned variables. This is called "*eliminating* the variables in question *between* the equations used for the purpose." In performing the elimination we may, of course, use any legitimate process of derivation, but strict attention must always be paid to the question of equivalence.

Example. Given the system

$$x^2 + y^2 = 1 \qquad (1),$$
$$x + y = 1 \qquad (2),$$

it is required to eliminate y, that is, to deduce from (1) and (2) an equation involving x alone.

The general form of the graph, so far as the right-hand side of the axis of y is concerned, will be as in Fig. 2.

As regards negative values of x and the left-hand side of the axis of y, in the present case, it is merely necessary to notice that, if we put $x = -a$, the result, so far as $1-x^2$ is concerned, is the same as if we put $x = +a$; for $1-(-a)^2 = 1-(+a)^2$. Hence for every point P on the curve, whose abscissa and ordinate are $+$OM and $+$MP, there will be a point P′, whose abscissa and ordinate are $-$OM and $+$MP. P and P′ are the images of each other with respect to Y′Y; and the part AP′B′ of the graph is merely an image of the part APB with respect to the line Y′Y.

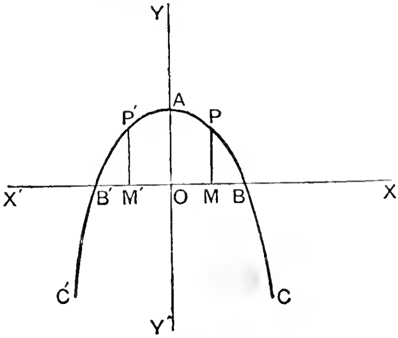


Fig. 2.

Let us see what the graph tells us regarding the function $1-x^2$.

First we see that the graph crosses the x-axis at two points and no more, those, namely, for which $x=+1$ and $x=-1$. Hence the function $1-x^2$ has only two roots, $+1$ and -1; in other words, the equation

$$1-x^2=0$$

has two real roots, $x=+1$, $x=-1$, and no more.

Secondly. Since the part BAB′ of the graph lies wholly above, and the parts C′B′, CB wholly below the x-axis, we see that, for all real values of x lying between -1 and $+1$, the function $1-x^2$ is positive, and for all other real values of x negative.

Thirdly. We see that the greatest positive value of $1-x^2$ is 1, corresponding to $x=0$; and that, by making x sufficiently great (numerically), we can give $1-x^2$ a negative value as large, numerically, as we please.

All these results could be obtained by direct discussion of the function, but the graph indicates them all to the eye at a glance.

§ 5.] Hitherto we have assumed that there are no breaks or discontinuities in the graph of the function. Such may, however, occur; and, as it is necessary, when we set to work to discuss by considering all possible cases, above all to be sure that no possible case has escaped our notice, we proceed now to consider the exceptions to the statement that the graph is in general a continuous curve.

I. *The function $f(x)$ may become infinite for a finite value of x.*

Example 1.

Consider the function $1/(1-x)$. When x is a very little less than $+1$, say $x=\cdot 99999$, then $y=1/(1-x)$ gives $y=+1/\cdot 00001=+100000$; that is to say, y is positive and very large; and it is obvious that, by bringing x sufficiently nearly up to $+1$, we can give y as large a positive value as we please.

On the other hand, if x be a very little greater than $+1$, say $x=+1\cdot 00001$, then $y=1/(-\cdot 00001)=-100000$; and it is obvious that, by making x exceed 1 by a sufficiently small quantity, y can be made as large a negative quantity as we please.

The graph of the function $1/(1-x)$ for values of x near $+1$ is therefore as follows:—

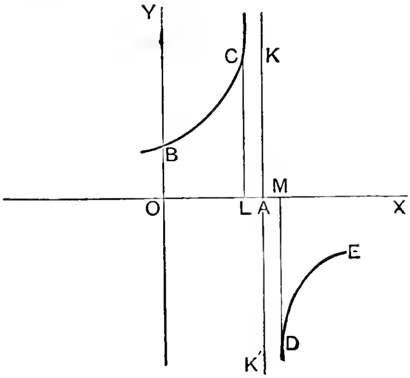


FIG. 3.

The branch BC ascends to an infinite distance along KAK′ (a line parallel to the y-axis at a distance from it $=+1$), continually coming nearer to KAK′, but never reaching it at any finite distance from the x-axis. The branch DE comes up from an infinite distance along the other side of KAK′ in a similar manner.

Here, if we cause x to increase from a value OL very little less than $+1$ to a value OM very little greater, the value of y will jump from a very large positive value $+$ LC to a very large negative value $-$ MD; and, in fact, the smaller we make the increase of x, provided always we pass from the one side of $+1$ to the other, the larger will be the jump in the value of y.

It appears then that, for $x=+1$, $1/(1-x)$ is both *infinite* and *discontinuous*.

Example 2.

$$y=1/(1-x)^2.$$

We leave the discussion to the reader. The graph is as in Fig. 4.

The function becomes infinite when $x=+1$; and, for a very small increment of x near this value, the increment of y is very large. In fact, if we increase or diminish x from the value $+1$ by an infinitely small amount, y will diminish by an infinitely great amount.

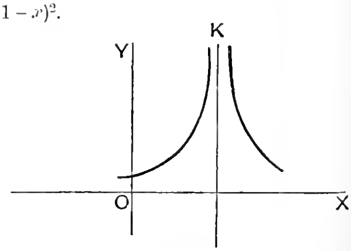


FIG. 4.

Here again we have infinite value of the function, and accompanying discontinuity.

II. *The value of the function may make a jump without becoming infinite.*

The graph for the neighbourhood of such a value would be of the nature indicated in Fig. 5, where, while x passes through the value OM, y jumps from MP to MQ.

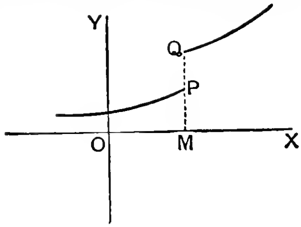


FIG. 5.

Such a case cannot, as we shall immediately prove, occur with integral functions of x. In fact it cannot occur with any algebraical function, so that we need not further consider it here.

The cases we have just considered lead us to give the following formal definition.

A function is said to be continuous when for an infinitely small change in the value of the independent variable the change in the value of the function is also infinitely small; and to be discontinuous when for an infinitely small change of the independent variable the change in the value of the function is either finite or infinitely great.

III. *It may happen that the value of a function, all of whose constants are real, becomes imaginary for a real value of its variable.*

Example.

This happens with the function $+\sqrt{(1-x^2)}$. If we confine ourselves to the positive value of the square root, so that we have a single-valued function to deal with, the graph is as in Fig. 6:—

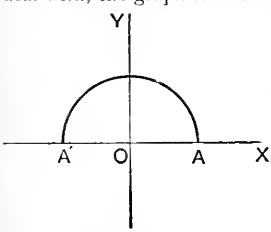


FIG. 6.

a semicircle, in fact, whose centre is at the origin.

For all values of $x > +1$, or < -1, the value of $y = +\sqrt{(1-x^2)}$ is imaginary; and the graphic points for them cannot be constructed in the kind of diagram we are now using.

The continuity of the function at A cannot, strictly speaking, be tested; since, if we attempt to increase x beyond $+1$, y becomes imaginary, and there can be no question of the magnitude of the increment, from our present point of view at least.*

No such case as this can arise so long as $f(x)$ is a rational algebraical function.

* See below, § 18.

We have now enumerated the exceptional cases of functional variation, so far at least as is necessary for present purposes. Graphic points, at which any of the peculiarities just discussed occur, may be generally referred to as *critical points.*

ON CERTAIN LIMITING CASES OF ALGEBRAICAL OPERATION.

§ 6.] We next lay down systematically the following propositions, some of which we have incidentally used already. The reader may, if he choose, take them as axiomatic, although, as we shall see, they are not all independent. The important matter is that they be thoroughly understood. To secure that they be so we shall illustrate some of them by examples. In the meantime we caution the reader that by "infinitely small" or "infinitely great" we mean, in mathematics, "smaller than any assignable fraction of unity," or "as small as we please," and "greater than any assignable multiple of unity," or "as great as we please." He must be specially on his guard against treating the symbol ∞, which is simply an abbreviation for "greater than any assignable magnitude," as a definite quantity. There is no justification for applying to it any of the laws of algebra, or for operating with it as we do with an ordinary symbol of quantity.

I. *If* P *be constant or variable, provided it does not become infinitely great when* Q *becomes infinitely small, then when* Q *becomes infinitely small* PQ *becomes infinitely small.*

Observe that nothing can be inferred without further examination in the case where P becomes infinitely great when Q becomes infinitely small. This case leads to the so-called indeterminate form $\infty \times 0$.*

Example 1.

Let us suppose, for example, that P is constant, $=100000$, say. Then, if we make $Q=1/100000$, we reduce PQ to 1 ; if we make $Q=1/100000000000$, we reduce PQ to $1/1000000$; and so on. It is abundantly evident, therefore, that by making Q sufficiently small PQ can be made as small as we please.

---

* Indeterminate forms are discussed in chap. xxv

Example 2.

$$\text{Let } P=x+1, \quad Q=x-1.$$

Here, when x is made to approach the value $+1$, P approaches the finite value $+2$, while Q approaches the value 0. Suppose, for example, we put $x=1+1/100000$, then

$$PQ=(2+1/100000)\times 1/100000,$$
$$=2/100000+1/10^{10},$$

and so on. Obviously, therefore, by sufficiently diminishing Q, we can make PQ as small as we please.

Example 3.

$$P=1/(x^2-1), \quad Q=x-1.$$

Here we have the peculiarity that, when Q is made infinitely small, P (see below, Proposition III.) becomes infinitely great. We can therefore no longer infer that PQ becomes infinitely small because Q does so. In point of fact, $PQ\equiv(x-1)/(x^2-1)\equiv 1/(x+1)$, which becomes $1/2$ when $x=1$.

II. *If* P *be either constant or variable, provided it do not become infinitely small when* Q *becomes infinitely great, then when* Q *becomes infinitely great* PQ *becomes infinitely great.*

The case where P becomes infinitely small when Q becomes infinitely great must be further examined; it is usually referred to as the indeterminate form $0\times\infty$.

Example 1.

Suppose $P=1/100000$. Then, by making $Q=100000$, we reduce PQ to 1; by making $Q=100000000000$ we reduce PQ to 1000000; and so on. It is clear, therefore, that by sufficiently increasing Q we could make PQ exceed any number, however great.

The student should discuss the following for himself:—

Example 2.

$$P=x+1, \quad Q=1/(x-1).$$
$$PQ=\infty \text{ when } x=1.$$

Example 3.

$$P=(x-1)^2, \quad Q=1/(x-1).$$
$$PQ=0 \text{ when } x=1.$$

III. *If* P *be either constant or variable, provided it do not become infinitely small when* Q *becomes infinitely small, then when* Q *becomes infinitely small* P/Q *becomes infinitely great.*

The case where P and Q become infinitely small for the same value of the variable requires further examination. This gives the so-called indeterminate form $\frac{0}{0}$.

Example 1.

Suppose P constant $=1/100000$. If we make $Q=1/100000$, P/Q becomes 1; if we make $Q=1/100000000000$, P/Q becomes 1000000; and so on. Hence we see that, if only we make Q small enough, we can make P/Q as large as we please.

The student should examine arithmetically the two following cases :—

Example 2.

$$P=x+1, \quad Q=x-1.$$
$$P/Q=\infty \text{ when } x=1.$$

Example 3.

$$P=x-1, \quad Q=x-1.$$
$$P/Q=1 \text{ when } x=1.$$

IV. *If* P *be either constant or variable, provided it do not become infinitely great when* Q *becomes infinitely great, then when* Q *becomes infinitely great* P/Q *becomes infinitely small.*

The case where P and Q become infinitely great together requires further examination. This gives the indeterminate form $\dfrac{\infty}{\infty}$.

Example 1.

Suppose P constant $=100000$. If we make $Q=100000$, P/Q becomes 1; if we make $Q=100000000000$, P/Q becomes 1/1000000; and so on. Hence by sufficiently increasing Q we can make P/Q less than any assignable quantity.

Example 2.

$$P=x+1, \quad Q=1/(x-1).$$
$$P/Q=0 \text{ when } x=1.$$

Example 3.

$$P=1/(x-1)^2, \quad Q=1/(x-1).$$
$$P/Q=\infty \text{ when } x=1.$$

V. *If* P *and* Q *each become infinitely small, then* P + Q *becomes infinitely small.*

For, let P be the numerically greater of the two for any value of the variable. Then, if the two have the same sign, and, *a fortiori*, if they have opposite signs, numerically

$$P+Q<2P.$$

Now 2 is finite, and, by hypothesis, P can be made as small as we please. Hence, by I. above, 2P can be made as small as we please. Hence P + Q can be made as small as we please.

VI. *If either* P *or* Q *become infinitely great, or if* P *and* Q *each*

become infinitely great and both have finally the same sign, then P + Q *becomes infinitely great.*

Proof similar to last.

The inference is not certain if the two have not ultimately the same sign. In this case there arises the indeterminate form $\infty - \infty$.

Example 1.

$$P=x^2/(x-1)^2, \quad Q=(2x-1)/(x-1)^2.$$

When $x=1$, we have $P=1/0=+\infty$, $Q=1/0=+\infty$. Also

$$P+Q=\frac{x^2}{(x-1)^2}+\frac{2x-1}{(x-1)^2}=\frac{x^2+2x-1}{(x-1)^2},$$

$$=\frac{2}{0}=\infty, \text{ when } x=1.$$

Example 2.

$$P=x^2/(x-1)^2, \quad Q=-(2x-1)/(x-1)^2.$$

Here $x=1$ makes $P=+\infty$, $Q=-\infty$, so that we cannot infer $P+Q=\infty$. In fact, in this case,

$$P+Q=\frac{x^2}{(x-1)^2}-\frac{2x-1}{(x-1)^2}=\frac{(x-1)^2}{(x-1)^2}=1$$

for all values of x, or, say, for any value of x as nearly $=+1$ as we please. In this case, therefore, by bringing x as near to $+1$ as we please, we cause the value of $P+Q$ to approach as near to $+1$ as we please.

§ 7.] The propositions stated in last paragraph are the fundamental principles of the theory of the limiting cases of algebraical operation. This subject will be further developed in the chapter on Limits in the second part of this work.

In the meantime we draw the following conclusions, which will be found useful in what follows :—

I. *If* $P=P_1P_2\ldots P_n$, *then* P *will remain finite if* $P_1, P_2, \ldots, P_n$ *all remain finite.*

P *will become infinitely small if one or more of the functions* $P_1, P_2, \ldots, P_n$ *become infinitely small, provided none of the remaining ones become infinitely great.*

P *will become infinitely great if one or more of the functions* $P_1, P_2, \ldots, P_n$ *become infinitely great, provided none of the remaining ones become infinitely small.*

II. *If* $S=P_1+P_2+\ldots+P_n$, *then* S *will remain finite if* $P_1, P_2, \ldots, P_n$ *each remain finite.*

S *will become infinitely small if* $P_1, P_2, \ldots, P_n$ *each become infinitely small.*

S *will become infinitely great if one or more of the functions* P_1, $P_2, \ldots, P_n$ *become infinitely great, provided all those that become infinitely great have the same sign.*

III. *Consider the quotient* P/Q.

$\dfrac{P}{Q}$ *will certainly be finite if both* P *and* Q *be finite,*

$$\begin{array}{rll} \textit{may be finite if } P = 0, & Q = 0, \\ \textit{or if } P = \infty, & Q = \infty. \end{array}$$

$$\begin{array}{rll} \dfrac{P}{Q} \textit{ will certainly } = 0 \textit{ if } P = 0, & Q \neq 0, \\ \textit{or if } P \neq \infty, & Q = \infty; \\ \textit{may } = 0 \textit{ if } P = 0, & Q = 0, \\ \textit{or if } P = \infty, & Q = \infty. \end{array}$$

$$\begin{array}{rll} \dfrac{P}{Q} \textit{ will certainly } = \infty \textit{ if } P = \infty, & Q \neq \infty, \\ \textit{or if } P \neq 0, & Q = 0; \\ \textit{may } = \infty \textit{ if } P = 0, & Q = 0, \\ \textit{or if } P = \infty, & Q = \infty. \end{array}$$

ON THE CONTINUITY OF FUNCTIONS, MORE ESPECIALLY OF RATIONAL FUNCTIONS.

§ 8.] We return now to the question of the continuity of functions.

By the increment of a function $f(x)$ *corresponding to an increment* h *of the independent variable* x *we mean* $f(x + h) - f(x)$.

For example, if $f(x) = x^2$, the increment is $(x+h)^2 - x^2 = 2xh + h^2$.
If $f(x) = 1/x$, the increment is $1/(x+h) - 1/x = -h/x(x+h)$.

The increments may be either positive or negative, according partly to choice and partly to circumstance. The increment of the independent variable x is of course entirely at our disposal; but when any value is given to it, and when x itself is also assigned, the increment of the function or dependent variable is determined.

Example.

Let the function be $1/x$, then if $x=1$, $h=3$, the corresponding increment of $1/x$ is $-3/1(1+3)=-3/4$. If $x=2$, $h=3$, the increment of $1/x$ is $-3/2(2+3)=-3/10$, and so on.

If P be a function of x, and p denote its increment when x is increased from x to $x+h$, then, by the definition of p, $\mathrm{P}+p$ is the value of P when x is altered from x to $x+h$.

We can now prove the following propositions :—

I. *The algebraic sum of any finite number of continuous functions is a continuous function.*

Let us consider $\mathrm{S}=\mathrm{P}-\mathrm{Q}+\mathrm{R}$, say. If the increments of P, Q, R, when x is increased by h, be p, q, r, then the value of S, when x is changed to $x+h$, is $(\mathrm{P}+p)-(\mathrm{Q}+q)+(\mathrm{R}+r)$; and the increment of S corresponding to h is $p-q+r$. Now, since P, Q, R are continuous functions, each of the increments, p, q, r, becomes infinitely small when h becomes infinitely small. Hence, by § 7, I., $p-q+r$ becomes infinitely small when h becomes infinitely small. Hence S is a continuous function.

The argument evidently holds for a sum of any number of terms, provided there be not an *infinite* number of terms.

II. *The product of a finite number of continuous functions is a continuous function so long as all factors remain finite.*

Consider, in the first place, PQ. Let the increments of P and Q, corresponding to the increment h of the independent variable x, be p and q respectively. Then when x is changed to $x+h$ PQ is changed to $(\mathrm{P}+p)(\mathrm{Q}+q)$, that is, to $\mathrm{PQ}+p\mathrm{Q}+q\mathrm{P}+pq$. Hence the increment of PQ corresponding to h is

$$p\mathrm{Q}+q\mathrm{P}+pq.$$

Now, since P and Q are continuous, p and q each become infinitely small when h becomes infinitely small. Hence by § 7, I. and II., it follows that $p\mathrm{Q}+q\mathrm{P}+pq$ becomes infinitely small when h is made infinitely small; at least this will certainly be so, provided P and Q remain finite for the value of x in question, which we assume to be the case.

It follows then that PQ is a continuous function.

Consider now a product of three continuous functions, say PQR. By what has just been established, PQ is a continuous

function, which we may denote by the single letter S; then PQR = SR where S and R are continuous. But, by last case, SR is a continuous function. Hence PQR is a continuous function.

Proceeding in this way, we establish the proposition for any finite number of factors.

Cor. 1. *If* A *be constant, and* P *a continuous function, then* AP *is a continuous function.*

This can either be established independently, or considered as a particular case of the main proposition, it being remembered that the increment of a constant is zero under all circumstances.

Cor 2. Ax^m, *where* A *is constant, and* m *a positive integer, is a continuous function.*

For $x^m = x \times x \times \ldots \times x$ (m factors), and x is continuous, being the independent variable itself. Hence, by the main proposition, x^m is continuous. Hence, by Cor. 1, Ax^m is a continuous function.

Cor. 3. *Every integral function of* x *is continuous; and cannot become infinite for a finite value of* x.

For every integral function of x is a sum of a finite number of terms such as Ax^m. Now each of these terms is a continuous function by Cor. 2. Hence, by Proposition I., the integral function is continuous. That an integral function is always finite for a finite value of its variable follows at once from § 7, I.

III. *If* P *and* Q *be integral functions of* x, *then* P/Q *is finite and continuous for all finite values of* x, *except such as render* Q = 0.

In the first place, if Q $\neq$ 0, then (see § 7, III.) P/Q can only become infinite if either P, or both P and Q, become infinite; but neither P nor Q can become infinite for a finite value of x, because both are integral functions of x. Hence P/Q can only become infinite, if at all, for values of x which make Q = 0.

If a value which makes Q = 0 makes P $\neq$ 0, then P/Q certainly becomes infinite for that value. But, if such a value makes both Q = 0 and also P = 0, then the matter requires further investigation.

Next, as to continuity, let the increments of P and Q corre-

sponding to h, the increment of x, be p and q as heretofore. Then the increment of P/Q is

$$\frac{P+p}{Q+q}-\frac{P}{Q}=\frac{pQ-qP}{Q(Q+q)}.$$

Now, by hypothesis, p and q each become infinitely small when h does so. Also P and Q remain finite. Hence $pQ-qP$ becomes infinitely small. It follows then that $(pQ-qP)/Q(Q+q)$ also becomes infinitely small when h does so, provided always (see § 6) that Q does not vanish for the value of x in question.

Example.

The increment of $1/(x-1)$ corresponding to the increment, h, of x is $1/(x+h-1)-1/(x-1)=-h/(x-1)(x+h-1)$. Now, if $x=2$, say, this becomes $-h/(1+h)$, which clearly becomes infinitely small when h is made infinitely small. On the other hand, if $x=1$, the increment is $-h/0h$, which is infinitely great so long as h has any value differing from 0 by ever so little.

§ 9.] When a function is finite and continuous between two values of its independent variable $x=a$ and $x=b$, its graph forms a continuous curve between the two graphic points whose abscissæ are a and b; that is to say, the graph passes from the one point to the other without break, and without passing anywhere to an infinite distance.

From this we can deduce the following important proposition:—

If $f(x)$ be continuous from $x=a$ to $x=b$, and if $f(a)=p$, $f(b)=q$, then, as x passes through every algebraical value between a and b, $f(x)$ passes at least once, and, if more than once, an odd number of times through every algebraical value between p and q.

Let P and Q be the graphic points corresponding to $x=a$ and $x=b$, AP and BQ their ordinates; then $AP=p$, $BQ=q$. We have supposed p and q both positive; but, if either were negative, we should simply have the graphic point below the x-axis, and the student will easily see by drawing the corresponding figure that this would alter nothing in the following reasoning.

Suppose now r to be any number between p and q, and draw a parallel UV to the x-axis at a distance from it equal to r units of the scale of ordinates, above the axis if r be positive, below if r be negative. The analytical fact that r is

intermediate to p and q is represented by the geometrical fact that the points P and Q lie on opposite sides of UV.

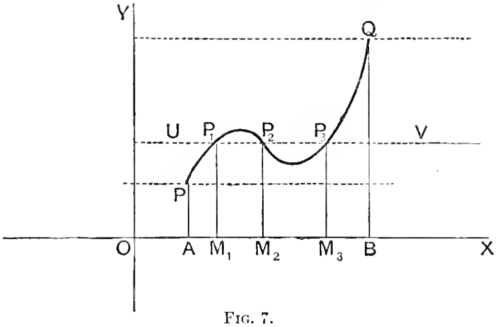


FIG. 7.

Now, since the graph passes continuously from P to Q, it must cross the intermediate line UV; and, since it begins on one side and ends on the other, it must do so either once, or thrice, or five times, or some odd number of times.

Every time the graph crosses UV the ordinate becomes equal to r; hence the proposition is proved.

Cor. 1. *If $f(a)$ be negative and $f(b)$ be positive, or vice versa, then $f(x)$ has at least one root, and, if more than one, an odd number of roots, between $x = a$ and $x = b$, provided $f(x)$ be continuous from $x = a$ to $x = b$.*

This is merely a particular case of the main proposition, for 0 is intermediate to any two values, one of which is positive and the other negative. Hence as x passes from a to b $f(x)$ must pass at least once, and, if more than once, an odd number of times through the value 0.

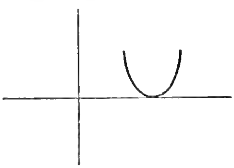


FIG. 8.

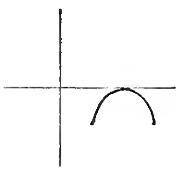


FIG. 9.

In fact, in this case, the axis of x plays the part of the parallel UV.

Observe, however, in regard to the converse of this proposition, that *a function may pass through the value 0 without changing its sign.* For the graph may just graze the x-axis as in Figs. 8 and 9.

Cor. 2. *If $f(a)$ and $f(b)$ have like signs, then, if there be any real roots of $f(x)$ between $x=a$ and $x=b$, there must be an even number, provided $f(x)$ be continuous between $x=a$ and $x=b$.*

Since an integral function is always finite and continuous for a finite value of its variable, the restriction in Cor. 1 is always satisfied, and we see that

Cor. 3. *An integral function can change sign only by passing through the value* 0.

Cor. 4. *If* P *and* Q *be integral functions of x algebraically prime to each other,* P/Q *can only change sign by passing through the values* 0 *or* ∞.

With the hint that the theorem of remainders will enable him to exclude the ambiguous case 0/0, we leave the reader to deduce Cor. 4 from Cor. 3.

Example 1.

When $x=0$, $1-x^2=+1$; and when $x=+2$, $1-x^2=-3$. Hence, since $1-x^2$ is continuous, for some value of x lying between 0 and $+2$ $1-x^2$ must become 0; for 0 is between $+1$ and -3. In point of fact, it becomes 0 once between the limits in question.

Example 2.

$$y=x^3-6x^2+11x-6.$$

When $x=0$, $y=-6$; and when $x=+4$, $y=+6$. Hence, between $x=0$ and $x=+4$ there must lie an odd number of roots of the equation

$$x^3-6x^2+11x-6=0.$$

It is easy to verify in the present case that this is really so; for $x^3-6x^2+11x-6\equiv(x-1)(x-2)(x-3)$; so that the roots in question are $x=1$, $x=2$, $x=3$.

The general form of the graph in the present case is as follows:—

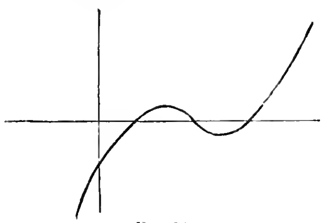


FIG. 10.

Example 3.

When $x=0$, $1/(1-x)=+1$; and when $x=+2$, $1/(1-x)=-1$; but since $1/(1-x)$ becomes infinite and discontinuous between $x=0$ and $x=+2$, namely, when $x=1$, we cannot infer that, for some value of x between 0 and $+2$,

$1/(1-x)$ will become 0, although 0 is intermediate to $+1$ and -1. In fact, $1/(1-x)$ does not pass through the value 0 between $x=0$ and $x=+2$.

§ 10.] It will be convenient to give here the following proposition, which is often useful in connection with the methods we are now explaining.

If $f(x)$ be an integral function of x, then by making x small enough we can always cause $f(x)$ to have the same sign as its lowest term, and by making x large enough we can always cause $f(x)$ to have the same sign as its highest term.

Let us take, for simplicity, a function of the 3rd degree, say

$$y = px^3 + qx^2 + rx + s.$$

If we suppose $s \neq 0$, then it is clear, since by making x small enough we can (see § 7, II.) make $px^3 + qx^2 + rx$ as small as we please, that we can, by making x small enough, cause y to have the same sign as s.

If $\qquad s = 0,$

then we have

$$\begin{aligned} y &= px^3 + qx^2 + rx, \\ &= (px^2 + qx + r)x. \end{aligned}$$

Here by making x small enough we can cause $px^2 + qx + r$ to have the same sign as r, and hence y to have the same sign as rx, which is the lowest existing term in y.

Again, we may write

$$y = x^3\left\{p + \frac{q}{x} + \frac{r}{x^2} + \frac{s}{x^3}\right\}.$$

Here by making x large enough we may make $q/x + r/x^2 + s/x^3$ as small as we please (see § 6, IV., and § 7, II.), that is to say, cause $p + q/x + r/x^2 + s/x^3$ to have the same sign as p. Hence by making x large enough we can cause y to have the same sign as px^3.

If we observe that, by chap. xiv., § 9, we can reduce every integral equation to the equivalent form

$$f(x) \equiv x^n + p_{n-1}x^{n-1} + \ldots + p_0 = 0,$$

and further notice that, in this case, if n be odd,

$$f(+\infty) = +\infty, \quad f(-\infty) = -\infty,$$

and, if n be even,

$$f(+\infty) = +\infty, \quad f(-\infty) = +\infty,$$

we have the following important conclusions.

Cor. 1. *Every integral equation of odd degree with real coefficients has at least one real root, and if it has more than one it has an odd number.*

Cor. 2. *If an integral equation of even degree with real coefficients has any real roots at all, it has an even number of such.*

Cor. 3. *Every integral equation with real coefficients, if it has any complex roots, has an even number of such.*

The student should see that he recognises what are the corresponding peculiarities in the graphs of integral functions of odd or of even degree.

Example.

Show that the equation

$$x^4 - 6x^3 + 11x^2 - x - 4 = 0$$

has at least two real roots.

Let $$y = x^4 - 6x^3 + 11x^2 - x - 4.$$

We have the following scheme of corresponding values:—

x	y
$-\infty$	$+\infty$
0	-4
$+\infty$	$+\infty$

Hence one root at least lies between $-\infty$ and 0, and one at least between 0 and $+\infty$. In other words, there are at least two real roots, one negative the other positive.

We can also infer that, if the remaining two of the possible four be also real, then they must be either both positive or both negative.

When the real roots of an integral equation are not very close together the propositions we have just established enable us very readily to assign upper and lower limits for each of them; and in fact to calculate them by successive approximation. The reader will thus see that the numerical solution of integral equations rests merely on considerations regarding continuity, and may be considered quite apart from the question of their formal solution by means of algebraical functions or otherwise. The application of this idea to the approximate determination of the real roots of an integral equation will be found at the end of the present chapter.

GENERAL PROPOSITIONS REGARDING MAXIMA AND MINIMA VALUES OF FUNCTIONS OF ONE VARIABLE.

§ 11.] *When $f(x)$ in passing through any value, $f(a)$ say, ceases to increase and begins to decrease, $f(a)$ is called a maximum value of $f(x)$.*

When $f(x)$ in passing through the value $f(a)$ ceases to decrease and begins to increase, $f(a)$ is called a minimum value of $f(x)$.

The points corresponding to maxima and minima values of the function are obviously superior and inferior culminating points on its graph, such as P_2 and P_9 in Fig. 1. They are also points where, in general, the tangent to the graph is parallel to the axis of x. It should be noticed, however, that points such as P and Q in Fig. 11 are maxima and minima points, according to our present definition, although it is not true in any proper sense that at them the tangent is parallel to OX. It

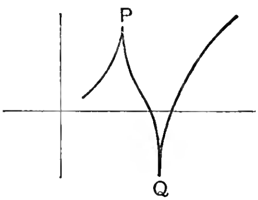


Fig. 11.

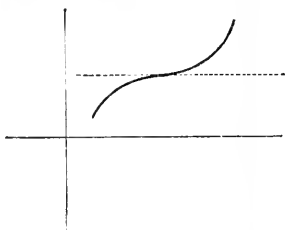


Fig. 12.

should also be observed that the tangent may be parallel to OX and yet the point may not be a true maximum or minimum point. Witness Fig. 12.

We shall include both maximum and minimum values as at present defined under the obviously appropriate name of *turning values*.

§ 12.] By considering an unbroken curve having maxima and minima points (see Fig. 1) the reader will convince himself graphically of the truth of the following propositions:—

I. *So long as $f(x)$ remains continuous its maxima and minima values succeed each other alternately.*

II. *If $x=a$, $x=b$ be two roots of $f(x)$ (a alg.$<b$), then, if $f(x)$ be not constant, but vary continuously between $x=a$ and $x=b$, there must*

be either at least one maximum or at least one minimum value of $f(x)$ *between* $x=a$ *and* $x=b$.

In particular, if $f(x)$ become positive immediately after x passes through the value a, then there must be at least one maximum before x reaches the value b; and, in like manner, if $f(x)$ become negative, at least one minimum.

§ 13.] It is obvious, from the definition of a turning value, and also from the nature of the graph in the neighbourhood of a culminating point, that *we can always find two values of the function on opposite sides of a turning value, which shall be as nearly equal as we please. These two values will be each less or each greater than the turning value according as the turning value is a maximum or minimum.*

Hence, if p be infinitely near a turning value of $f(x)$ (less in the case of a maximum, greater in the case of a minimum), then two roots of $f(x)-p$ will be infinitely nearly equal to one another. It follows, therefore, that, *if* p *be actually equal to a turning value of* $f(x)$, *the function* $f(x)-p$ *will have two of its roots equal.* This criterion may be used for finding turning values, as will be seen in a later chapter.

CONTINUITY AND GRAPHICAL REPRESENTATION OF A FUNCTION OF TWO INDEPENDENT VARIABLES.

§ 14.] Let the function be denoted by $f(x, y)$, and let us denote the dependent variable by z; so that

$$z=f(x, y).$$

We confine ourselves entirely to the case where $f(x, y)$ *is an integral function, and we suppose all the constants to be real, and consider only real values of* x *and* y. *The value of* z *will therefore be always real.*

Since there are now two independent variables, x and y, there are two independent increments, say h and k, to consider. Hence the increment of z, that is, $f(x+h, y+k)-f(x, y)$, now depends on four quantities, x, y, h, k. Since, however, $f(x, y)$ consists of a sum of terms such as $Ax^m y^n$, it can easily be shown by reasoning, like that used in the case of $f(x)$, that *the increment of* z *always becomes infinitely small when* h *and* k *are made infinitely*

small. Hence, *as x and y pass continuously from one given pair of values, say (a, b), to another given pair, say (a', b'), z passes continuously from one value, say c, to another, say c'.*

§ 15.] There is, however a distinct peculiarity in the case now in hand, inasmuch as there are an infinity of different ways in which (x, y) may pass from (a, b) to (a', b'). In fact we require now *a two-dimensional diagram to represent the variations of the independent variables.* Let X′OX, Y′OY be two lines in a

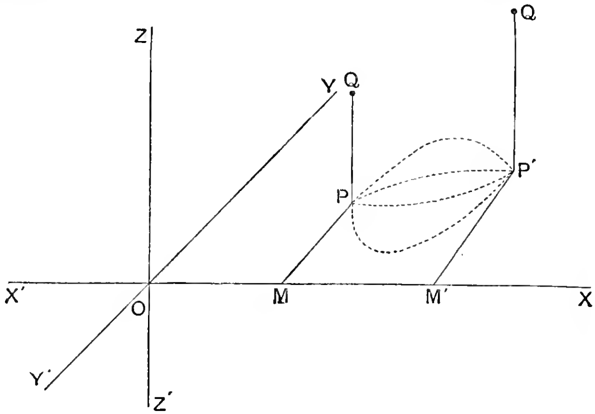


FIG. 13.

horizontal plane drawn from west to east and from south to north respectively. Consider any point P in that plane, whose abscissa and ordinate, with the usual understanding as to sign, are x and y. Then P, which we may call the *variable point*, gives us a graphic representation of the variables (x, y).

Let us suppose that for P $x=a$, $y=b$, and that for another point P′ $x=a'$, $y=b'$. Then it is obvious that, if we pass along any continuous curve whatever from P to P′, x will vary continuously from a to a', and y will vary continuously from b to b'; and, conversely, that any imaginable combination of a continuous variation of x from a to a' with a continuous variation of y from b to b' will correspond to the passage of a point from P to P′ along some continuous curve.

It is obvious, therefore, that the continuous variation of

(x, y) from (a, b) to (a', b') may be accomplished in an infinity of ways. We may call the path in which the point which represents the variables travels the *graph of the variables*.

To represent the value of the function $z = f(x, y)$ we draw through P, the variable point representing (x, y), a vertical line PQ, containing z units of any fixed scale of length that may be convenient, upwards if z be positive, downwards if z be negative. Q is then the graphic point which represents the value of the function $z = f(x, y)$.

To every variable point in the plane XOY there corresponds a graphic point, such as Q; and the assemblage of graphic points constitutes a surface which we call the *graphic surface* of the function $f(x, y)$

When the variable point travels along any particular curve S in the plane XOY, the graphic point of the function travels along a particular curve Σ on the graphic surface; and it is obvious that S is the orthogonal projection of Σ on the plane XOY.

§ 16.] If we seek for values of the variables which correspond to a given value c of the function, we have to draw a horizontal plane U, c units above or below XOY according as c is positive or negative; and find the curve Σ where this plane U meets the graphic surface. This line Σ is what is usually called a *contour line of the graphic surface*. In this case the orthogonal projection S of Σ upon XOY will be simply Σ itself transferred to XOY, and may be called the *contour line of the function for the value c*. All the variable points upon S correspond to pairs of values of (x, y), for which $f(x, y)$ has the given value c.

If we take a number of different values, $c_1, c_2, c_3, \ldots, c_n$, we get a system of as many contour lines. Suppose, for example, that the graph of the function were a rounded conical peak, then the system of contour lines would be like Fig. 14, where the successive curves narrow in towards a point which corresponds to a maximum value of the function.

Any reader who possesses a one-inch contoured Ordnance Survey map has to hand an excellent example of the graphic representation of a function. In this case x and y are the dis-

tances east from the left-hand side of the map, and north from the lower side; and the function z is the elevation of the land

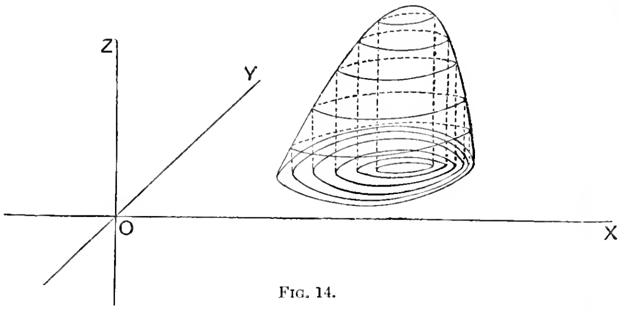


FIG. 14.

at any point above the sea level. The study of such a map from the present point of view will be an excellent exercise both in geometry and in analysis.

An important particular case is that where we seek the values of x and y which make $f(x, y) = 0$. In this case the plane U is the plane XOY. This plane cuts the graphic surface in a continuous curve S (*zero contour line*), every point on which has for its abscissa and ordinate a pair of values that satisfy $f(x, y) = 0$.

The curve S in this case divides the plane into regions, such that in any region $f(x, y)$ has always either the sign + or the sign −, and S always forms the boundary between two regions in which $f(x, y)$ has opposite signs.

If we draw a continuous curve from a point in a + region to a point in a − region, it must cross the boundary S an odd number of times. This corresponds to the analytical statement that *if $f(a, b)$ be positive and $f(a', b')$ be negative, then, if (x, y) vary continuously from (a, b) to (a', b'), $f(x, y)$ will pass through the value 0 an odd number of times.*

The fact just established, that all the "variable points" for which $f(x, y) = 0$ lie on a continuous curve, gives us a beautiful geometrical illustration of the fact established in last chapter, that the equation $f(x, y) = 0$ has an infinite number of solutions, and gives us the fundamental idea of co-ordinate geometry, namely, that

a plane curve can be analytically represented by means of a single equation connecting two variables.

Example.

Consider the function $z=x^2+y^2-1$. If we describe, with O as centre, a circle whose radius is unity, it will be seen that for all points inside this circle z is negative, and for all points outside z is positive. Hence this circle is the zero contour line, and for all points on it we have

$$x^2+y^2-1=0.$$

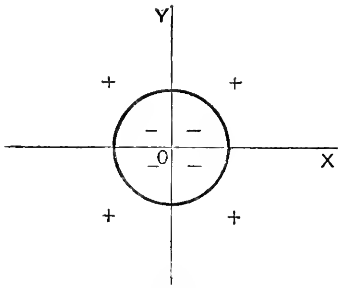


FIG. 15.

INTEGRAL FUNCTIONS OF A SINGLE COMPLEX VARIABLE.

§ 17.] *Here we confine ourselves to integral functions, but no longer restrict either the constants of the function or its independent variable x to be real.*

Let us suppose that $x=\xi+\eta i$, and let us adopt Argand's method of representing $\xi+\eta i$ graphically, so that, if $\Omega M=\xi$, $MP=\eta$, in the diagram of Fig. 16, then P represents $\xi+\eta i$.

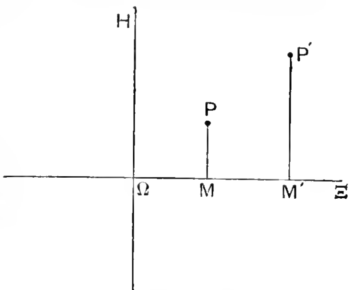


FIG. 16.

If P *move continuously from any position* P *to another* P′, *the complex variable is said to vary continuously.* If the values of (ξ, η) at P and P′ be (α, β) and (α', β') respectively, this is the same as saying that $\xi+\eta i$ *is said to vary continuously from the value* $\alpha+\beta i$ *to the value* $\alpha'+\beta' i$, *when* ξ *varies continuously from* α *to* α', *and* η *varies continuously from* β *to* β'. There are of course

an infinite number of ways in which this variation may be accomplished.

§ 18.] Suppose now we have any integral function of x whose constants may or may not be real. Then we have $f(x) = f(\xi + \eta i)$; but this last can, by the rules of chap. xii., always be reduced to the form $\xi' + \eta' i$, where ξ' and η' are integral functions of ξ and η whose constants are real (say real integral functions of (ξ, η)). Now, by § 14, ξ' and η' are finite and continuous so long as (ξ, η) are finite. *Hence $f(\xi + \eta i)$ varies continuously when $\xi + \eta i$ varies continuously.*

A graphic representation of the function $f(\xi + \eta i)$ can be obtained by constructing another diagram for the complex number $\xi' + \eta' i$. Then the continuity of $f(\xi + \eta i)$ is expressed by saying that, *when the graph of the independent variable is a continuous curve* S, *the graph of the dependent variable is another continuous curve* S′.

Example.

Let $$y = +\sqrt{(1 - x^2)}.$$

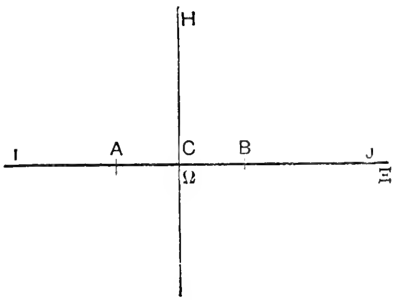


Fig. 17.

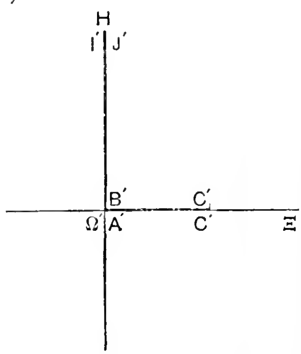


Fig. 18.

For simplicity, we shall confine ourselves to a variation of x which admits only real values; in other words, we suppose η always $=0$.

The path of the independent variable is then IACBJ, the whole extent of the ξ-axis. In the diagram we have taken CA = CB = 1; so that A and B mark the points in the path for which the function begins to have, and ceases to have, a real value.

Let Fig. 18 be the diagram of the dependent variable, $y = \xi' + \eta' i$. If $A'C' = 1$ (A', B', and Ω' are all coincident), then the path of the dependent

variable is the whole of the η'-axis above Ω, together with $A'C'$, each reckoned twice over. The pieces of the two paths correspond as follows :—

Independent Variable.	Dependent Variable.
IA	I'A'
AC	A'C'
CB	C'B'
BJ	B'J'

§ 19.] ξ' and η' being functions of ξ and η, we may represent this fact to the eye by writing

$$\xi' = \phi(\xi, \eta), \quad \eta' = \psi(\xi, \eta).$$

If we seek for values of (ξ, η) that make $\xi' = 0$, that is the same as seeking for values of (ξ, η) that make $\phi(\xi, \eta) = 0$. All the points in the diagram of the independent variable corresponding to these will lie (by § 16) on a curve S.

Similarly all the points that correspond to $\eta' = 0$, that is, to $\psi(\xi, \eta) = 0$, lie on another curve T.

The points for which both $\xi' = 0$ and $\eta' = 0$,—in other words, the points corresponding to roots of $f(\xi + \eta i)$,—must therefore be the intersections of the two curves S and T.

Example.

$$y = ix^2 + 8.$$

If we put $x = \xi + \eta i$, and $y = \xi' + \eta' i$, we have

$$\xi' + \eta' i = i(\xi + \eta i)^2 + 8,$$
$$= 2(4 - \xi\eta) + (\xi^2 - \eta^2)i.$$

Hence $\qquad \xi' = 2(4 - \xi\eta), \qquad \eta' = \xi^2 - \eta^2.$

Hence the S and T curves, above spoken of, are given by the equations

$$2(4 - \xi\eta) = 0 \qquad (S),$$
$$\xi^2 - \eta^2 = 0 \qquad (T).$$

These are equivalent to

$$\eta = \frac{4}{\xi} \qquad (S),$$

$$\left.\begin{array}{l}\eta = +\xi \\ \eta = -\xi\end{array}\right\} \qquad (T).$$

The student should have no difficulty in constructing these. The diagram that results is

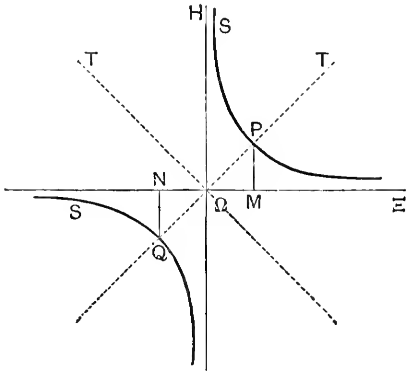


FIG. 19.

The S curve (a rectangular hyperbola as it happens) is drawn thick. The T curve (two straight lines bisecting the angles between the axes) is dotted. The intersections are P and Q.

Corresponding to P we have $\xi=2$, $\eta=2$; corresponding to Q, $\xi=-2$, $\eta=-2$.

It appears therefore that the roots of the function are $+2+2i$ and $-2-2i$. The student may verify that these values do in fact satisfy the equation

$$ix^2+8=0.$$

HORNER'S METHOD FOR APPROXIMATING TO THE VALUES OF THE REAL ROOTS OF AN INTEGRAL EQUATION.

§ 20.] In the following paragraphs we shall show how the ideas of §§ 8-10 lead to a method for calculating digit by digit the numerical value of any real root of an integral equation. It will be convenient in the first place to clear the way by establishing a few preliminary results upon which the method more immediately depends.

§ 21.] *To deduce from the equation*

$$p_0x^n + p_1x^{n-1} + . \quad . \quad . + p_{n-1}x + p_n = 0 \qquad (1)$$

another equation each of whose roots is m times a corresponding root

of (1). Let x be any root of (1); and let $\xi = mx$. Then $x = \xi/m$. Hence, from (1), we have

$$p_0(\xi/m)^n + p_1(\xi/m)^{n-1} + \ldots + p_{n-1}(\xi/m) + p_n = 0.$$

If we multiply by the constant m^n, we deduce the equivalent equation

$$p_0\xi^n + p_1 m\xi^{n-1} + \ldots + p_{n-1}m^{n-1}\xi + p_n m^n = 0 \qquad (2),$$

which is the equation required.

Cor. *The equation whose roots are those of* (1) *with the signs changed is*

$$p_0\xi^n - p_1\xi^{n-1} + \ldots + (-)^{n-1}p_{n-1}\xi + (-)^n p_n = 0 \quad (3).$$

This follows at once by putting $m = -1$ in (2). We thus see that the calculation of a negative real root of any equation can always be reduced to the calculation of a positive real root of a slightly different equation.

Example. The equation whose roots are 10 times the respective roots of

$$3x^3 - 15x^2 + 5x + 6 = 0$$

is

$$3x^3 - 150x^2 + 500x + 6000 = 0.$$

§ 22.] *To deduce from the equation* (1) *of* § 21 *another, each of whose roots is less by* a *than a corresponding root of* (1).

Let x denote any root of (1); ξ the corresponding root of the required equation; so that $\xi = x - a$, and $x = \xi + a$. Then we deduce at once from (1)

$$p_0(\xi + a)^n + p_1(\xi + a)^{n-1} + \ldots + p_{n-1}(\xi + a) + p_n = 0 \quad (4).$$

If we arrange (4) according to powers of ξ, we get

$$p_0\xi^n + q_1\xi^{n-1} + \ldots + q_{n-1}\xi + q_n = 0 \qquad (5),$$

which is the equation required.

It is important to have a simple systematic process for calculating the coefficients of (5). This may be obtained as follows.

Since $\xi = x - a$, we have, by comparing the left-hand sides of (1) and (5),

$$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$$
$$\equiv p_0(x - a)^n + q_1(x - a)^{n-1} + \ldots + q_{n-1}(x - a) + q_n.$$

The problem before us is, therefore, simply to expand the function $f(x) \equiv p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$ in powers of $(x-a)$. Hence, as we have already seen in chap. v., § 21, q_n is the remainder when $f(x)$ is divided by $x-a$; q_{n-1} the remainder when the integral quotient of the last division is divided by $x-a$; and so on. The calculation of the remainder in any particular case is always carried out by means of the synthetic process of chap. v., § 13.

It should be observed that the last coefficient, q_n, is the value of $f(a)$.

Example. To diminish the roots of

$$5x^3 - 11x^2 + 10x - 2 = 0$$

by 1.

We simply reproduce the calculation of § 21, Example 1, in a slightly modified form; thus—

$$\begin{array}{rrrrl}
5 & -11 & +10 & -2 & (1 \\
 & 5 & -6 & 4 & \\
\hline
 & -6 & 4 & \boxed{2} & \\
 & 5 & -1 & & \\
\hline
 & -1 & \boxed{3} & & \\
 & 5 & & & \\
\hline
 & \boxed{4} & & &
\end{array}$$

Hence the required equation is

$$5\xi^3 + 4\xi^2 + 3\xi + 2 = 0.$$

§ 23.] *If one of the roots of the equation* (1) *of* § 21 *be small, say between* 0 *and* + 1, *then an approximate value of that root is* $-p_n/p_{n-1}$. For, if x denote the root in question, we have, by (1),

$$x = -\frac{p_n}{p_{n-1}} - \frac{x^2}{p_{n-1}}(p_{n-2} + p_{n-3}x + \ldots + p_0x^{n-2}) \qquad (6).$$

Hence, if x be small, we have approximately $x = -p_n/p_{n-1}$.

It is easy to assign an upper limit to the error. We have, in fact,

$$x = -p_n/p_{n-1} - \epsilon,$$

where
$$\operatorname{mod} \epsilon < \frac{p_r x^2}{p_{n-1}}(1 + x + \ldots + x^{n-2}),$$

p_r being the numerically greatest among the coefficients $p_0, \ldots, p_{n-3}, p_{n-2}$. Hence, since $x \not> 1$, we have

$$\operatorname{mod} \epsilon < (n-1)p_r/p_{n-1}.$$

It would be easy to assign a closer limit for the error; but in the applications which we shall make of the theorem we have indirect means of estimating the sufficiency of the approximation; all that is really wanted for our purpose is a suggestion of the approximation.

Example. The equation

$$x^3+8192x^2+16036288x-5969856=0$$

has a root between 0 and 1, find a first approximation to that root.

By the above rule, we have for the root in question

$$x=5969856/16036288-\epsilon$$
$$=\cdot37227-\epsilon$$

where $$\epsilon<2\times8192/16036288<\cdot00103.$$

Hence $x=\cdot372$, with an error of not more than 1 in the last digit.

In point of fact, since $x<\cdot4$, we have

$$\epsilon<\{(\cdot4)^3+8192\times(\cdot4)^2\}/16036288,$$
$$<1311/16036288<1600/16000000,$$
$$<\cdot0001;$$

so that the approximation is really correct to the 4th place of decimals.

§ 24.] *Horner's Method.* Suppose that we have an equation $f(x)=0$, having a positive root $235\cdot67$ . . . This root would be calculated, according to Horner's method, as follows:—First we determine, by examining the sign of $f(x)$, that $f(x)=0$ has one root, and only one,* lying between 200 and 300: the first digit is therefore 2. Then we diminish the roots of $f(x)=0$ by 200, and thus obtain the first subsidiary equation, say $f_1(x)=0$. Then $f_1(x)=0$ has a root lying between 0 and 100. Also since the absolute term of $f_1(x)$ is $f(200)$, and no root of $f(x)=0$ lies between 0 and 200, the absolute term of $f(x)$ (that is, $f(0)$) and the absolute term of $f_1(x)$ must have the same sign. By examining the sign of $f_1(x)$ for $x=0, 10, 20, \ldots, 90$, we determine that this root lies between 30 and 40: the next digit of the root of the original equation is therefore 3. The labour of this last process is, in practice, shortened by using the rule of § 23. Let us suppose that 30 is thus suggested; to test whether this

---

* For a discussion of the precautions necessary when an equation has two roots which commence with one or more like digits, see *Burnside and Panton's Theory of Equations*, § 104.

is correct we proceed to diminish the roots of $f_1(x)=0$ by 30,—to deduce, in fact, the second subsidiary equation $f_2(x)=0$. Since the roots of $f(x)$ have now been diminished by 230, the absolute term of $f_2(x)$ is $f(230)$. Hence the absolute term of $f_2(x)$ must have the same sign as the absolute term of $f_1(x)$, unless the digit 3 is too large. In other words, if the digit 3 is too large, we shall be made aware of the fact by a change of sign in the absolute term. In practice, it does not usually occur (at all events in the later stages of the calculation) that the digit suggested by the rule of § 23 is too small; but, if that were so, we should become aware of the error on proceeding to calculate the next digit which would exceed 9.

The second subsidiary equation is now used as before to find the third digit 5.

The third subsidiary equation would give ·6. To avoid the trouble and possible confusion arising from decimal points, we multiply the roots of the third (and of every following) subsidiary equation by 10; or, what is equivalent, we multiply the second coefficient of the equation in question by 10, the third coefficient by 100, and so on; and then proceed as before, observing, however, that, if the trial division for the next digit be made after this modification of the subsidiary equation, that digit will appear as 6, and not as ·6, because the last coefficient has been multiplied by 10^n and the second last by 10^{n-1}.

The fundamental idea of Horner's method is therefore simply to deduce a series of subsidiary equations, each of which is used to determine one digit of the root. The calculation of the coefficients of each of these subsidiary equations is accomplished by the method of § 22. After a certain number of the digits of the root have been found, a number more may be obtained by a contraction of the process above described, the nature of which will be easily understood from the following particular case.

Example. Find an approximation to the least positive root of

$$f(x)\equiv x^3+2x^2-5x-7=0 \qquad (1).$$

Since $\qquad f(0)=-7,\quad f(1)=-9,\quad f(2)=-1,\quad f(3)=+23,$

the root in question lies between 2 and 3. The first digit is therefore 2.

We now diminish the roots of (1) by 2. The calculation of the coefficients runs thus:—

$$
\begin{array}{cccl}
1 & +2 & -5 & -7\ (2 \\
 & 2 & 8 & 6 \\
\hline
 & 4 & 3 & {}^{2}|\,-1 \\
 & 2 & 12 & \\
\hline
 & 6 & {}^{2}|\,15 & \\
 & 2 & & \\
\hline
 & {}^{2}|\,8 & &
\end{array}
$$

where the prefix ${}^{2}|$ is used to mark coefficients of the first subsidiary equation. The first subsidiary equation is therefore

$$x^3+8x^2+15x-1=0.$$

Since the next digit follows the decimal point, we multiply the roots of this equation by 10. The resulting equation is then

$$x^3+80x^2+1500x-1000=0 \qquad (2).$$

Since $1000/1500<1$, it is suggested that the next digit is 0. We therefore multiply the roots of (2) by 10, and deduce

$$x^3+800x^2+150000x-1000000=0 \qquad (3).$$

Since $1000000/150000=6\cdot\ .\ .\ .$, the next digit suggested is 6. We now diminish the roots of (3) by 6.

$$
\begin{array}{cccl}
1 & +800 & +150000 & -1000000\ (06 \\
 & 6 & 4836 & 929016 \\
\hline
 & 806 & 154836 & {}^{4}|\,-70984000 \\
 & 6 & 4872 & \\
\hline
 & 812 & {}^{4}|\,15970800 & \\
 & 6 & & \\
\hline
 & {}^{4}|\,8180 & &
\end{array}
$$

The resulting subsidiary equation, after the multiplication of its roots by 10, is

$$x^3+8180x^2+15970800x-70984000 \qquad (4).$$

Since $70984000/15970800=4\cdot\ .\ .\ .$, the next digit suggested is 4. The reader should notice that, owing to the continual multiplication of the roots by 10, the coefficients towards the right increase in magnitude much more rapidly than those towards the left: it is for this reason that the rule of § 23 becomes more and more accurate as the operation goes on. Thus, even at the present stage, the quotient 70984000/15970800 would give correctly more than one of the following digits, as may be readily verified.

We now diminish the roots of (4) by 4; and add the zeros to the coefficients as before.

1	+8180	+15970800	−70984000 (4
	4	32736	64014144
	8184	16003536	−6969856000
	4	32752	
	8188	1603628800	
	4		
	81920		

Then we have the subsidiary equation

$$x^3+81920x^2+1603628800x-6969856000 \qquad (5).$$

It will be observed that throughout the operation, so far as it has gone, the two essential conditions for its accuracy have been fulfilled, namely, that the last coefficient shall retain the same sign, and that each digit shall come out not greater than 9. It will also be observed that the number of the figures in the working columns increases much more rapidly than their utility in determining the digits of the root. All that is actually necessary for the suggestion of the next digit at any step, and to make sure of the accuracy of the suggestion, is to know the first two or three figures of the last two coefficients.

Unless, therefore, a very large number of additional digits of the root is required, we may shorten the operation by neglecting some of the figures in (5). If, for example, we divide all the coefficients of (5) by 1000, we get the equivalent equation *

$$\cdot 001x^3+81\cdot 92x^2+1603628\cdot 8x-6969856=0 \qquad (5').$$

Hence, retaining only the integral parts of the coefficients, we have

$$0x^3+81x^2+1603628x-6969856=0 \qquad (5'').$$

It will be noticed that the result is the same as if, instead of adding zeros, as heretofore, we had cut off one figure from the second last coefficient, two from the third last, and so on.†

Since $6969856/1603628=4\cdot$ . . ., we have for the next digit 4. We then diminish the roots of (5″) by 4. In the necessary calculation the first working column now disappears owing to the disappearance of the coefficient of x^3; we have in its place simply 81 standing unaltered. It is advisable, however, in multiplying the contracted coefficients by 4 to carry the nearest number of tens from the last figure cut off (just as in ordinary contracted multiplication and division and for the same reason).

* If the reader find any difficulty in following the above explanation of the contracted process, he can satisfy himself of its validity by working out the above calculation to the end in full and then running his pen through the unnecessary figures.

† In many cases it may not be advisable to carry the contraction so far at each step as is here done. We might, for instance, divide the coefficients of 5 by 100 only. The resulting subsidiary equation would then be

$$0x^3+819x^2+16036288x-69698560,$$

with which we should proceed as before.

The next step, therefore, runs thus:—

81	+1603628	−6969856 (4
	328	6415828
	1603957	^{6}| −554028
	328	
	^{6}| 1604285	

The corresponding subsidiary equation is

$$81x^2+16042850x-55402800=0 \qquad (6);$$

or, contracted,

$$0x^2+160428x-554028=0 \qquad (6').$$

The next digit is 3; and, as the coefficient of x^2, namely 0·81, still has a slight effect on the second working column, the calculation runs thus:—

0	+160428	−554028 (3
	2	481293
	160431	^{7}| −72735
	2	
	^{7}| 160433	

The resulting subsidiary equation after contraction is

$$16043x-72735=0 \qquad (7).$$

The rest of the operation now coincides with the ordinary process of contracted division; it represents, in fact, the solution of the linear equation (7), that is (see chap. xvi., § 1), the division of 72735 by 16043.

The whole calculation may be arranged in practice as below. But the prefixes $^{2}|$, $^{3}|$, &c., which indicate the coefficients of the various equations, may be omitted. Also the record may be still farther shortened by performing the multiplications and additions or subtractions mentally, and only recording the figures immediately below the horizontal lines in the following scheme. The advisability of this last contraction depends of course on the arithmetical power of the calculator.

1	+2	−5	−7 (2·064434533
	2	8	6
	4	3	^{3}| −1000000
	2	12	929016
	6	$^{2,\,3}$| 150000	^{4}| −70984000
	2	4836	64014144
	$^{2,\,3}$| 800	154836	^{5}| −6969856
	6	4872	6415828
	806	^{4}| 15970800	^{6}| −554028
	6	32736	481293
	812	16003536	^{7}| −72735
	6	32752	64173
	^{4}| 8180	^{5}| 16036288	^{8}| −8562
	4	328	8022
	8184	1603957	^{9}| −540
	4	328	481
	8188	^{6}| 1604285	^{10}| −59
	4	2	48
	^{5}| 8192	160431	−11
		2	
		^{7}| 160433	

The number of additional digits obtained by the contracted process is less by two than the number of digits in the second last coefficient at the beginning of the contraction. Owing to the uncertainty of the carriages the last digit is uncertain, but the next last will in such a case as the present be absolutely correct. In fact, by substituting in the original equation, it is easily verified that the root lies between 2·064434534 and 2·064434535 ; so that the last digit given above errs in defect by 1 only. The number of accurate figures obtained by the contracted process will occasionally be considerably less than in this example ; and the calculator must be on his guard against error in this respect (see Horner's Memoir, cited below).

§ 25.] Since the extraction of the square, cube, fourth, . . . roots of any number, say 7, is equivalent to finding the positive real root of the equations, $x^2 + 0x - 7 = 0$, $x^3 + 0x^2 + 0x - 7 = 0$, $x^4 + 0x^3 + 0x^2 + 0x - 7 = 0$, . . . respectively, it is obvious that by Horner's method we can find to any desired degree of approximation the root of any order of any given number whatsoever. In fact, the process, given in chap. xi., § 13, for extracting the square root, and the process, very commonly given in arithmetical text-books, for extracting the cube root will be found to be contained in the scheme of calculation described in § 24.*

---

* Horner's method was first published in the *Transactions of the Philosophical Society of London* for 1819. Considering the remarkable elegance, generality, and simplicity of the method, it is not a little surprising that it has not taken a more prominent place in current mathematical text-books. Although it has been well expounded by several English writers (for example, De Morgan, Todhunter, Burnside and Panton), it has scarcely as yet found a recognised place in English curricula. Out of five standard Continental text-books where one would have expected to find it we found it mentioned in only one, and there it was expounded in a way that showed little insight into its true character. This probably arises from the mistaken notion that there is in the method some algebraic profundity. As a matter of fact, its spirit is purely arithmetical ; and its beauty, which can only be appreciated after one has used it in particular cases, is of that indescribably simple kind which distinguishes the use of position in the decimal notation and the arrangement of the simple rules of arithmetic. It is, in short, one of those things whose invention was the creation of a commonplace. For interesting historical details on this subject, see De Morgan—*Companion to British Almanack*, for 1839 ; Article "Involution and Evolution," *Penny Cyclopædia* ; and *Budget of Paradoxes*, pp. 292, 374.

Exercises XXII.

[The student should trace some at least of the curves required in the following graphic exercises by laying them down correctly to some convenient scale. He will find this process much facilitated by using paper ruled into small squares, which is sold under the name of Plotting Paper.]

Discuss graphically the following functions :—

(1.) $y=\frac{1}{x}$. (2.) $y=\frac{1}{x^2+1}$. (3.) $y=\frac{1}{(x-1)^2}$.

(4.) $y=\frac{1}{(x-1)^3}$. (5.) $y=\frac{x-1}{x-2}$. (6.) $y=\frac{x^2}{x^2-9}$.

(7.) Construct to scale the graph of $y=-x^2+8x-9$; and obtain graphically the roots of the equation $x^2-8x+9=0$ to at least three places of decimals.

(8.) Solve graphically the equation

$$x^3-16x^2+71x-129=0.$$

(9.) Discuss graphically the following question. Given that y is a continuous function of x, does it follow that x is a continuous function of y?

(10.) Show that when h, the increment of x, is very small, the increment of

$$p_nx^n+p_{n-1}x^{n-1}+\ .\ \ .\ \ .+p_1x+p_0$$

is

$$(np_nx^{n-1}+(n-1)p_{n-1}x^{n-2}+\ .\ \ .\ \ .+1.p_1)h.$$

(11.) If h be very small, and $x=1$, find the increment of $2x^3-9x^2+12x+5$.

(12.) If an equation of even degree have its last term negative, it has at least two real roots which are of opposite signs.

(13.) Indicate roughly the values of the real roots of

$$10x^3-17x^2+x+3=0.$$

(14.) What can you infer regarding the roots of

$$x^3-5x+8=0\ ?$$

(15.) Show by considerations of continuity alone that $x^n-1=0$ cannot have more than one real root, if n be odd.

(16.) If $f(x)$ be an integral function of x, and if $f(a)=-p$, $f(b)=+q$, where p and q are both small, show that $x=(qa+pb)/(p+q)$ is an approximation to a root of the equation $f(x)=0$.

Draw a series of contour lines for the following functions, including in each case the zero contour line :—

(17.) $z=xy$. (18.) $z=\frac{x}{y}$. (19.) $z=x^2-y^2$. (20.) $z=\frac{x^2+y^2}{x}$.

Is the proposition of § 16 true for the last of these ?

Draw the Argand diagram of the dependent variable in the following cases, the path of the independent variable being in each case a circle of radius unity whose centre is Ω :—

(21.) $y=\frac{1}{x}$. (22.) $y=+\sqrt{x}$. (23.) $y=\sqrt[n]{x}$. (24.) $y=1-x^2$.

Find by Horner's method the positive real roots of the following equations in each case to at least seven places of decimals :—*

(25.) $x^3-2=0$.

(26.) $x^3-2x-5=0$.

(27.) $x^3+x-1000=0$.

(28.) $x^3-46x^2-36x+18=0$.

(29.) $x^4+x^3+x^2+x-127694=0$.

(30.) $x^4-80x^3+24x^2-6x-80379639=0$.

(31.) $x^5-4x^4+7x^3-863=0$.

(32.) $x^5-7=0$.

(33.) $x^5-4x-2000=0$.

(34.) $4x^6+7x^5+9x^4+6x^3+5x^2+3x-792=0$.

(35.) Find to twenty decimal places the negative root of $2x^4+3x^3-6x-8=0$.

(36.) Continue the calculation on p. 344 two stages farther in its uncontracted form; and then estimate how many more digits of the root could be obtained by means of the trial division alone.

---

* Most of these exercises are taken from a large selection given in De Morgan's *Elements of Arithmetic* (1854).

CHAPTER XVI.

Equations and Functions of the First Degree.

EQUATIONS WITH ONE VARIABLE.

§ 1.] It follows by the principles of chap. xiv. that every integral equation of the 1st degree can be reduced to an equivalent equation of the form

$$ax + b = 0 \qquad (1);$$

this may therefore be regarded as a general form, including all such equations. As a particular case b may be zero; but we suppose, for the present at least, that a is neither infinitely great nor infinitely small.

Since $a \neq 0$, we may write (1) in the form

$$a\left\{x - \left(-\frac{b}{a}\right)\right\} = 0 \qquad (2);$$

whence we see that one solution is $x = -b/a$. We know already, by the principles of chap. xiv., § 6, that *an integral equation of the* 1*st degree in one variable has one and only one solution.* Hence we have completely solved the given equation (1).

It may be well to add another proof that the solution is unique.

Let us suppose that there are two distinct solutions, $x=\alpha$ and $x=\beta$, of (1). Then we must have

$$a\alpha + b = 0,$$

$$a\beta + b = 0.$$

From these, by subtraction, we derive

$$a(\alpha - \beta) = 0.$$

Now, by hypothesis, $a \neq 0$, therefore we must have $\alpha - \beta = 0$, that is, $\alpha = \beta$; in other words, the two solutions are not distinct.

§ 2.] *Two equations of the 1st degree in one variable will in general be inconsistent.*

If the equations be $$ax + b = 0 \quad (1),$$
$$a'x + b' = 0 \quad (2),$$
the necessary and sufficient condition for consistency is
$$ab' - a'b = 0 \quad (3).$$

The solution of (1) is $x = -b/a$, and the solution of (2) is $x = -b'/a'$. These will not in general be the same; hence the equations (1) and (2) will in general be inconsistent.

The necessary and sufficient condition that (1) and (2) *be* consistent is
$$-\frac{b}{a} = -\frac{b'}{a'} \quad (4).$$

Since $a \neq 0$, $a' \neq 0$, (4) is equivalent to
$$a'b = ab',$$
or
$$ab' - a'b = 0.$$

Obs. 1. If $b = 0$ and $b' = 0$, then the condition of consistency is satisfied. In this case the equations become $ax = 0$, $a'x = 0$; and these have in fact the common solution $x = 0$.

Obs. 2. When two equations of the 1st degree in one variable are consistent, the one is derivable by multiplying the other by a constant. In fact, since $a \neq 0$, if we also suppose $b \neq 0$, we derive from (3), by dividing by ab and then transposing,
$$\frac{a'}{a} = \frac{b'}{b}, \text{ each } = k, \text{ say};$$
hence
$$a' = ka, \quad b' = kb,$$
so that
$$a'x + b' \equiv kax + kb,$$
$$\equiv k(ax + b).$$

If, then, (3) be satisfied, (2) is nothing more or less than
$$k(ax + b) = 0$$
where k is a constant.

This might have been expected, for, transpositions apart, the only way of deriving from a single equation another perfectly equivalent is to multiply the given equation by a constant.

EXERCISES XXIII.

Solve the following equations :—

(1.) $$\frac{18x+7}{2}-\left(2x-\frac{2x-1}{7}\right)=36.$$

(2.) $$\frac{1-\frac{1+(1-x)/2}{3}}{4}=1.$$

(3.) $$\frac{51}{3x-1}=\frac{62}{12x+5}$$

(4.) $$5+\frac{2}{3-\frac{1}{4-x}}=\frac{29}{5}.$$

(5.) $$\cdot 68(\cdot 32x-\cdot 5)+\frac{x-5}{1\cdot 368}=3\cdot 694x,$$

find x to three places of decimals.

(6.) $$a/(1-bx)=b/(1-ax).$$

(7.) $$(a+x)(b+x)-a(b+c)=(ca^2+bx^2)/b.$$

(8.) $$\frac{x-a}{b-a}+\frac{x-c}{b-c}=2.$$

(9.) $$\frac{x^2-a^2}{x-a}+\frac{x^2-b^2}{x-b}+\frac{x^2-c^2}{x-c}=a+b+c-3x.$$

(10.) $$(a^3+b^3)x+a^3-b^3=a^4-b^4+ab(a^2+b^2).$$

(11.) $$\frac{x^2-a^2}{b+a}-\frac{x^2-c^2}{b+c}=c-a.$$

(12.) $$(x-1)(x+2)(2x-2)=(2x-1)(2x+1)(x/2+1).$$

(13.) $$\frac{1}{x+1}-\frac{2}{x+2}=\frac{3}{x+3}-\frac{4}{x+4}.$$

(14.) $$\frac{x-1}{x-2}-\frac{x-3}{x-4}=\frac{x-2}{x-3}-\frac{x-4}{x-5}.$$

(15.) $$\frac{11}{12x+11}+\frac{5}{6x+5}=\frac{7}{4x+7}.$$

(16.) $$\frac{3-x}{1-x}-\frac{5-x}{7-x}=1-\frac{x^2-2}{7-8x+x^2}.$$

(17.) $$\frac{2}{x+2}+\frac{14}{x+10}=\frac{10}{x+6}+\frac{6}{x+14}.$$

(18.) $$\frac{x^2-4x+5}{x^2+6x+10}-\left(\frac{x-2}{x+3}\right)^2=0.$$

(19.) $$\frac{1}{x+1}+\frac{1}{x+2}-\frac{3(x+5)}{(x+1)(x+2)}=6-\frac{6x+17}{x+2}.$$

(20.) $$\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}=\frac{4ab}{4b^2-x^2}.$$

(21.) $$\frac{(a+b)x+c}{(a-b)x+d}-\frac{(a-b)x+c}{(a+b)x+f}=\frac{4ab}{(a+b)(a-b)}$$

(22.) $$\frac{a+b}{x-a}+\frac{a-b}{x-b}=\frac{a}{x-a}-\frac{b}{x-b}.$$

(23.) $$\frac{a}{x-a}+\frac{b}{x-b}=\frac{a^2+b^2}{x(x-a-b)+ab}.$$

(24.) $$\frac{x}{x+a-b}+\frac{x}{x+b-c}=2.$$

(25.) $$\frac{1}{(x-a)(x-b)}-\frac{2}{(x-a)(x-c)}+\frac{1}{(x-b)(x-c)}$$
$$=\frac{1}{(x+a)(x+b)}-\frac{2}{(x+a)(x+c)}+\frac{1}{(x+b)(x+c)}.$$

EQUATIONS WITH TWO VARIABLES.

§ 3.] *A single equation of the* 1*st degree in two variables has a one-fold infinity of solutions.*

Consider the equation

$$ax+by+c=0 \qquad (1).$$

Assign to y any constant value we please, say β, then (1) becomes

$$ax+b\beta+c=0 \qquad (2).$$

We have now an equation of the 1st degree in one variable, which, as we have seen, has one and only one solution, namely, $x=-(b\beta+c)/a$.

We have thus obtained for (1) the solution $x=-(b\beta+c)/a$, $y=\beta$, where β may have any value we please. In other words, we have found an infinite number of solutions of (1).

Since the solution involves the *one* arbitrary constant β, we say that the equation (1) has a one-fold infinity (sometimes symbolised by ∞^1) of solutions.

Example.

$$3x-2y+1=0,$$

the solutions are given by

$$x=\frac{2\beta-1}{3}, \quad y=\beta;$$

we have, for example, for $\beta=-2$, $\beta=-1$, $\beta=0$, $\beta=+\frac{1}{2}$, $\beta=+1$, $\beta=+2$, the following solutions :—

β	-2	-1	0	$+\frac{1}{2}$	$+1$	$+2$
x	$-\frac{5}{3}$	-1	$-\frac{1}{3}$	0	$+\frac{1}{3}$	$+1$
y	-2	-1	0	$+\frac{1}{2}$	$+1$	$+2$

And so on.

§ 4.] We should expect, in accordance with the principles of chap. xiv., § 5, that a system of two equations each of the 1st degree in two variables admits of definite solution.

The process of solution consists in deducing from the given system an equivalent system of two equations in which the variables are separated; that is to say, a system such that x alone appears in one of the equations and y alone in the other.

We may arrive at this result by any method logically consistent with the general principles we have laid down in chap. xiv., for the derivation of equations. The following proposition affords one such method :—

If l, l', m, m' be constants, any one of which may be zero, but which are such that $lm'-l'm\neq 0$, then the two systems

$$ax+by+c=0 \qquad (1),$$
$$a'x+b'y+c'=0 \qquad (2),$$

and

$$l(ax+by+c)+l'(a'x+b'y+c')=0 \qquad (3),$$
$$m(ax+by+c)+m'(a'x+b'y+c')=0 \qquad (4),$$

are equivalent.

It is obvious that any solution of (1) and (2) will satisfy (3) and (4); for any such solution reduces both $ax+by+c$ and $a'x+b'y+c'$ to zero, and therefore also reduces the left-hand sides of both (3) and (4) to zero.

Again, any solution of (3) and (4) is obviously a solution of

$$\begin{aligned} &m'\{\ l(ax+by+c)+\ \ l'(a'x+b'y+c')\} \\ -&l'\ \{m(ax+by+c)+m'(a'x+b'y+c')\}=0 \qquad (5), \\ -&m\ \{\ l(ax+by+c)+\ \ l'(a'x+b'y+c')\} \\ +&l\ \ \{m(ax+by+c)+m'(a'x+b'y+c')\}=0 \qquad (6). \end{aligned}$$

Now (5) and (6) reduce to

$$(lm'-l'm)\ (ax\ +by\ +c\ )=0 \qquad (7),$$

$$(lm'-l'm)\ (a'x+b'y+c')=0 \qquad (8),$$

and, provided $lm'-l'm \neq 0$, (7) and (8) are equivalent to

$$ax\ +by\ +c\ =0,$$

$$a'x+b'y+c'=0.$$

We have therefore shown that every solution of (1) and (2) is a solution of (3) and (4); and that every solution of (3) and (4) is a solution of (1) and (2).

All we have now to do is to give such values to l, l', m, m' as shall cause y to disappear from (3), and x to disappear from (4). This will be accomplished if we make

$$l=+b', \quad l'=-b,$$

$$m=-a', \quad m'=+a;$$

so that $$lm'-l'm=ab'-a'b.$$

The system (3) and (4) then reduces to

$$(ab'-a'b)x+cb'-c'b=0 \qquad (3'),$$

$$(ab'-a'b)y+c'a-ca'=0 \qquad (4');$$

and this new system (3'), (4') will be equivalent to (1), (2), provided

$$ab'-a'b \neq 0 \qquad (9).$$

But (3') and (4') are each equations of the 1st degree in one variable, and, since $ab'-a'b \neq 0$, they each have one and only one solution, namely—

$$\left.\begin{aligned} x&=-\frac{cb'-c'b}{ab'-a'b} \\ y&=-\frac{ac'-a'c}{ab'-a'b} \end{aligned}\right\} \qquad (10).$$

It therefore follows that the system

$$ax + by + c = 0 \qquad (1),$$
$$a'x + b'y + c' = 0 \qquad (2)$$

has one and only one definite solution, namely, (10), *provided*

$$ab' - a'b \neq 0 \qquad (9).$$

The method of solution just discussed goes by the name of *cross multiplication*, because it consists in taking the coefficient of y from the second equation, multiplying the first equation therewith; then taking the coefficient of y from the first equation, multiplying the second therewith; and finally subtracting the two equations, with the result that a new equation appears not containing y.

The following *memoria technica* for the values of x and y will enable the student to recollect the values in (10).

The denominators are the same, namely, $ab' - a'b$, formed from the coefficients of x and y thus

$$\begin{matrix} a & & b \\ & \times & \\ a' & & b' \end{matrix}$$

the line sloping down from left to right indicating a positive product, that from right to left a negative product.

The numerator of x is formed from its denominator by putting c and c' in place of a and a' respectively.

The numerator of y by putting c and c' in place of b and b'.

Finally, negative signs must be affixed to the two fractions.

Another way which the reader may prefer is as follows:—

Observe that we may write (10) thus,

$$x = \frac{bc' - b'c}{ab' - a'b}, \quad y = \frac{ca' - c'a}{ab' - a'b} \qquad (11),$$

where the common denominator and the two numerators are formed according to the scheme

$$\begin{matrix} a & & b & & c & & a \\ & \times & & \times & & \times & \\ a' & & b' & & c' & & a'. \end{matrix}$$

It is very important to remark that (1) and (2) are collaterally symmetrical with respect to $\begin{pmatrix} x, & y \\ a, & b \\ a', & b' \end{pmatrix}$, see chap. iv., § 20. Hence, if we know the value of x, we can derive the value of y by putting everywhere b for a, a for b, b' for a', and a' for b'. In

fact the value of y thus derived from the value of x in (10) is $-(ca'-c'a)/(ba'-b'a)$; and this is equal to $-(ac'-a'c)/(ab'-a'b)$, which is the value of y given in (10).

Example 1.

$$3x+2y-3=0 \qquad (\alpha),$$
$$-9x+4y+5=0 \qquad (\beta).$$

Proceeding by direct application of (11), we have

$$\begin{array}{cccc} +3 & +2 & -3 & +3 \\ -9 & +4 & +5 & -9 \end{array}$$

$$x=\frac{10+12}{12+18}=\frac{11}{15}, \quad y=\frac{27-15}{12+18}=\frac{2}{5}.$$

Or thus: multiply (α) by 2, and we have the equivalent system

$$6x+4y-6=0,$$
$$-9x+4y+5=0;$$

whence, by subtraction,

$$15x-11=0,$$

which gives

$$x=\frac{11}{15}.$$

Again multiplying (α) by 3, and then adding (β), we have

$$10y-4=0,$$

which gives

$$y=\frac{4}{10}=\frac{2}{5}.$$

Example 2.

$$\frac{x}{\alpha}+\frac{y}{\beta}=1,$$
$$\frac{x}{\alpha^2}+\frac{y}{\beta^2}=\frac{1}{\gamma}.$$

Multiplying the first of these equations by $\frac{1}{\beta}$, and subtracting the second, we obtain

$$\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)x=\frac{1}{\beta}-\frac{1}{\gamma},$$

that is,

$$\frac{\alpha-\beta}{\alpha^2\beta}x=\frac{\gamma-\beta}{\beta\gamma},$$

whence

$$x=\frac{\alpha^2(\gamma-\beta)}{\gamma(\alpha-\beta)}.$$

Since the equations are symmetrical in $\begin{pmatrix} x, & y \\ \alpha, & \beta \end{pmatrix}$ we get the value of y by interchanging α and β, namely,

$$y=\frac{\beta^2(\gamma-\alpha)}{\gamma(\beta-\alpha)}.$$

Sometimes, before proceeding to apply the above method, it is convenient to replace the given system by another which is equivalent to it but simpler.

Example 3.

$$a^2x+b^2y=2ab(a+b) \qquad (\alpha),$$
$$b(2a+b)x+a(a+2b)y=a^3+a^2b+ab^2+b^3 \qquad (\beta).$$

By adding, we deduce from (α) and (β)

$$(a+b)^2x+(a+b)^2y=(a+b)^3,$$

which is equivalent to

$$x+y=a+b \qquad (\gamma).$$

It is obvious that (α) and (γ) are equivalent to (α) and (β). Multiplying (γ) by b^2 and subtracting, we have

$$(a^2-b^2)x=2a^2b+ab^2-b^3,$$
$$=b(2a-b)(a+b).$$

Hence
$$x=\frac{b(2a-b)}{a-b}.$$

Since the original system is symmetrical in $\begin{pmatrix} x, & y \\ a, & b \end{pmatrix}$, we have

$$y=\frac{a(2b-a)}{b-a}.$$

§ 5.] Under the theory of last paragraph a variety of particular cases in which one or more of the constants a, b, c, a', b', c' involved in the two equations

$$ax+by+c=0,$$
$$a'x+b'y+c'=0$$

become zero are admissible; all cases, in short, which do not violate the condition $ab'-a'b \neq 0$.

Thus we have the following *admissible cases*:—

$a=0$	(1),	$b'=0$	(4),
$b=0$	(2),	$a=0$ and $b'=0$	(5),
$a'=0$	(3),	$a'=0$ and $b=0$	(6).

The following are *exceptional* cases, because they involve $ab'-a'b=0$:—

$a=0$ and $a'=0$ (I.),

$a=0$ and $b=0$ (II.),

$b'=0$ and $a'=0$ (III.),

$b'=0$ and $b=0$ (IV.),

a, b, a', b' all different from 0, but such that $ab'-a'b=0$ (V.).

We shall return again to the consideration of the exceptional cases. In the meantime the reader should verify that the formulæ (10) do really give the correct solution in cases (1) to (6), as by theory they ought to do.

Take case (1), for example. The equations in this case reduce to

$$by+c=0, \quad a'x+b'y+c'=0.$$

The first gives $y=-c/b$, and this value of y reduces the second to

$$a'x-\frac{b'c}{b}+c'=0,$$

which gives

$$x=\frac{b'c-bc'}{a'b}.$$

It will be found that (10) gives the same result, if we put $a=0$.

There is one special case that deserves particular notice, that, namely, where $c=0$ and $c'=0$; so that the two equations are *homogeneous*, namely,

$$ax+by=0 \qquad (\alpha),$$

$$a'x+b'y=0 \qquad (\beta).$$

If $ab'-a'b \neq 0$, these formulæ (10) give $x=0$, $y=0$ as the only possible solution. If $ab'-a'b=0$, these formulæ are no longer applicable; what then happens will be understood if we reflect that, provided $y \neq 0$, (α) and (β) may be written

$$az+b=0 \qquad (\alpha'),$$

$$a'z+b'=0 \qquad (\beta'),$$

where $z=x/y$.

We now have two equations of the 1st degree in z, which are consistent (see § 2), since $ab'-a'b=0$. Each of them gives the same value of z, namely, $z=-b/a$, or $z=-b'/a'$ (these two being equal by the condition $ab'-a'b=0$).

If then $ab'-a'b \neq 0$, *the only solution of* (α) *and* (β) *is* $x=0$, $y=0$; *if* $ab'-a'b=0$, x *and* y *may have any values such that the ratio* $x/y=-b/a=-b'/a'$.

§ 6.] There is another way of arranging the process of solution, commonly called *Bezout's method*,* which is in reality merely a variety of the method of § 4.

* For an account of Bezout's methods, properly so called, see Muir's papers on the "History of Determinants;" *Proc. R.S.E.*, 1886.

If λ *be any finite constant quantity whatever,* then any solution of the system*

$$ax+by+c=0, \quad a'x+b'y+c'=0 \qquad (1)$$

is a solution of the equation

$$(ax+by+c)+\lambda(a'x+b'y+c')=0 \qquad (2),$$

that is to say, of
$$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')=0 \qquad (3).$$

Now, since λ is at our disposal, we may so choose it that y shall disappear from (3); then must

$$\lambda b'+b=0 \qquad (4),$$

and (3) will reduce to
$$(a+\lambda a')x+(c+\lambda c')=0 \qquad (5).$$

From (4) we have $\lambda=-b/b'$, and, using this value of λ, we deduce from (5)

$$x=-\frac{c+\lambda c'}{a+\lambda a'}=-\frac{b'c-bc'}{ab'-a'b},$$

which agrees with (10).

The value of y may next be obtained by so determining λ that x shall disappear from (3). We thus get

$$\lambda a'+a=0 \qquad (6),$$

$$(b+\lambda b')y+(c+\lambda c')=0 \qquad (7),$$

and so on.

To make this method independent and complete, theoretically, it would of course be necessary to add a proof that the values of x and y obtained do in general actually satisfy (1) and (2); and to point out the exceptional case.

§ 7.] There is another way of proceeding, which is interesting and sometimes practically useful.

The systems

$$\left.\begin{aligned} ax+by+c&=0\\ a'x+b'y+c'&=0\end{aligned}\right\} \qquad (1)$$

and

$$\left.\begin{aligned} y&=-\frac{ax+c}{b}\\ a'x+b'y+c'&=0\end{aligned}\right\} \qquad (2)$$

are equivalent, provided $b\neq 0$, for the first equation of (2) is derived from the first of (1) by the reversible processes of transposition and multiplication by a constant factor.

Also, since any solution of (2) makes y identically equal to $-(ax+c)/b$, we may replace y by this value in the second equation of (2). We thus deduce the equivalent system,

---

* So far as logic is concerned λ might be a function of the variables, but for present purposes it is taken to be constant. A letter introduced in this way is usually called an "indeterminate multiplier"; more properly it should be called an "undetermined multiplier."

$$\left.\begin{aligned} y &= -\frac{ax+c}{b} \\ a'x - \frac{b'(ax+c)}{b} + c' &= 0 \end{aligned}\right\} \quad (3).$$

Now, since $b \neq 0$, the second of the equations (3) gives

$$(a'b - ab')x + (bc' - b'c) = 0 \quad (4).$$

If $a'b - ab' \neq 0$, (4) has one and only one solution, namely,

$$x = \frac{bc' - b'c}{ab' - a'b} \quad (5),$$

this value of x reduces the first of the equations (3) to

$$y = -\frac{1}{b}\left\{\frac{a(bc' - b'c)}{ab' - a'b} + c\right\},$$

$$= -\frac{abc' - a'bc}{b(ab' - a'b)},$$

that is, to

$$y = \frac{ca' - c'a}{ab' - a'b} \quad (6).$$

The equations (5) and (6) are equivalent to the system (3), and therefore to the original system (1). Hence we have proved that, if $ab' - a'b \neq 0$ and $b \neq 0$, the system (1) has one and only one solution.

We can remove the restriction $b \neq 0$; for if $b = 0$ the first of the equations (1) reduces to $ax + c = 0$. Hence (if $a \neq 0$, which must be, since, if both $a = 0$ and $b = 0$, then $ab' - a'b = 0$) we have $x = -c/a$, and this value of x reduces the second of equations (1) to

$$-\frac{a'c}{a} + b'y + c' = 0,$$

which gives (since b' cannot in the present case be 0 without making $ab' - a'b = 0$) $y = (ca' - c'a)/ab'$. Now these values of x and y are precisely those given by (5) and (6) when $b = 0$.

The excepted case $b = 0$ is therefore included; and the only exceptional cases excluded are those that come under the condition $ab' - a'b = 0$.

The method of this paragraph may be called solution by *substitution.* The above discussion forms a complete and independent logical treatment of the problem in hand. The student may, on account of its apparent straightforwardness and theoretical simplicity, prefer it to the method of § 4. The defect of the method lies in its want of symmetry; the practical result of which is that it often introduces needless detail into the calculations.

Example.

$$3x+2y-3=0 \qquad (\alpha),$$
$$-9x+4y+5=0 \qquad (\beta).$$

From (α) we have

$$y=\frac{-3x+3}{2} \qquad (\gamma).$$

Using (γ), we reduce (β) to

$$-9x+2(-3x+3)+5=0,$$

that is,

$$-15x+11=0;$$

whence

$$x=\frac{11}{15}.$$

This value of x reduces (γ) to

$$y=\frac{-3\times\frac{11}{15}+3}{2},$$
$$=\frac{2}{5}.$$

The solution of the system (α) and (β) is therefore

$$x=\frac{11}{15}, \quad y=\frac{2}{5}.$$

§ 8.] *Three equations of the 1st degree in two variables, say*

$$ax+by+c=0, \quad a'x+b'y+c'=0, \quad a''x+b''y+c''=0 \qquad (1),$$

will not be consistent unless

$$a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)=0 \qquad (2);$$

and they will in general be consistent if this condition be satisfied.

We suppose, for the present, that none of the three functions $ab'-a'b$, $a''b-ab''$, $a'b''-a''b'$ vanishes.* This is equivalent to supposing that every pair of the three equations has a determinate finite solution.

If we take the first two equations as a system, they have the definite solution

* See below, § 25.

$$x = \frac{bc' - b'c}{ab' - a'b}, \quad y = \frac{ca' - c'a}{ab' - a'b}.$$

The necessary and sufficient condition for the consistency of the three equations is that this solution should satisfy the third equation; in other words, that

$$a''\frac{bc' - b'c}{ab' - a'b} + b''\frac{ca' - c'a}{ab' - a'b} + c'' = 0.$$

Since $ab' - a'b \neq 0$, this is equivalent to

$$a''(bc' - b'c) + b''(ca' - c'a) + c''(ab' - a'b) = 0 \qquad (3).$$

The reader should notice that this condition may be written in any one of the following forms by merely rearranging the terms:—

$$a(b'c'' - b''c') + b(c'a'' - c''a') + c(a'b'' - a''b') = 0 \qquad (4),$$
$$a'(bc'' - b''c) + b'(ca'' - c''a) + c'(ab'' - a''b) = 0 \qquad (5),$$
$$a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c) = 0 \qquad (6),$$
$$b(c'a'' - c''a') + b'(c''a - ca'') + b''(ca' - c'a) = 0 \qquad (7),$$
$$c(a'b'' - a''b') + c'(a''b - ab'') + c''(ab' - a'b) = 0 \qquad (8),$$
$$ab'c'' - ab''c' + bc'a'' - bc''a' + ca'b'' - ca''b' = 0 \qquad (9).$$

The forms (4) and (5) could have been obtained directly by taking the solution of the two last equations and substituting in the first, and by solving the first and last and substituting in the second, respectively. Each of these processes is obviously logically equivalent to the one actually adopted above.

The forms (6), (7), (8) would result as the condition of the consistency of the three equations

$$ax + a'y + a'' = 0, \quad bx + b'y + b'' = 0, \quad cx + c'y + c'' = 0 \quad (10).$$

We have therefore the following interesting side result:—

Cor. *If the three equations* (1) *be consistent, then the three equations* (10) *are consistent.*

If the reader will now compare the present paragraph with § 2, he will see that the function

$$ab' - a'b$$

plays the same part for the system

$$ax + b = 0, \quad a'x + b' = 0$$

as does the function

$$a(b'c''-b''c')+b(c'a''-c''a')+c(a'b''-a''b')$$

for the system

$$ax+by+c=0, \quad a'x+b'y+c'=0, \quad a''x+b''y+c''=0.$$

These functions are called the *determinants* of the respective systems of equations. They are often denoted by the notations

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \text{ for } ab'-a'b \qquad (11);$$

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \text{ for } ab'c''-ab''c'+bc'a''-bc''a'+ca'b''-ca''b' \qquad (12).$$

The reader should notice—

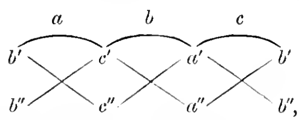
1st. That the determinant is of the 1st degree in the constituents of any one row or of any one column of the square symbol above introduced.

2nd. That, if all the constituents be considered, its degree is equal to the number of equations in the system.

A special branch of algebra is nowadays devoted to the *theory of determinants*, so that it is unnecessary to pursue the matter in this treatise. For the sake of more advanced students we have here and there introduced results of this theory, but always in such a way as not to interfere with the progress of such as may be unacquainted with them.

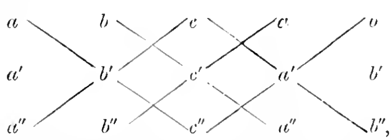
The reader may find the following *memoriæ technicæ* useful in enabling him to remember the determinant of a system of three equations :—

For the form (4),



to be interpreted like the similar scheme in § 4.

For the form (9),



where the letters in the diagonal lines are to form products with the signs + or −, according as the diagonals slope downwards from left to right or from right to left.

Example.

To show that the equations

$$3x+5y-2=0, \quad 4x+6y-1=0, \quad 2x+4y-3=0$$

are consistent.

Solving the first two equations, we have $x=-7/2$, $y=5/2$. These values

reduce $2x+4y-3$ to $-7+10-3$, which is zero. Hence the solution of the first two equations satisfies the third; that is, the three are consistent.

We might also use the general results of the above paragraph.

Since $3\times 6-5\times 4=-2$, $5\times 2-3\times 4=-2$, $4\times 4-2\times 6=+4$, each pair of equations has by itself a definite solution. Again, calculating the determinant of the system by the rule given above, we have, for the value of the determinant, $-54-10-32+24+12+60=0$. Hence the system is consistent.

$$\begin{array}{cccccccccc} +3 & + & 5 & - & 2 & + & 3 & + & 5 \\ \\ +4 & + & 6 & - & 1 & + & 4 & + & 6 \\ \\ +2 & + & 4 & - & 3 & + & 2 & + & 4. \end{array}$$

Exercises XXIV.

Solve the following :—

(1.) $\frac{1}{2}x+\frac{1}{3}y=6,\quad \frac{1}{3}x+\frac{1}{2}y=6\frac{1}{2}.$

(2.) $2x+3y=18,\quad 3x-2y=9.$

(3.) $\cdot 123x+\cdot 685y=3\cdot 34,\quad \cdot 893x-\cdot 593y=3\cdot 71,$

find x and y to five places of decimals.

(4.) $x+y:x-y=5:3,\quad x+5y=36.$

(5.) $3x+1=2y+1=3y+2x.$

(6.) $(x+3)(y+5)=(x-1)(y+2),\quad 8x+5=9y+2.$

(7.) $x+y=a+b,\quad (x+a)/(y+b)=b/a.$

(8.) $\dfrac{x}{la}+\dfrac{y}{mb}=1,\quad \dfrac{x}{2ma}+\dfrac{y}{3lb}=1.$

(9.) $ax+by=0,\quad (a-b)x+(a+b)y=2c$

(10.) $(a+b)x-(a-b)y=c,\quad (a-b)x+(a+b)y=c.$

(11.) $(a+b)x+(a-b)y=a^2+2ab-b^2,\quad (a-b)x+(a-b)y=a^2+b^2.$

(12.) $\dfrac{x}{a^2-b^2}-\dfrac{y}{a^2+ab+b^2}=ab,\quad \dfrac{x}{a^2+b^2}+\dfrac{y}{a^2-ab+b^2}=a(2a+b).$

(13.) $(ap^m+bq^m)x+(ap^{m+1}+bq^{m+1})y=ap^{m+2}+bq^{m+2},$
$(ap^n+bq^n)x+(ap^{n+1}+bq^{n+1})y=ap^{n+2}+bq^{n+2}.$

(14.) Find λ and μ so that $x^3+\lambda x^2+\mu x+abc$ may be exactly divisible by $x-b$ and by $x-c$.

(15.) If $\lambda\neq 0$, and if $x-y=a-b$, $\dfrac{x}{a+\lambda}+\dfrac{y}{b+\lambda}=1$, $\dfrac{x}{a-\lambda}+\dfrac{y}{b-\lambda}=1$, be consistent, show that $\lambda=\pm\sqrt{ab}$.

(16.) If the system $(b+c)x+(c+a)y+(a+b)=0$, $(c+a)x+(a+b)y+(b+c)=0$, $(a+b)x+(b+c)y+(c+a)=0$, be consistent, then $a^3+b^3+c^3-3abc=0$.

(17.) Find the condition that $ax+by=c$, $a^2x+b^2y=c^2$, $a^3x+b^3y=c^3$ be consistent.

(18.) Find an integral function of x of the 1st degree whose values shall be $+9$ and $+10$ when x has the values -3 and $+2$ respectively.

(19.) Find an integral function of x of the 2nd degree, such that the coefficient of its highest term is 1, and that it vanishes when $x=2$ and when $x=-3$.

(20.) Find an integral function of x of the 2nd degree which vanishes when $x=0$, and has the values -1 and $+2$ when $x=-1$ and $x=+3$ respectively.

EQUATIONS WITH THREE OR MORE VARIABLES.

§ 9.] *A single equation of the* 1*st degree in three variables admits of a two-fold infinity of solutions.*

For in any such equation, say

$$ax+by+cz+d=0,$$

we may assign to two of the variables any constant values we please, say $y=\beta$, $z=\gamma$, then the equation becomes an equation of the 1st degree in one variable, which has one and only one solution, namely,

$$x=-\frac{b\beta+c\gamma+d}{a}.$$

We thus have the solution

$$x=-\frac{b\beta+c\gamma+d}{a},\quad y=\beta,\quad z=\gamma.$$

Since there are here two arbitrary constants, to each of which an infinity of values may be given, we say that there is a two-fold infinity (∞^2) of solutions. A symmetric form is given for this doubly indeterminate solution in Exercises XXV., 27.

§ 10.] *A system of two equations of the* 1*st degree in three variables admits in general of a one-fold infinity of solutions.*

Consider the equations

$$ax+by+cz+d=0,\quad a'x+b'y+c'z+d'=0 \qquad (1).$$

We suppose that the functions $bc'-b'c$, $ca'-c'a$, $ab'-a'b$ do not all vanish, say $ab'-a'b\neq 0$.

If we give to z any arbitrary constant value whatever, say $z=\gamma$, then the two given equations give definite values for x and y. We thus obtain the solution

$$x = \frac{(bc' - b'c)\gamma + (bd' - b'd)}{ab' - a'b}, \quad y = \frac{(ca' - c'a)\gamma + (da' - d'a)}{ab' - a'b}, \quad z = \gamma \quad (2).$$

Since we have here one arbitrary constant, there is a one-fold infinity of solutions.

Cor. There is an important particular case of the above that often occurs in practice, that, namely, where $d = 0$ and $d' = 0$.

We then have, from (2),

$$x = \frac{bc' - b'c}{ab' - a'b}\gamma, \quad y = \frac{ca' - c'a}{ab' - a'b}\gamma, \quad z = \gamma.$$

This result can be written as follows :—

$$\frac{x}{bc' - b'c} = \frac{\gamma}{ab' - a'b},$$

$$\frac{y}{ca' - c'a} = \frac{\gamma}{ab' - a'b},$$

$$\frac{z}{ab' - a'b} = \frac{\gamma}{ab' - a'b}.$$

Now, γ being entirely at our disposal, we can so determine it that $\gamma/(ab' - a'b)$ shall have any value we please, say ρ. *Hence, ρ being entirely arbitrary, we have, as the solution of the system,*

$$\left.\begin{aligned} ax + by + cz &= 0 \\ a'x + b'y + c'z &= 0 \end{aligned}\right\} \quad (3),$$

$$x = \rho(bc' - b'c), \quad y = \rho(ca' - c'a), \quad z = \rho(ab' - a'b) \quad (4).$$

It will be observed that, although the individual values of x, y, z depend on the arbitrary constant ρ, the ratios of x, y, z are perfectly determined, namely, we have from (4)

$$x : y : z = (bc' - b'c) : (ca' - c'a) : (ab' - a'b).$$

Example 1.

$$2x + 3y + 4z = 0,$$
$$3x - 2y - 6z = 0,$$

$$\begin{array}{ccccccc} & x & & y & & z & \\ 3 & & 4 & & 2 & & 3 \\ -2 & & -6 & & 3 & & -2 \end{array}$$

give
$$x : y : z = -10 : 24 : -13 ;$$

or, which is the same thing,

$$x=-10\rho, \quad y=24\rho, \quad z=-13\rho,$$

ρ being any quantity whatsoever.

Example 2.

$$ax+by+cz=0,$$
$$a^2x+b^2y+c^2z=0,$$

give

$$x=(bc^2-b^2c)\rho=-bc\rho(b-c),$$
$$y=(ca^2-c^2a)\rho=-ca\rho(c-a),$$
$$z=(ab^2-a^2b)\rho=-ab\rho(a-b).$$

If we choose, we may replace $-abc\rho$ by σ, say, and we then have

$$x=\sigma(b-c)/a, \quad y=\sigma(c-a)/b, \quad z=\sigma(a-b)/c,$$

where σ is arbitrary.

In other words, we have

$$x:y:z=(b-c)/a:(c-a)/b:(a-b)/c.$$

§ 11.] *A system of three equations of the 1st degree in three variables, say*

$$ax+by+cz+d=0 \qquad (1),$$
$$a'x+b'y+c'z+d'=0 \qquad (2),$$
$$a''x+b''y+c''z+d''=0 \qquad (3),$$

has one and only one solution, provided

$$ab'c''-ab''c'+bc'a''-bc''a'+ca'b''-ca''b' \neq 0 \qquad (4).$$

The three coefficients c, c', c'' cannot all vanish, otherwise we should have a system of three equations in two variables, x and y, a case already considered in § 8.

Let us suppose that $c \neq 0$, then the following system

$$ax+by+cz+d=0 \qquad (5),$$
$$c'(ax+by+cz+d)-c(a'x+b'y+c'z+d')=0 \qquad (6),$$
$$c''(ax+by+cz+d)-c(a''x+b''y+c''z+d'')=0 \qquad (7),$$

is obviously equivalent to (1), (2), and (3). Matters are so arranged that z disappears from (6) and (7); and if, for shortness, we put

$$A=ac'-a'c, \quad B=bc'-b'c, \quad C=dc'-d'c,$$
$$A'=ac''-a''c, \quad B'=bc''-b''c, \quad C'=dc''-d''c,$$

we may write the system (5), (6), (7) as follows:—

$$ax+by+cz+d=0 \qquad (5'),$$
$$Ax+By+C=0 \qquad (6'),$$
$$A'x+B'y+C'=0 \qquad (7').$$

Now, provided $$AB' - A'B \neq 0 \qquad (8),$$

(6′) and (7′) have the unique solution

$$x = \frac{BC' - B'C}{AB' - A'B} \qquad (9),$$

$$y = \frac{CA' - C'A}{AB' - A'B} \qquad (10).$$

These values of x and y enable us to derive from (5′)

$$z = -\frac{a(BC' - B'C) + b(CA' - C'A) + d(AB' - A'B)}{c(AB' - A'B)} \qquad (11).$$

(9), (10), and (11) being equivalent to (5′), (6′), (7′), that is, to (1), (2), (3), constitute a unique solution of the three given equations.

It only remains to show that the condition (8) is equivalent to (4).

We have

$$AB' - A'B \equiv (ac' - a'c)(bc'' - b''c) - (ac'' - a''c)(bc' - b'c),$$
$$\equiv c(ab'c'' - ab''c' + bc'a'' - bc''a' + ca'b'' - ca''b') \qquad (12).$$

Hence, since $c \neq 0$, (8) is equivalent to (4).

Although, in practice, the general formulæ are very rarely used, yet it may interest the student to see the values of x, y, z given by (9), (10), (11) expanded in terms of the coefficients. We have

$$-(BC' - B'C) \equiv (dc' - d'c)(bc'' - b''c) - (dc'' - d''c)(bc' - b'c).$$

Comparing with (12), we see that $-(BC' - B'C)$ differs from $AB' - A'B$ merely in having d written everywhere in place of a (the dashes being imagined to stand unaltered). Hence

$$-(BC' - B'C) \equiv c(db'c'' - db''c' + bc'd'' - bc''d' + cd'b'' - cd''b').$$

So that we may write

$$x = -\frac{d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)} \qquad (13).$$

We obtain the values of y and z by interchanging a and b and a and c respectively, namely,

$$y = -\frac{d(a'c'' - a''c') + d'(a''c - ac'') + d''(ac' - a'c)}{b(a'c'' - a''c') + b'(a''c - ac'') + b''(ac' - a'c)} \qquad (14),$$

$$z = -\frac{d(b'a'' - b''a') + d'(b''a - ba'') + d''(ba' - b'a)}{c(b'a'' - b''a') + c'(b''a - ba'') + c''(ba' - b'a)} \qquad (15).$$

Written in determinant notation these would become

$$x = -\begin{vmatrix} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \end{vmatrix} \div \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \qquad (13'),$$

$$y = -\begin{vmatrix} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \end{vmatrix} \div \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \qquad (14'),$$

$$z = -\begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix} \div \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \qquad (15').$$

§ 12.] In the special case where $d = 0$, $d' = 0$, $d'' = 0$, the equations (1), (2), (3) of last paragraph become

$$ax + by + cz = 0 \qquad (1),$$
$$a'x + b'y + c'z = 0 \qquad (2),$$
$$a''x + b''y + c''z = 0 \qquad (3),$$

which are homogeneous in x, y, z.

If the determinant of the system, namely, $a''(bc' - b'c) + b''(ca' - c'a) + c''(ab' - a'b)$, *do not vanish, we see from* § 11 (9), (10), (11) (or more easily from (13), (14), and (15) of the same section) *that*

$$x = 0, \quad y = 0, \quad z = 0.$$

If the determinant does vanish, this conclusion does not necessarily follow.

In fact, if we write (1), (2), (3) in the form

$$a\frac{x}{z} + b\frac{y}{z} + c = 0 \qquad (1'),$$

$$a'\frac{x}{z} + b'\frac{y}{z} + c' = 0 \qquad (2'),$$

$$a''\frac{x}{z} + b''\frac{y}{z} + c'' = 0 \qquad (3'),$$

and regard x/z and y/z as variables, these equations are consistent, since

$$a''(bc' - b'c) + b''(ca' - c'a) + c''(ab' - a'b) = 0 \qquad (4),$$

and any two of them determine the ratios x/z, y/z; so that we have

$$\begin{aligned} x : y : z &= bc' - b'c : ca' - c'a : ab' - a'b, \\ &= bc'' - b''c : ca'' - c''a : ab'' - a''b, \\ &= b'c'' - b''c' : c'a'' - c''a' : a'b'' - a''b'. \end{aligned}$$

These different values of the ratios are in agreement, by virtue of (4), as the student should verify by actual calculation.

Hence, *if the determinant of a system of three homogeneous equations of the 1st degree in x, y, z vanish, the values of x, y, z are indeterminate (there being a one-fold infinity of solutions), but their ratios are determinate.*

§ 13.] Knowing, as we now do, that a system of three equations of the 1st degree in x, y, z has in general one definite solution and no more, we may take any logically admissible method of obtaining the solution that happens to be convenient. (1) We may guess the solution, or, as it is put, solve by inspection, verifying if necessary. (2) We may carry out, in the special case, the process of § 11; this is perhaps the most generally useful plan. (3) We may solve by substitution. (4) We may use Bezout's method. (5) We may derive from the given system another which happens to be simpler, and then solve the derived system. The following examples illustrate these different methods:—

Example 1.

$$x+y+z=a+b+c, \quad (b-c)x+(c-a)y+(a-b)z=0, \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3.$$

A glance shows us that this system is satisfied by $x=a$, $y=b$, $z=c$; and, since the system has only one solution, nothing more is required.

Example 2.

$$3x+5y-7z-2=0 \qquad (\alpha),$$
$$4x+8y-14z+3=0 \qquad (\beta),$$
$$3x+6y-8z-3=0 \qquad (\gamma).$$

Multiplying (α) by 4 and (β) by 3, and subtracting, we obtain

$$4y-14z+17=0 \qquad (\delta).$$

From (α) and (γ), by subtraction,

$$y-z-1=0 \qquad (\epsilon).$$

Multiplying (ϵ) by 4, and subtracting (δ), we have, finally,

$$10z-21=0;$$

whence

$$z=2\cdot1.$$

Using this value of z in (ϵ), we find

$$y=3\cdot1;$$

and, putting $y=3\cdot1$, $z=2\cdot1$ in (α), we find

$$x=\cdot4.$$

The solution of the system (α), (β), (γ) is therefore

$$x=\cdot4, \quad y=3\cdot1, \quad z=2\cdot1.$$

Example 3.

Taking the equations (α), (β), (γ) of last example, we might proceed by substitution, as follows :—

From (α)

$$x = -\frac{5}{3}y + \frac{7}{3}z + \frac{2}{3}.$$

This value of x reduces (β) to

$$-\frac{20}{3}y + \frac{28}{3}z + \frac{8}{3} + 8y - 14z + 3 = 0,$$

which is equivalent to

$$4y - 14z + 17 = 0 \qquad (\delta').$$

Substituting the same value of x as before in (γ), we deduce

$$y - z - 1 = 0 \qquad (\epsilon').$$

Now (ϵ') gives

$$y = z + 1,$$

and this value of y reduces (δ') to

$$-10z + 21 = 0,$$

hence

$$z = 2{\cdot}1.$$

The values of y and x can now be obtained by using first (ϵ') and then (α).

Example 4.

Taking once more the equations (α), (β), (γ) of Example 2, we might proceed by Bezout's method.

If λ and μ be two arbitrary multipliers, we derive from (α), (β), (γ),

$$(3x + 5y - 7z - 2) + \lambda(4x + 8y - 14z + 3) + \mu(3x + 6y - 8z - 3) = 0 \qquad (\delta').$$

Suppose that we wish to find the value of x. We determine λ and μ so that (δ') shall contain neither y nor z. We thus have

$$8\lambda + 6\mu + 5 = 0 \qquad (\epsilon'),$$

$$-14\lambda - 8\mu - 7 = 0 \qquad (\zeta'),$$

$$(3 + 4\lambda + 3\mu)x - 2 + 3\lambda - 3\mu = 0 \qquad (\eta').$$

If we solve (ϵ') and (ζ'), we obtain

$$\lambda = -{\cdot}1, \quad \mu = -{\cdot}7.$$

The last equation (η') thus becomes

$$(3 - {\cdot}4 - 2{\cdot}1)x - 2 - {\cdot}3 + 2{\cdot}1 = 0,$$

that is,

$${\cdot}5x - {\cdot}2 = 0\,;$$

whence

$$x = {\cdot}2/{\cdot}5 = {\cdot}4.$$

The values of y and z may be obtained by a similar process.

Example 5.

$$ax + by + cz = 0 \qquad (\alpha),$$

$$(b + c)x + (c + a)y + (a + b)z = 0 \qquad (\beta),$$

$$a^2x + b^2y + c^2z = a^2(b - c) + b^2(c - a) + c^2(a - b) \qquad (\gamma).$$

From (α) and (β) we derive, by addition,

$$(a + b + c)(x + y + z) = 0,$$

which, provided $a + b + c \neq 0$, is equivalent to

$$x + y + z = 0 \qquad (\delta).$$

We can now, if we please, replace (α) and (β) by the equivalent simpler pair (α) and (δ).

Now (see § 10), by virtue of (α) and (δ), we have

$$\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b} \qquad (\epsilon).$$

If none of the three, $b-c$, $c-a$, $a-b$, vanish, we may write (γ) in the form

$$a^2(b-c)\frac{x}{b-c}+b^2(c-a)\frac{y}{c-a}+c^2(a-b)\frac{z}{a-b}=a^2(b-c)+b^2(c-a)+c^2(a-b).$$

Using (ϵ) we can replace $y/(c-a)$ and $z/(a-b)$ by $x/(b-c)$, and the last equation becomes

$$\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}\frac{x}{b-c}=a^2(b-c)+b^2(c-a)+c^2(a-b);$$

and, since $a^2(b-c)+b^2(c-a)+c^2(a-b)\equiv-(b-c)(c-a)(a-b)$, which does not vanish, if our previous assumptions be granted, it follows that

$$\frac{x}{b-c}=1.$$

Hence $x=b-c$, and, by symmetry, $y=c-a$, $z=a-b$.

This solution might of course have been obtained at once by inspection.

Example 6.

$$\left.\begin{array}{l} x+ay+a^2z+a^3=0 \\ x+by+b^2z+b^3=0 \\ x+cy+c^2z+c^3=0 \end{array}\right\} \qquad (\alpha).$$

From the identity

$$\xi^3+p\xi^2+q\xi+r\equiv(\xi-a)(\xi-b)(\xi-c),$$

(see chap. iv., § 9), where

$$p=-a-b-c, \quad q=bc+ca+ab, \quad r=-abc,$$

we have

$$\left.\begin{array}{l} r+aq+a^2p+a^3=0 \\ r+bq+b^2p+b^3=0 \\ r+cq+c^2p+c^3=0 \end{array}\right\} \qquad (\beta).$$

It appears, therefore, from (β) that

$$x=r, \quad y=q, \quad z=p$$

is a solution of (α). Hence, since (α) has only one solution, that solution is

$$x=-abc, \quad y=bc+ca+ab, \quad z=-a-b-c.$$

This result may be generalised and extended in various obvious ways.

§ 14.] *A system of more than three equations of the 1st degree in three variables will in general be inconsistent. To secure consistency one condition must in general be satisfied for every equation beyond three.* This may be seen by reflecting that the first three equations will in general uniquely determine the variables, and that the values thus found must satisfy each of the remaining equations. Thus, in the case of four equations, there will be one condition for consistency. The equation expressing this condition could easily be found in its most general form; but its expression would be cumbrous and practically useless without the use of

determinantal or other abbreviative notation. There is, however, no difficulty in working out the required result directly in any special case.

Example.

Determine the numerical constant p, so that the four equations

$$2x-3y+5z=18, \quad 3x-y+4z=20, \quad 4x+2y-z=5,$$
$$(p+1)x+(p+2)y+(p+3)z=76$$

shall be consistent.

If we take the first three equations, they determine the values of x, y, z, namely,
$$x=1, \quad y=3, \quad z=5.$$

These values must satisfy the last equation; hence we must have
$$(p+1)+(p+2)3+(p+3)5=76,$$
which is equivalent to
$$9p=54.$$
Hence
$$p=6.$$

§ 15.] If the reader will now reconsider the course of reasoning through which we have led him in the cases of equations of the 1st degree in one, two, and three variables respectively, he will see that the spirit of that reasoning is general; and that, by pursuing the same course step by step, we should arrive at the following general conclusions:—

I. *A system of $n-r$ equations of the* 1st *degree in n variables has in general a solution involving r arbitrary constants; in other words, has an r-fold infinity of different solutions.*

II. *A system of n equations of the* 1st *degree in n variables has a unique determinate solution, provided a certain function of the coefficients of the system, which we may call the determinant of the system, does not vanish.*

III. *A system of $n+r$ equations of the* 1st *degree in n variables will in general be inconsistent. To secure consistency r different conditions must in general be satisfied.*

There would be no great difficulty in laying down a rule for calculating step by step the function spoken of above as the determinant of a system of n equations of the 1st degree in n variables; but the final form in which it would thus be obtained would be neither elegant nor luminous. Experience has shown that it is better to establish independently the theory of a certain class of functions called determinants, and then to apply the properties of these functions to the general theory of equations of the 1st degree. A brief sketch of this way of proceeding is given in the next paragraph, and will be quite intelligible to those acquainted with the elements of the theory of determinants.

GENERAL SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY MEANS OF DETERMINANTS.

§ 16.] Consider the system

$$\begin{aligned} U_1 &\equiv a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + c_1 = 0 \qquad (1),\\ U_2 &\equiv a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + c_2 = 0 \qquad (2),\\ &\ldots\ldots\\ U_n &\equiv a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + c_n = 0 \qquad (n), \end{aligned}$$

where there are n variables, $x_1, x_2, \ldots, x_n$, and n equations to determine them.

Let

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix};$$

and let $\Delta_1, \Delta_2, \ldots, \Delta_n$ denote the determinants obtained from Δ by replacing the constituents of the 1st, 2nd . . . nth columns respectively, by the set $c_1, c_2, \ldots, c_n$.

Also let the co-factors of $a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}$, &c., Δ be denoted by $A_{11}, A_{12}, \ldots, A_{1n}, A_{21}, A_{22}, \ldots, A_{2n}$, &c., as usual.

Then, by the theory of determinants, we have

$$\left.\begin{aligned} a_{11}A_{11} + a_{21}A_{21} + \ldots + a_{n1}A_{n1} &= \Delta \\ a_{12}A_{11} + a_{22}A_{21} + \ldots + a_{n2}A_{n1} &= 0 \\ \ldots\ldots \\ a_{1n}A_{11} + a_{2n}A_{21} + \ldots + a_{nn}A_{n1} &= 0 \\ c_1A_{11} + c_2A_{21} + \ldots + c_nA_{n1} &= \Delta_1 \end{aligned}\right\} \qquad (a),$$

and so on.

If the determinant Δ, which we call the determinant of the system of equations, does not vanish, then $A_{11}, A_{21}, \ldots, A_{n1}$ cannot all vanish. Let us suppose that $A_{11} \neq 0$. Then, by chap. xiv., § 10, the system

$$\begin{gathered} A_{11}U_1 + A_{21}U_2 + \ldots + A_{n1}U_n = 0, \\ U_2 = 0, \quad U_3 = 0, \quad \ldots, \quad U_n = 0, \end{gathered}$$

is equivalent to the system (1), (2) . . . (n). If we collect the

coefficients of the variables $x_1, x_2, \ldots, x_n$ in the first of these equations, and attend to the relations (α), that equation reduces to

$$\Delta x_1 + \Delta_1 = 0.$$

Since $\Delta \neq 0$, this is equivalent to

$$x_1 = -\Delta_1/\Delta.$$

By exactly similar reasoning we could show that $x_2 = -\Delta_2/\Delta, \ldots$ $x_n = -\Delta_n/\Delta$. Hence the solution, and the only solution, of (1), (2), . . ., (n) is

$$x_1 = -\Delta_1/\Delta, \quad x_2 = -\Delta_2/\Delta, \quad \ldots, \quad x_n = -\Delta_n/\Delta \quad (\beta).$$

Although, from the way we have conducted the demonstration, it is not necessary to verify that (β) does in fact satisfy (1), (2), . . ., (n), yet the reader should satisfy himself by substitution that this is really the case.

We have thus shown that a system of n equations of the 1st degree in n variables has a unique determinate solution, provided its determinant does not vanish.

Next, let us suppose that, in addition to the equations (1), (2), . . ., (n) above, we had another, namely,

$$a_{n+1,1}x_1 + a_{n+1,2}x_2 + \ldots + a_{n+1,n}x_n + c_{n+1} = 0, \qquad (n+1),$$

the system of $n+1$ equations thus obtained will in general be inconsistent.

The necessary and sufficient condition for consistency is that the solution of the first n shall satisfy the $n+1$th, namely, if $\Delta \neq 0$, $-a_{n+1,1}\Delta_1 - a_{n+1,2}\Delta_2 - \ldots - a_{n+1,n}\Delta_n + c_{n+1}\Delta = 0$, that is,

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} & c_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} & c_n \\ a_{n+1,1} & a_{n+2,2} & \ldots & a_{n+2,n} & c_{n+1} \end{vmatrix} = 0 \qquad (\gamma).$$

Lastly, let us consider the *particular case of n homogeneous equations of the 1st degree in n variables.* In other words, let us suppose that, in equations (1), (2), . . ., (n) above, we have $c_1 = 0,\ c_2 = 0,\ \ldots,\ c_n = 0$.

1st. Suppose $\Delta \neq 0$, then, since now $\Delta_1 = 0$, $\Delta_2 = 0$, . . ., $\Delta_n = 0$, (β) gives $x_1 = 0$, $x_2 = 0$, . . ., $x_n = 0$.

2nd. Suppose $\Delta = 0$.

We may write the equations in the form

$$a_{11}\frac{x_1}{x_n} + a_{12}\frac{x_2}{x_n} + \ldots + a_{1n} = 0,$$

$$a_{21}\frac{x_1}{x_n} + a_{22}\frac{x_2}{x_n} + \ldots + a_{2n} = 0,$$

$$\cdots\cdots\cdots$$

$$a_{n1}\frac{x_1}{x_n} + a_{n2}\frac{x_2}{x_n} + \ldots + a_{nn} = 0.$$

These may be regarded as a system of n equations of the 1st degree in the $n-1$ variables $x_1/x_n, x_2/x_n, \ldots, x_{n-1}/x_n$; and, since $\Delta = 0$, they are a consistent system. Using only the last $n-1$ of them, we find

$$\frac{x_1}{x_n} = -\begin{vmatrix} a_{2n} & a_{22} & \ldots & a_{2,n-1} \\ a_{3n} & a_{32} & \ldots & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{nn} & a_{n2} & \ldots & a_{n,n-1} \end{vmatrix} \div \begin{vmatrix} a_{21} & a_{22} & \ldots & a_{2,n-1} \\ a_{31} & a_{32} & \ldots & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{n,n-1} \end{vmatrix}$$

$$= -(-1)^{2n-3}\frac{A_{11}}{A_{1n}} = \frac{A_{11}}{A_{1n}}.$$

In a similar way we prove that

$$x_2/x_n = A_{12}/A_{1n} \quad \ldots, \quad x_{n-1}/x_n = A_{1,n-1}/A_{1n}.$$

Hence we have

$$x_1 : x_2 : \ldots : x_n = A_{11} : A_{12} : \ldots : A_{1n};$$

and, by parity of reasoning,

$$x_1 : x_2 : \ldots : x_n = A_{r1} : A_{r2} : \ldots : A_{rn}$$

where $r = 2, = 3, \ldots, = n$, as we please. In other words, the ratios of the variables are determinate, but their actual values are indeterminate, there being a one-fold infinity of different solutions.

Exercises XXV.

Solve the following systems :—

(1.) $\frac{x}{3}+\frac{y}{4}+\frac{z}{6}=36$, $\frac{x}{15}+\frac{y}{20}+\frac{z}{9}=10$, $\frac{x}{2}+\frac{y}{10}+\frac{z}{4}=43$.

(2.) $2x+3y+4z=29,\quad 3x+2y+5z=32,\quad 4x+3y+2z=25.$

(3.) $$\cdot 3x+1\cdot 2y+6\cdot 8z=1,\quad 3\cdot 3x-2\cdot 5y-3\cdot 82z=\cdot 5,$$
$$\cdot 01x-\cdot 003y-\cdot 301z=\cdot 013\,;$$
calculate x, y, z to four places of decimals.

(4.) $$x+y+z=26,\quad x-y=4,\quad x-z=6.$$

(5.) If $$\frac{(x+1)^2}{(x+2)(x^2+x+1)}\equiv\frac{A}{x+2}+\frac{Bx+C}{x^2+x+1},$$
determine the numerical constants A, B, C.

(6.) Find a linear function of x and y, which shall vanish when $x=x'$, $y=y'$, and also when $x=x''$, $y=y''$, and which shall have the value $+1$ when $x=x'''$, $y=y'''$.

(7.) An integral function of x of the 2nd degree vanishes when $x=2$, and when $x=3$, and has the value -1 when $x=-2$; find the function.

Solve the following systems:—

(8.) $$y+z=a,\quad z+x=b,\quad x+y=c.$$

(9.) $$\frac{y}{b+c}+\frac{z}{b-c}=2a,\quad \frac{z}{c+a}+\frac{x}{c-a}=2b,\quad \frac{x}{a+b}+\frac{y}{a-b}=2c.$$

(10.) An integral function of x of the 2nd degree takes the values A, B, C, when x has the values a, b, c respectively; find the function.

Solve the following systems:—

(11.) $$bc(b-c)x+ca(c-a)y+ab(a-b)z=0,$$
$$(a+b-c)x+(b+c-a)y+(c+a-b)z=a^2+b^2+c^2,$$
$$b^2c^2x+c^2a^2y+a^2b^2z=abc(bc+ca+ab).$$

(12.) If $$\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=1,$$
$$\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=1,$$
$$\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1,$$
then
$$\frac{x}{(a+\alpha)(a+\beta)^2}+\frac{y}{(b+\alpha)(b+\beta)^2}+\frac{z}{(c+\alpha)(c+\beta)^2}=\frac{\gamma-\beta}{(a+\beta)(b+\beta)(c+\beta)}.$$

(13.) $$ax+by+cz=a+b+c,$$
$$a^2x+b^2y+c^2z=(a+b+c)^2,$$
$$bcx+cay+abz=0.$$

(14.) $$ax+cy+bz=cx+by+az=bx+ay+cz=a^3+b^3+c^3-3abc.$$

(15.) $$lx+my+nz=mn+nl+lm,$$
$$x+y+z=l+m+n,$$
$$(m-n)x+(n-l)y+(l-m)z=0.$$

(16.) $$lx+my+nz=0,$$
$$(m+n)x+(n+l)y+(l+m)z=l+m+n,$$
$$l^2x+m^2y+n^2z=p^2.$$

(17.) Show that $(b-c)x+by-cz=0$, $(c-a)y+cz-ax=0$, $(a-b)z+ax-by=0$, are consistent.

(18.) Show that the system $cy-bz=f$, $az-cx=g$, $bx-ay=h$ has no finite solution unless $af+bg+ch=0$, in which case it has an infinite number of solutions.

Find a symmetrical form for the indeterminate solution involving one arbitrary constant.

Solve the following systems :—

(19.) $3x-2y+3u=0$, $x-y+z=0$, $3y+3z-2u=0$, $x+2y+3z+4u=8$.

(20.) If $ax=p-r$, $by=p-s$, $cz=r-s$, $d(y+z)=s-q$, $e(z+x)=q-r$, $f(x+y)=q-p+g$, find z in terms of a, b, c, d, e, f, g.

(21.) $$2y+x=\frac{4z+x}{3}=\frac{3u+x}{4}=\frac{5v+x}{8}$$
$$=\frac{x+y+z+u+v-1}{4}=\frac{5x+4y+3z+2u+v+2}{9}.$$

(22.) $ax+by=1$, $cx+dz=1$, $ez+fu=1$, $gu+hv=1$, $x+y+z+u+v=0$.

(23.) Prove that, with a certain exception, the system $U=0$, $V=0$, $W=0$, and $\lambda U+\mu V+\nu W=0$, $\lambda' U+\mu' V+\nu' W=0$, $\lambda'' U+\mu'' V+\nu'' W=0$ are equivalent.

(24.) If $$x=by+cz+du,\quad y=ax+cz+du,$$
$$z=ax+by+du,\quad u=ax+by+cz,$$
then $$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{d}{d+1}=1.$$

(25.) Show that the system $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$, $a''x+b''y+c''z+d''=0$ will be equivalent to only two equations if the system $ax+a'y+a''=0$, $bx+b'y+b''=0$, $cx+c'y+c''=0$, $dx+d'y+d''=0$ be consistent, that is, if
$$\frac{b'c''-b''c'}{a'd''-a''d'}=\frac{b''c-bc''}{a''d-ad''}=\frac{bc'-b'c}{ad'-a'd}.$$

Show that in the case of the system
$$x+y+z=a+b+c,\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\quad \frac{x}{a^3}+\frac{y}{b^3}+\frac{z}{c^3}=0,$$
the above two conditions reduce to one only, namely,
$$bc+ca+ab=0.$$

(26.) Show that the three equations
$$x=A+A'u+A''v,\quad y=B+B'u+B''v,\quad z=C+C'u+C''v,$$
where u and v are variable, are equivalent to a single linear equation connecting x, y, z; and find that equation.

(27.) If $ax+by+cz+d=0$, show that
$$x=\left(\frac{p}{a}+q\right)(b-c)-\frac{d}{3a},$$
$$y=\left(\frac{p}{b}+q\right)(c-a)-\frac{d}{3b},$$
$$z=\left(\frac{p}{c}+q\right)(c-a)-\frac{d}{3c},$$
where p and q are arbitrary constants.

(28.) If $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$, show that

$$x=p(bc'-b'c)+\{(b'-c')d-(b-c)d'\}/\{a(b'-c')+b(c'-a')+c(a'-b')\},$$
$$y=p(ca'-c'a)+\{(c'-a')d-(c-a)d'\}/\{a(b'-c')+b(c'-a')+c(a'-b')\},$$
$$z=p(ab'-a'b)+\{(a'-b')d-(a-b)d'\}/\{a(b'-c')+b(c'-a')+c(a'-b')\},$$

where p is an arbitrary constant.

EXAMPLES OF EQUATIONS WHOSE SOLUTION IS EFFECTED BY MEANS OF LINEAR EQUATIONS.

§ 17.] We have seen in chap. xiv. that every system of algebraical equations can be reduced to a system of rational integral equations such that every solution of the given system will be a solution of the derived system, although the derived system may admit of solutions, called "extraneous," which do not satisfy the original system. It may happen that the derived system is linear, or that it can, by the process of factorisation, be replaced by equivalent alternative linear systems. In such cases all we have to do is to solve these linear systems, and then satisfy ourselves, either by substitution or by examining the reversibility of the steps of the process, which, if any, of the solutions obtained are extraneous. The student should now re-examine the examples worked out in chap. xiv., find, wherever he can, all the solutions of the derived equations, and examine their admissibility as solutions of the original system. We give two more instances here.

Example 1.

$$\frac{1}{\sqrt{\{x+\sqrt{(x^2-1)}\}}}+\frac{1}{\sqrt{\{x-\sqrt{(x^2-1)}\}}}=\sqrt{\{2(x^3+1)\}} \qquad (\alpha).$$

(Positive values to be taken for all the square roots.)

If we rationalise the two denominators on the left, we deduce from (α) the equivalent equation,

$$\sqrt{\{x-\sqrt{(x^2-1)}\}}+\sqrt{\{x+\sqrt{(x^2-1)}\}}=\sqrt{\{2(x^3+1)\}} \qquad (\beta).$$

From (β) we derive, by squaring both sides,

$$2x+2\sqrt{\{x^2-(x^2-1)\}}=2(x^3+1),$$

that is, $$2x+2=2x^3+2 \qquad (\gamma).$$

Now (γ) is equivalent to $$x^3-x=0,$$

that is, to $$x(x-1)(x+1)=0 \qquad (\delta).$$

Again (δ) is equivalent to the alternatives $\left(\begin{array}{c} x=0 \\ x-1=0 \\ x+1=0 \end{array}\right)$,

that is to say, its solutions are $x=0$, $x=1$, $x=-1$.

Since, however, the step from (β) to (γ) is irreversible, it is necessary to examine which of these solutions actually satisfy (α).

Now $x=0$ gives $\sqrt{-i}+\sqrt{+i}=\sqrt{2}$,

that is (see chap. xii., § 17, Example 3),

$$\frac{1-i}{\sqrt{2}}+\frac{1+i}{\sqrt{2}}=\sqrt{2},$$

which is correct.

Also, $x=1$ obviously satisfies (α).

But $x=-1$ gives $2i=0$, which is not true, hence $x=-1$ is not a solution of (α).

Remark that $x=-1$ is a solution of the slightly different equation,

$$\frac{1}{\sqrt{\{x+\sqrt{(x^2-1)}\}}}-\frac{1}{\sqrt{\{x-\sqrt{(x^2-1)}\}}}=\sqrt{2(x^3+1)}.$$

Example 2.

$$x^2-y^2=x-y, \quad 2x+3y-1=0 \qquad (\alpha).$$

Since the first of these equations is equivalent to $(x-y)(x+y-1)=0$, the system (α) is equivalent to

$$\begin{pmatrix} x-y=0, \text{ and } 2x+3y-1=0 \\ x+y-1=0, \text{ and } 2x+3y-1=0 \end{pmatrix}$$

now the solution of $x-y=0$, $2x+3y-1=0$ is $x=1/5$, $y=1/5$; and the solution of $x+y-1=0$, $2x+3y-1=0$ is $x=2$, $y=-1$. Hence the solutions of (α) are

x	y
1/5	1/5
2	−1

§ 18.] The solution of linear systems is sometimes facilitated by the introduction of *Auxiliary Variables*, or, as it is sometimes put, by *changing the variables*. This artifice sometimes enables us to abridge the labour of solving linear systems, and occasionally to use methods appropriate to linear systems in solving systems which are not themselves linear. The following are examples :—

Example 1.

$$\frac{(x-a)^3}{(x+b)^3}=\frac{x-2a-b}{x+a+2b} \qquad (\alpha).$$

Let $x+b=z$, so that $x=z-b$; and, for shortness, let $c=a+b$. Then (α) may be written

$$\frac{(z-c)^3}{z^3}=\frac{z-2c}{z+c} \qquad (\beta).$$

From (β) we derive

$$(z-c)^3(z+c)=z^3(z-2c),$$

that is,

$$z^4-2z^3c+2zc^3-c^4=z^4-2z^3c,$$

which is equivalent to

$$2c^3z-c^4=0 \qquad (\gamma).$$

Now (γ) has the unique solution $z=c/2$, which evidently satisfies (α) Hence $x=c/2-b$, that is, $x=(a-b)/2$, is the only finite solution of (α).

Example 2.

$$a(x+y)+b(x-y)+c=0, \quad a'(x+y)+b'(x-y)+c'=0 \qquad (\alpha).$$

Let $\xi=x+y$, $\eta=x-y$, then the system (α) may be written

$$a\xi+b\eta+c=0, \quad a'\xi+b'\eta+c'=0 \qquad (\alpha').$$

Now (α') is a linear system in ξ and η, and we have, by § 4,

$$\xi=\frac{bc'-b'c}{ab'-a'b}, \quad \eta=\frac{ca'-c'a}{ab'-a'b} \qquad (\beta).$$

Replacing ξ and η by their values, we have

$$x+y=\frac{bc'-b'c}{ab'-a'b}, \quad x-y=\frac{ca'-c'a}{ab'-a'b} \qquad (\gamma).$$

From (γ), by first adding and then subtracting, we obtain

$$x=\frac{bc'-b'c+ca'-c'a}{2(ab'-a'b)}, \quad y=\frac{bc'-b'c-ca'+c'a}{2(ab'-a'b)}.$$

Example 3.

$$cy+bz=az+cx=bx+ay=abc.$$

Dividing by bc, by ca, and by ab, we may write the given system in the following equivalent form

$$\frac{y}{b}+\frac{z}{c}=a \qquad (\alpha).$$

$$\frac{z}{c}+\frac{x}{a}=b \qquad (\beta),$$

$$\frac{x}{a}+\frac{y}{b}=c \qquad (\gamma).$$

Now, if we add the equations (β) and (γ), and subtract (α), we have

$$\left(\frac{x}{a}+\frac{y}{b}\right)+\left(\frac{z}{c}+\frac{x}{a}\right)-\left(\frac{y}{b}+\frac{z}{c}\right)=b+c-a,$$

that is,

$$2\frac{x}{a}=b+c-a\,;$$

whence

$$x=\frac{a(b+c-a)}{2}.$$

By symmetry, we have

$$y=b(c+a-b)/2, \quad z=c(a+b-c)/2.$$

Here we virtually regard x/a, y/b, z/c as the variables, although we have not taken the trouble to replace them by new letters.

Example 4.

$$\frac{x-a}{p}=\frac{y-b}{q}=\frac{z-c}{r} \qquad (\alpha),$$
$$lx+my+nz=d \qquad (\beta).$$

Represent each of the three equal functions in (α) by ρ. Then we have

$$(x-a)/p=\rho, \quad (y-b)/q=\rho, \quad (z-c)/r=\rho,$$

which are equivalent to

$$x=a+p\rho, \quad y=b+q\rho, \quad z=c+r\rho \qquad (\gamma).$$

Using (γ), we reduce (β) to

$$l(a+p\rho)+m(b+q\rho)+n(c+r\rho)=d,$$

for which we obtain, for the value of the auxiliary ρ,

$$\rho=\frac{d-la-mb-nc}{lp+mq+nr} \qquad (\delta).$$

From (γ) and (δ) we have, finally,

$$x=a+p\frac{d-la-mb-nc}{lp+mq+nr},$$
$$=\frac{m(aq-bp)+n(ar-cp)+pd}{lp+mq+nr}.$$

The values of y and z can be similarly found, or they can be written down at once by considering the symmetry of the original system.

Example 5.

$$x-2y+3z=0 \qquad (\alpha),$$
$$2x-3y+4z=0 \qquad (\beta),$$
$$4x^3+3y^3+z^3-xyz=216 \qquad (\gamma).$$

From (α) and (β) we have (see § 10 above)

$$x/1=y/2=z/1=\rho, \text{ say.}$$

Hence $\qquad x=\rho, \quad y=2\rho, \quad z=\rho \qquad (\delta).$

By means of (δ) we deduce from (γ)

$$27\rho^3=216,$$

which is equivalent to $\qquad \rho^3=8. \qquad (\epsilon).$

Now the three cube roots of 8 are (see chap. xii., § 20, Example 1)

$$2, \quad 2(-1+\sqrt{3}i), \quad 2(-1-\sqrt{3}i).$$

Hence the solutions of (ϵ) are

$$\rho=2, \quad \rho=2(-1+\sqrt{3}i), \quad \rho=2(-1-\sqrt{3}i).$$

Hence, by (δ), we obtain the three following solutions of (α), (β), (γ) :—

x	y	z
2	4	2
$-1+\sqrt{3}i$	$2(-1+\sqrt{3}i)$	$-1+\sqrt{3}i$
$-1-\sqrt{3}i$	$2(-1-\sqrt{3}i)$	$-1-\sqrt{3}i$

Since, by chap. xiv., § 6, the system in question has only three solutions, we have obtained the complete solution.

N.B.—In general, if $u_1, u_2, \ldots, u_{n-1}$ be homogeneous functions of the 1st degree in n variables, and v a homogeneous function of the nth degree in the same variables, the solution of the system

$$u_1=0, \quad u_2=0, \quad \ldots, \quad u_{n-1}=0, \quad v=0$$

may be effected by solving a system of $n-1$ linear equations in $n-1$ variables, and then extracting an nth root. See in this connection § 16 above.

Example 6.

$$ax^2+by^2+c=0, \quad a'x^2+b'y^2+c'=0.$$

If we regard x^2 and y^2 as the variables, we have to do with a linear system, and we obtain as heretofore,

$$x^2=(bc'-b'c)/(ab'-a'b), \quad y^2=(ca'-c'a)/(ab'-a'b).$$

Hence

$$x=\pm\sqrt{(bc'-b'c)/(ab'-a'b)}, \quad y=\pm\sqrt{(ca'-c'a)/(ab'-a'b)}.$$

Since either of the one pair of double signs may go with either of the other pair, we thus obtain the full number of $2\times 2=4$ solutions.

Example 7.

$$ay+bx+cxy=0, \quad a'y+b'x+c'xy=0.$$

These two equations evidently have the solution $x=0$, $y=0$.

Setting these values aside, we may divide each of the two equations by xy. We thus deduce the system

$$a\frac{1}{x}+b\frac{1}{y}+c=0, \quad a'\frac{1}{x}+b'\frac{1}{y}+c'=0,$$

which is linear, if we regard $1/x$ and $1/y$ as the variables. Solving from this point of view, we obtain

$$\frac{1}{x}=\frac{bc'-b'c}{ab'-a'b}, \quad \frac{1}{y}=\frac{ca'-c'a}{ab'-a'b};$$

for which we have

$$x=(ab'-a'b)/(bc'-b'c), \quad y=(ab'-a'b)/(ca'-c'a).$$

We have thus found two out of the four solutions of the given system. There are no more finite solutions.

Exercises XXVI.

Solve the following by means of linear systems:—

(1.) $$\frac{\sqrt{ax}+\sqrt{b}}{\sqrt{ax}-\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}.$$

(2.) $$\frac{\sqrt{x}+4m}{\sqrt{x}+3n}=\frac{\sqrt{x}+2m}{\sqrt{x}+n}.$$

(3.) $$\sqrt{(x+22)}-\sqrt{(x+11)}=1.$$

(4.) $$\sqrt{x}+\sqrt{(x+3)}=12/\sqrt{(x+3)}.$$

(5.) $\sqrt{(x+2)}+\sqrt{(x-2)}=5/\sqrt{(x+2)}$.

(6.) $\sqrt{x}+\sqrt{(\alpha+\beta)}-\sqrt{(\alpha-\beta)}=\sqrt{(x+2\beta)}$.

(7.) $\sqrt{(x+q-r)}+\sqrt{(x+r-p)}+\sqrt{(x+p-q)}=0$.

(8.) $$\frac{1}{\sqrt{(x-p)}-\sqrt{p}}+\frac{1}{\sqrt{(x-p)}+\sqrt{p}}=\sqrt{(x-p)}.$$

(9.) $x=\sqrt{\{a^2-x\sqrt{(b^2+x^2-a^2)}\}}+a$.

(10.) $\sqrt{(\sqrt{x}+\sqrt{a})}+\sqrt{(\sqrt{x}-\sqrt{a})}=\sqrt{(2\sqrt{x}+2\sqrt{b})}$.

(11.) $\sqrt{x}+\sqrt{\{3-\sqrt{(2x+x^2)}\}}=\sqrt{3}$.

(12.) $$\frac{1}{\sqrt{(x-3)}}-\frac{2}{\sqrt{(y-2)}}=\frac{1}{6},\quad \sqrt{(y-2)}=\frac{5}{2}\sqrt{(x-3)}.$$

(13.) $\sqrt{(a-x)}-\sqrt{(y-x)}=\sqrt{y},\quad \sqrt{(b-x)}+\sqrt{(y-x)}=\sqrt{y}$.

(14.) $$\sqrt{x}-\sqrt{y}=\frac{1}{4},\quad x-y=\frac{17}{48}.$$

(15.) $(x-a)^2-(y-b)^2=0,\quad (x-b)(y-a)=a(2b-a)$.

(16.) $x-y=3,\quad x^2-y^2=45$.

(17.) $x/y=a/b,\quad x^3-y^3=d$.

(18.) $$\begin{gathered} x+ay+a^2z+a^3u+a^4=0,\\ x+by+b^2z+b^3u+b^4=0,\\ x+cy+c^2z+c^3u+c^4=0,\\ x+dy+d^2z+d^3u+d^4=0. \end{gathered}$$

(19.) $$\begin{gathered} x+y+z=0,\quad ax+by+cz=0,\\ bcx+cay+abz=(b^2-c^2)(c^2-a^2)(a^2-b^2). \end{gathered}$$

(20.) $$\begin{gathered} \frac{mx-ly}{lm(a-b)}=\frac{ny-mz}{mn(b-c)}=\frac{lz-nx}{nl(c-a)}=\frac{1}{lmn},\\ mnx+nly+lmz=a+b+c. \end{gathered}$$

(21.) $$\frac{x+y-z}{b+c}=\frac{y+z-x}{c+a}=\frac{z+x-y}{a+b}=a+b+c.$$

(22.) $\Sigma(b-c)x=0,\quad \Sigma a(b^2-c^2)x=0,\quad \Sigma a(b-c)x=\Pi(b-c)$.

(23.) $\Sigma x=1,\quad \Sigma x/(b-c)=0,\quad \Sigma x/(b^2-c^2)=0$.

(24.) $$\begin{gathered} ax+k(y+z+u)=0,\quad by+k(z+u+x)=0,\\ cz+k(u+x+y)=0,\quad du+k(x+y+z)=0. \end{gathered}$$

(25.) $x+y+z=a,\quad y+z+u=b,\quad z+u+x=c,\quad u+x+y=d$.

(26.) $3/x-2/y=\frac{1}{4},\quad 4x+7y=1\frac{2}{3}xy$.

(27.) $$\frac{1}{x+5}+\frac{1}{y+7}=\frac{12}{35},\quad \frac{xy+2x+3y+2}{xy+1}=2.$$

(28.) $$\frac{1}{7x}+\frac{y}{9}=11,\quad \frac{1}{9x}+\frac{y}{2}=16.$$

(29.) $$\frac{\lambda^2}{x^2-y^2}+\frac{\mu^2}{x^2+y^2}=1,\quad \frac{\lambda^2+\mu^2}{x^2-y^2}-\frac{\lambda^2-\mu^2}{x^2+y^2}=1.$$

(30.) $$a(b-y)+b(a-x)=c(a-x)(b-y),$$ $$a^2(b-y)+b^2(a-x)=c^2(a-x)(b-y).$$

(31.) $$\frac{a}{c+a+x}+\frac{b}{c+b+y}=1,\quad \frac{a^2}{c+a+x}+\frac{b^2}{c+b+y}=a+b.$$

(32.) $$\frac{b^2}{y^2}+\frac{c^2}{z^2}=1,\quad \frac{c^2}{z^2}+\frac{a^2}{x^2}=1,\quad \frac{a^2}{x^2}+\frac{b^2}{y^2}=1.$$

(33.) $$\frac{a^2}{yz}+\frac{b^2}{zx}=1,\quad \frac{b^2}{zx}+\frac{c^2}{xy}=1,\quad \frac{c^2}{xy}+\frac{a^2}{yz}=1.$$

(34.) $ayz-bzx-cxy=-ayz+bzx-cxy=-ayz-bzx+cxy=xyz.$

(35.) Show that $(1+lx)(1+ay)=1+lz$, $(1+mx)(1+by)=1+mz$, $(1+nx)(1+cy)=1+nz$ are not consistent unless

$$(b-c)a/l+(c-a)b/m+(a-b)c/n=0.$$

If this condition be satisfied, then $x=(c/n-b/m)/(b-c)$; and particular solutions for y and z are $y=-1/a$, $z=-1/l$.

GRAPHICAL DISCUSSION OF LINEAR FUNCTIONS OF ONE AND OF TWO VARIABLES.

§ 19.] *The graph of a linear function of one variable is a straight line.*

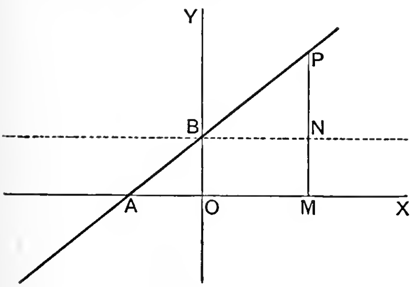


FIG. 1.

Consider the function $y=ax+b$. To find the point where its graph cuts OY, that is, to find the point for which $x=0$, we have to measure $OB=b$ upwards or downwards, according as b is positive or negative (Figs. 1 and 2). Through B draw a line parallel to the x-axis.

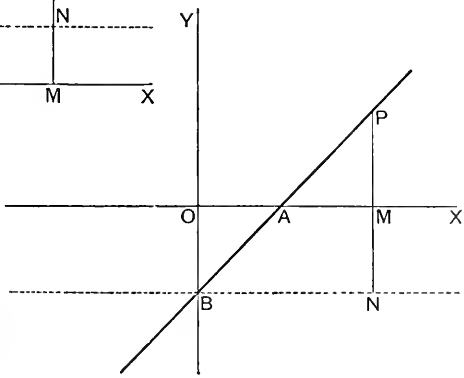


FIG. 2.

Let OM represent any positive value of x, and MP the corresponding value of y.

By the equation to the graph, we have $(y-b)/x=a$. Now, since $b= +\text{OB}= +\text{MN}$ in Fig. 1, $= -\text{OB}= -\text{MN}$ in Fig. 2, we have

$$y-b=\text{PM}-\text{MN}=\text{PN} \text{ in Fig. 1,}$$
$$y-b=\text{PM}+\text{MN}=\text{PN} \text{ in Fig. 2.}$$

Hence we have in both cases

$$\frac{\text{PN}}{\text{BN}}=\frac{\text{PN}}{\text{OM}}=\frac{y-b}{x}=a.$$

In other words, the ratio of PN to BN is constant; hence, by elementary geometry, the locus of P is a straight line. If a be positive, then PN and BN must have the same sign, and the line will slope upwards, from left to right, as in Figs. 1 and 2; if a be negative, the line will slope downwards, from left to right, as in Figs. 3 and 4. The student will easily complete the discussion by considering negative values of x.

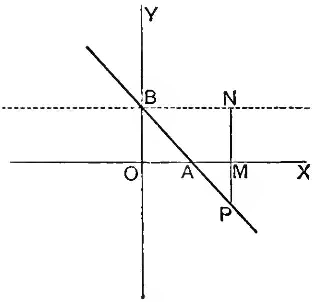


FIG. 3.

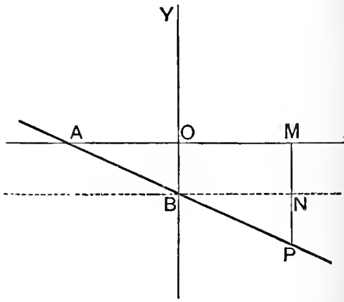


FIG. 4.

§ 20.] So long as the graphic line is not parallel to the axis of x, that is, so long as $a \neq 0$, it will meet the axis in one point, A, and in one only. In analytical language, the equation $ax+b=0$ has one root, and one only.

Also, since a straight line has no turning points, a linear function can have no turning values. In other words, if we increase x continuously from $-\infty$ to $+\infty$, $ax+b$ either increases continuously from $-\infty$ to $+\infty$, or decreases continuously from $+\infty$ to $-\infty$; the former happens when a is positive, the latter when a is negative.

Since $ax+b$ passes only once through every value between

$+\infty$ and $-\infty$, it can pass only once through the value 0. We have thus another proof that the equation $ax+b=0$ has only one root.

A purely analytical proof that $ax+b$ has no turning values may be given as follows:—Let the increment of x be h, then the increment of $ax+b$ is

$$\{a(x+h)+b\}-\{ax+b\}=ah.$$

Now ah is independent of x, and, if h be positive, is always positive or always negative, according as a is positive or negative. Hence, if a be positive, $ax+b$ always increases as x is increased; and if a be negative, $ax+b$ always decreases as x is increased.

§ 21.] We may investigate graphically the condition that the two functions $ax+b$, $a'x+b'$ shall have the same root; in other words, that the equations, $ax+b=0$, $a'x+b'=0$, shall be consistent. Denote $ax+b$ and $a'x+b'$ by y and y' respectively, so that the equations of the two graphs are $y=ax+b$, $y'=a'x+b'$. If both functions have the same root, the graphs must meet OX in the same point A. Now, if P′M PM be ordinates of the two graphs corresponding to the same abscissa OM, and if the graphs meet OX in the same point A, it is obvious that the ratio P′M/PM is constant. Conversely, if P′M/PM is constant, then P′M must vanish when PM vanishes; that is, the graphs must meet OX in the same point. Hence the necessary and sufficient analytical condition is that $(a'x+b')/(ax+b)$ shall be constant, $=k$ say. In other words, we must have

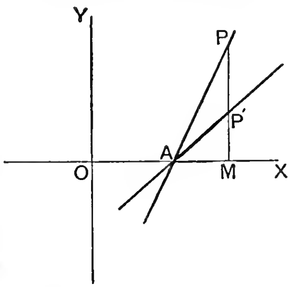


Fig. 5.

$$a'x+b'\equiv k(ax+b).$$

From this it follows that

$$a'=ka, \quad b'=kb,$$

and

$$ab'-a'b=0.$$

These agree with the results obtained above in § 2.

§ 22.] By means of the graph we can illustrate various limiting cases, some of which have hitherto been excluded from consideration.

I. Let $b=0$, $a\neq 0$. In this case $OB=0$, and B coincides with O; that is to say, the graph passes through O (see Figs.

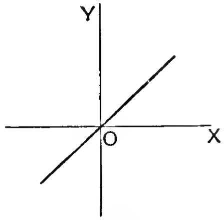


Fig. 6.

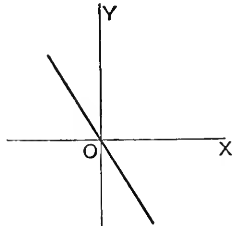


Fig. 7.

6 and 7). Here the graph meets OX at O, and the root of $ax=0$ is $x=0$, as it should be.

II. Let $b\neq 0$ and $a=0$. In this case the equation to the graph is $y=b$, which represents a line parallel to the x-axis (see Figs. 8 and 9). In this case the point of intersection of the

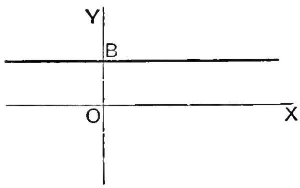


Fig. 8.

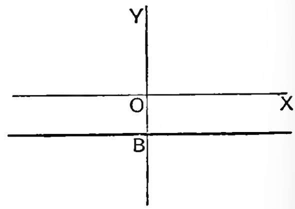


Fig. 9.

graph with OX is at an infinite distance, and $OA=\infty$. If we agree that the solution of the equation $ax+b=0$ shall in all cases be $x=-b/a$, then, when $b\neq 0$, $a=0$, this will give $x=\infty$, in agreement with the conclusion just derived by considering the graph.

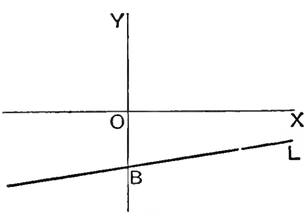


Fig. 10.

This case will be best understood by approaching it, both geometrically and analytically, as a limit. Let us suppose that $b=-1$, and that a is very small, $=1/100000$, say. Then the graph corresponding to $y=x/100000-1$ is something like Fig. 10, where

the intersection of BL with the axis of x is very far to the right of O; that is to say, BL is nearly parallel to OX.

On the other hand, the equation

$$\frac{x}{100000} - 1 = 0$$

gives $x = 100000$, a very large value of x. The smaller we make a the more nearly will BL become parallel to OX, and the greater will be the root of the equation $ax + b = 0$.

If, therefore, in any case where an equation of the 1st degree in x was to be expected, we obtain the paradoxical equation

$$b = 0,$$

where b is a constant, this indicates that the root of the equation has become infinite.

III. If $a = 0$, $b = 0$, the equation to the graph becomes $y = 0$, which represents the axis of x itself. The graph in this case coincides with OX, and its point of intersection with OX becomes indeterminate. If we take the analytical solution of $ax + b = 0$ to be $x = -b/a$ in all cases, it gives us, in the present instance, $x = 0/0$, an indeterminate form, as it ought to do, in accordance with the graphical result.

§ 23.] The graphic surface of a linear function of two independent variables x and y, say $z = ax + by + c$, is a plane. It would not be difficult to prove this, but, for our present purposes, it is unnecessary to do so. We shall confine ourselves to a discussion of the contour lines of the function.

The contour lines of the function $z = ax + by + c$ are a series of parallel straight lines.

For, if k be any constant value of z, the corresponding contour line has for its equation (see chap. xv., § 16)

$$ax + by + c = k \qquad (1).$$

Now (1) is equivalent to

$$y = \left(-\frac{a}{b}\right)x + \frac{k-c}{b} \qquad (2).$$

But (2), as we have seen in § 19 above, represents a straight line, which meets the axes of x and y in A and B, so that

$$OB=\frac{k-c}{b},\quad OA=\frac{k-c}{b}\Big/\frac{a}{b}=\frac{k-c}{a}.$$

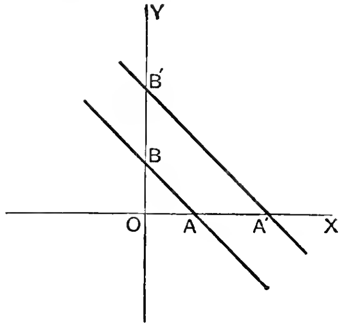


FIG. 11.

Let k' be any other value of z, then the equation to the corresponding contour line is

$$ax+by+c=k' \qquad (3),$$

or

$$y=\left(-\frac{a}{b}\right)x+\frac{k'-c}{b} \qquad (4).$$

Hence, if this second contour line meet the axes in A′ and B′ respectively, we have

$$OB'=\frac{k'-c}{b},\quad OA'=\frac{k'-c}{a}.$$

Hence

$$\frac{OA}{OB}=\frac{b}{a}=\frac{OA'}{OB'},$$

which proves that AB is parallel to A′B′.

The zero contour line of $z=ax+by+c$ is given by the equation

$$ax+by+c=0 \qquad (5).$$

This straight line divides the plane XOY into two regions, such that the values of x and y corresponding to any point in one of them render $ax+by+c$ positive, and the values of x and y corresponding to any point in the other render $ax+by+c$ negative.

§ 24.] Let us consider the zero contour lines, L and L′, of two linear functions, $z=ax+by+c$ and $z'=a'x+b'y+c'$. Since the co-ordinates of every point on L satisfy the equation

$$ax+by+c=0 \qquad (1),$$

and the co-ordinates of every point on L′ satisfy the equation

$$a'x + b'y + c' = 0 \qquad (2),$$

it follows that the co-ordinates of the point of intersection of L and L′ will satisfy both (1) and (2); in other words, the co-ordinates of the intersection will be a solution of the system (1), (2).

Now, any two straight lines L and L′ in the same plane have one and only one finite point of intersection, provided L and L′ be neither parallel nor coincident. Hence we infer that the linear system (1), (2) has in general one and only one solution.

It remains to examine the two exceptional cases.

I. Let L and L′ (Fig. 11) be parallel, and let them meet the axes of X and Y in A, B and in A′, B′ respectively. In this case the point of intersection passes to an infinite distance, and both its co-ordinates become infinite.

The necessary and sufficient condition that L and L′ be parallel is OA/OB = OA′/OB′. Now, OA $= -c/a$, OB $= -c/b$; and OA′ $= -c'/a'$, OB′ $= -c'/b'$. Hence the necessary and sufficient condition for parallelism is $b/a = b'/a'$, that is, $ab' - a'b = 0$.

We have thus fallen upon the excepted case of §§ 4 and 5. If we assume that the results of the general formulæ obtained for the case $ab' - a'b \neq 0$, namely,

$$x = \frac{bc' - b'c}{ab' - a'b}, \quad y = \frac{ca' - c'a}{ab' - a'b},$$

hold also when $ab' - a'b = 0$, we see that in the present case neither of the numerators $bc' - b'c$, $ca' - c'a$, can vanish. For if, say, $bc' - b'c = 0$, then $-c/b = -c'/b'$, that is, OB = OB′; and the two lines AB, A′B′, already parallel, would coincide, which is not supposed.

It follows, then, that

$$x = \frac{bc' - b'c}{0} = \infty, \quad y = \frac{ca' - c'a}{0} = \infty;$$

and the analytical result agrees with the graphical.

II. Let L and L′ be coincident, then the intersection becomes indeterminate. The conditions for coincidence are

$$OA = OA', \quad OB = OB',$$

whence $$-c/a = -c'/a', \quad -c/b = -c'/b'.$$

These give $$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'},$$

which again give

$$bc' - b'c = 0, \quad ca' - c'a = 0, \quad ab' - a'b = 0.$$

We thus have once more the excepted case of §§ 4 and 5, but this time with the additional peculiarity that $bc' - b'c = 0$ and $ca' - c'a = 0$.

If we assert the truth of the general analytical solution in this case also, we have

$$x = \frac{0}{0}, \quad y = \frac{0}{0},$$

that is, the values of x and y are indeterminate, as they ought to be, in accordance with the graphical result.

§ 25.] Since three straight lines taken at random in a plane have not in general a common point of intersection, it follows that the three equations,

$$ax + by + c = 0, \quad a'x + b'y + c' = 0, \quad a''x + b''y + c'' = 0 \quad (1),$$

have not in general a common solution. When these have a common solution their three graphic lines, L, L′, L″, will have a common intersection. We found the analytical condition for this to be

$$ab'c'' - ab''c' + bc'a'' - bc''a' + ca'b'' - ca''b' = 0 \qquad (2).$$

In our investigation of this condition we left out of account the cases where any one of the three functions, $ab' - a'b$, $a''b - ab''$, $a'b'' - a''b'$, vanishes.

We propose now to examine graphically the excepted cases.

First, we remark that if two of the functions vanish, the third will also vanish; so that we need only consider (I.) the case where two vanish, (II.) the case where only one vanishes.

I. $$ab' - a'b = 0, \quad a''b - ab'' = 0.$$

This involves that L and L′ are parallel, and that L and L″ are parallel; so that all three, L, L′, L″, are parallel; and we have, in addition to the two given conditions, also $a'b'' - a''b' = 0$.

Hence, since the condition (2) may be written

$$c(a'b'' - a''b') + c'(a''b - ab'') + c''(ab' - a'b) = 0,$$

it appears that the general analytical condition for a common solution is satisfied.

This agrees with the graphical result, for three parallel straight lines may be regarded as having a common intersection at infinity.

In the present case is of course included the two cases where two of the lines coincide, or all three coincide. The corresponding analytical peculiarities in the equations will be obvious to the reader.

II. $$ab' - a'b = 0.$$

Here two of the graphic lines, L and L′, are parallel, and the third, L″, is supposed to be neither coincident with nor parallel to either.

Looking at the matter graphically, we see that in this case the three lines cannot have a common intersection unless L and L′ coincide, that is, unless

$$a' = ka, \quad b' = kb, \quad c' = kc,$$

where k is some constant.

Let us see whether the condition (2) also brings out this result, as it ought to do.

Since $$ab' - a'b = 0,$$

we have $$\frac{a'}{a} = \frac{b'}{b} = k, \text{ say.}$$

Hence $$a' = ka, \quad b' = kb.$$

Now, by virtue of these results, (2) reduces to

$$a''(bc' - b'c) + b''(ca' - c'a) = 0,$$

that is, to

$$a''(bc' - kbc) + b''(cka - c'a) = 0,$$

that is, to

$$(a''b - ab'')(c' - kc) = 0,$$

which gives, since $a''b - ab'' \neq 0$,

$$c' - kc = 0,$$

that is, $$c' = kc.$$

Hence the agreement between the analysis and the geometry is complete.*

§ 26.] It would lead us too far if we were to attempt here to take up the graphical discussion of linear functions of three variables. We should have, in fact, to go into a discussion of the disposition of planes and lines in space of three dimensions.

We consider the subject, so far as we have pursued it, an essential part of the algebraic training of the student. It will help to give him clear ideas regarding the generality and coherency of analytical expression, and will enable him at the same time to grasp the fundamental principles of the application of algebra to geometry. The two sciences mutually illuminate each other, just as two men each with a lantern have more light when they walk together than when each goes a separate way.

Exercises XXVII.

Draw to scale the graphs of the following linear functions of x:—

(1.) $y=x+1$.	(4.) $y=2x+3$.
(2.) $y=-x+1$.	(5.) $y=-\frac{1}{2}x-\frac{1}{3}$.
(3.) $y=-x-1$.	(6.) $y=-3(x-1)$.

(7.) Draw the graphs of the two functions, $3x-5$ and $5x+7$; and by means of them solve the equation $3x-5=5x+7$.

(8.) Draw to scale the contour lines of $z=2x-3y+1$, corresponding to

$$z=-2,\quad z=-1,\quad z=0,\quad z=+1,\quad z=+2.$$

(9.) Draw the zero contour lines of $z=5x+6y-3$ and $z'=8x-9y+1$; and by means of them solve the system

$$5x+6y-3=0,\quad 8x-9y+1=0.$$

---

* It may be well to warn the reader explicitly that he must be careful to use the limiting cases which we have now introduced into the theory of equations with a proper regard to accompanying circumstances. Take, for instance, the case of the paradoxical equation $b=0$, out of which we manufactured a linear equation by writing it in the form $0x+b=0$; and to which, accordingly, we assigned one infinite root. Nothing in the equation itself prevents us from converting it in the same way into a quadratic equation, for we might write it $0x^2+0x+b=0$, and say (see chap. xviii., § 5) that it has two infinite roots. Before we make any such assertion we must be sure beforehand whether a linear, or a quadratic or other equation was, generally speaking, to be expected. This must, of course, be decided by the circumstances of each particular case.

Also show that the two contour lines divide the plane into four regions, such that in two of them $(5x+6y-3)(8x-9y+1)$ is always positive, and in the other two the same function is always negative.

(10.) Is the system

$$3x-4y+2=0, \quad 6x-8y+3=0, \quad x-\tfrac{4}{3}y+1=0$$

consistent or inconsistent?

(11.) Determine the value of c in order that the system

$$2x+y-1=0, \quad 4x+2y+3=0, \quad (c+1)x+(c+2)y+5=0$$

may be consistent.

(12.) Prove graphically that, if $ab'-a'b=0$, then the infinite values of x and y, which constitute the solution of

$$ax+by+c=0, \quad a'x+b'y+c'=0,$$

have a finite ratio, namely,

$$x/y=(bc'-b'c)/(ca'-c'a).$$

(13.) If $(ax+by+c)/(a'x+b'y+c')$ be independent of x and y, show that

$$ab'-a'b=0, \quad ca'-c'a=0, \quad bc'-b'c=0;$$

and that two of these conditions are sufficient.

(14.) Illustrate graphically the reasoning in the latter part of § 5 of the preceding chapter.

(15.) Explain graphically the leading proposition in § 6.

CHAPTER XVII.

Equations of the Second Degree.

EQUATIONS OF THE SECOND DEGREE IN ONE VARIABLE.

§ 1.] Every equation of the 2nd degree (*Quadratic Equation*) in one variable, can be reduced to an equivalent equation of the form

$$ax^2 + bx + c = 0 \qquad (1).$$

Either or both of the coefficients b and c may vanish; but we cannot (except as a limiting case, which we shall consider presently) suppose $a = 0$ without reducing the degree of the equation.

By the general proposition of chap. xii., § 23, when a, b, c are given, two values of x and no more can be found which shall make the function $ax^2 + bx + c$ vanish; that is, *the equation* (1) *has always two roots and no more.* The roots may be equal or unequal, real or imaginary, according to circumstances.

The general theory of the solution of quadratic equations is thus to a large extent already in our hands. It happens, however, that the *formal solution* of a quadratic equation is always obtainable; so that we can verify the general proposition by actually finding the roots as closed functions of the coefficients a, b, c.

§ 2.] We consider first the following particular cases :—

I. $\qquad c = 0.$

The equation (1) reduces to

$$ax^2 + bx = 0,$$

that is, since $a \neq 0$,

$$ax\left(x + \frac{b}{a}\right) = 0,$$

which is equivalent to

$$\left\{\begin{array}{c} x = 0 \\ x + \dfrac{b}{a} = 0 \end{array}\right\}.$$

Hence the roots are $x = 0$, $x = -b/a$.

II. $$b = 0, \quad c = 0.$$

The equation (1) now reduces to

$$ax \times x = 0,$$

which, since $a \neq 0$, is equivalent to

$$\left\{\begin{array}{c} x = 0 \\ x = 0 \end{array}\right\}.$$

Hence the roots are $x = 0$, $x = 0$. This might also be deduced from I.

Here the roots are equal. We might of course say that there is only one root, but it is more convenient, in order to maintain the generality of the proposition regarding the number of the roots of an integral equation of the nth degree in one variable, to say that there are two equal roots.

III. $$b = 0.$$

The equation (1) reduces to

$$ax^2 + c = 0,$$

that is, since $a \neq 0$, to

$$a\left(x + \sqrt{-\frac{c}{a}}\right)\left(x - \sqrt{-\frac{c}{a}}\right) = 0,$$

which is equivalent to

$$\left\{\begin{array}{c} x + \sqrt{-\dfrac{c}{a}} = 0 \\ x - \sqrt{-\dfrac{c}{a}} = 0 \end{array}\right\}.$$

Hence the roots are $x = -\sqrt{(-c/a)}$, $x = +\sqrt{(-c/a)}$; that is, the roots are equal, but of opposite sign. If c/a be negative,

both roots will be real; if c/a be positive, both roots will be imaginary, and we may write them in the more appropriate form $x = -i\sqrt{(c/a)}$, $x = +i\sqrt{(c/a)}$.

§ 3.] The general case, where all the three coefficients are different from zero, may be treated in various ways; but a little examination will show the student that all the methods amount to reducing the equation

$$ax^2 + bx + c = 0 \qquad (1)$$

to an equivalent form, $a(x+\lambda)^2 + \mu = 0$, which is treated like the particular case III. of last paragraph.

1*st Method.*—The most direct method is to take advantage of the identity of chap. vii., § 5. We have

$$ax^2 + bx + c \equiv a\left\{x - \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}\right\}\left\{x - \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}\right\};$$

hence the equation (1) is equivalent to

$$a\left\{x - \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}\right\}\left\{x - \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}\right\} = 0,$$

that is, to

$$\left\{\begin{array}{l} x - \dfrac{-b + \sqrt{(b^2 - 4ac)}}{2a} = 0 \\[2ex] x - \dfrac{-b - \sqrt{(b^2 - 4ac)}}{2a} = 0 \end{array}\right\}.$$

The roots of (1) *are therefore* $\{-b + \sqrt{(b^2 - 4ac)}\}/2a$, and $\{-b - \sqrt{(b^2 - 4ac)}\}/2a$.

2*nd Method.*—We may also adopt the ordinary process of "*completing the square.*" We may write (1) in the equivalent form

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad (2),$$

and render the left-hand side of (2) a complete square by adding $(b/2a)^2$ to both sides. We thus deduce the equivalent equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a},$$

$$= \frac{b^2 - 4ac}{4a^2} \qquad (3).$$

The equation (3) is obviously equivalent to

$$\left.\begin{aligned} x + \frac{b}{2a} &= + \sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)} \\ x + \frac{b}{2a} &= - \sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)} \end{aligned}\right\},$$

from which we deduce

$$x = \{-b + \sqrt{(b^2 - 4ac)}\}/2a, \quad x = \{-b - \sqrt{(b^2 - 4ac)}\}/2a,$$

as before.

3rd *Method*.—By changing the variable, we can always make (1) depend on an equation of the form $az^2 + d = 0$. Let us assume that $x = z + h$, where h is entirely at our disposal, and z is to be determined by means of the derived equation. Then, by (1), we have

$$a(z + h)^2 + b(z + h) + c = 0 \qquad (4).$$

It is obvious that this equation is equivalent to (1), provided x be determined in terms of z by the equation $x = z + h$.

Now (4) may be written

$$az^2 + (2ah + b)z + (ah^2 + bh + c) = 0 \qquad (5).$$

Since h is at our disposal, we may so determine it that $2ah + b = 0$; that is, we may put $h = -b/2a$. The equation (5) then becomes

$$az^2 + a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = 0,$$

that is,

$$az^2 - \frac{b^2 - 4ac}{4a} = 0 \qquad (6).$$

From (6) we deduce $z = +\sqrt{(b^2 - 4ac)}/2a$, $z = -\sqrt{(b^2 - 4ac)}/2a$. Hence, since $x = z + h = -b/2a + z$, we have

$$x = \{-b + \sqrt{(b^2 - 4ac)}\}/2a, \quad x = \{-b - \sqrt{(b^2 - 4ac)}\}/2a,$$

as before.

In solving any particular equation the student may either quote the forms $\{-b \pm \sqrt{(b^2 - 4ac)}\}/2a$, which give the roots in all cases, and substitute the values which a, b, c happen to have in the particular case, or he may work through the process of

the 2nd method in the particular case. The latter alternative will often be found the more conducive to accuracy.

§ 4.] In distinguishing the various cases that may arise when the coefficients a, b, c are real rational numbers, we have merely to repeat the discussion of chap. vii., § 7, on the nature of the factors of an integral quadratic function.

We thus see that the roots of

$$ax^2 + bx + c = 0,$$

(1) Will be real and unequal if $b^2 - 4ac$ be positive.

(2) Will be real and equal if $b^2 - 4ac = 0$.

(3) Will be two conjugate complex numbers if $b^2 - 4ac$ be negative. The appropriate expressions in this case are $\{-b + i\sqrt{(4ac - b^2)}\}/2a$, $\{-b - i\sqrt{(4ac - b^2)}\}/2a$.

(4) The roots will be rational if $b^2 - 4ac$ be positive and the square of a rational number.

(5) The roots will be conjugate surds of the form $A \pm \sqrt{B}$ in the case where $b^2 - 4ac$ is positive, but not the square of a rational number.

(6) If the coefficients a, b, c be rational functions of any given quantities $p, q, r, s, \ldots$ then the roots will or will not be rational functions of $p, q, r, s, \ldots$ according as $b^2 - 4ac$ is or is not the square of a rational function of $p, q, r, s, \ldots$

It should be noticed that the conditions given as characterising the above cases are not only sufficient but also necessary.

The cases where a, b, c are either irrational real numbers, or complex numbers of the general form $a + a'i$, are not of sufficient importance to require discussion here.

Example 1.

$$2x^2 - 3x = 0.$$

By inspection we see that the roots are $x = 0$, $x = 3/2$.

Example 2.

$$2x^2 + 8 = 0.$$

This equation is equivalent to $x^2 + 4 = 0$, whose roots are $x = 2i$, $x = -2i$.

Example 3.

$$35x^2 - 2x - 1 = 0.$$

The equation is equivalent to

$$x^2 - \frac{2}{35}x = \frac{1}{35},$$

that is, to $$\left(x-\frac{1}{35}\right)^2=\frac{1}{35^2}+\frac{1}{35}=\frac{36}{35^2}.$$

Hence $$x-\frac{1}{35}=\pm\frac{6}{35}.$$

Hence $$x=\frac{1\pm6}{35}.$$

The roots are, therefore, $+1/5$ and $-1/7$.

Example 4.
$$x^2-2x-2=0.$$

The roots are $1+\sqrt{3}$ and $1-\sqrt{3}$.

Example 5.
$$3x^2+24x+48=0.$$

The given equation is equivalent to
$$x^2+8x+16=0,$$
that is, to $$(x+4)^2=0.$$
Hence $x=-4\pm0$; that is to say, the two roots are each equal to -4.

Example 6.
$$x^2-4x+7=0.$$
This is equivalent to
$$x^2-4x+4=-3,$$
that is, to $$(x-2)^2=3i^2.$$
Hence the roots are $2+\sqrt{3}i$, $2-\sqrt{3}i$.

Example 7.
$$x^2-2(p+q)^2x+2p^4+12p^2q^2+2q^4=0.$$

This equation is equivalent to
$$\begin{aligned}\{x-(p+q)^2\}^2&=(p+q)^4-2p^4-12p^2q^2-2q^4,\\ &=-(p-q)^4,\\ &=(p-q)^4i^2.\end{aligned}$$

Hence the roots are $(p+q)^2+(p-q)^2i$, $(p+q)^2-(p-q)^2i$.

Exercises XXVIII.

(1.) $x^2+x=0.$

(2.) $(2x-1)(3x-2)=0.$

(3.) $(x+1)(x-1)+1=0.$

(4.) $(x-1)^2+3(x-1)=0.$

(5.) $(x-1)^2+(x-2)^2=0.$

(6.) $3(x-1)^2-2(x-2)^2=0.$

(7.) $p(x+\alpha)^2-q(x+\beta)^2=0.$

(8.) $(px+q)^2+(qx+p)^2=0.$

(9.) $2x^2+3x+5=3x^2+4x+1.$

(10.) $x^3+8x^2+16x-1=(x+3)^3.$

(11.) $255x^2-431x+182=0.$

(12.) $4x^2-40x+107=0.$

(13.) $x^2-22x+170=0.$

(14.) $x^2-201x+200=0.$

(15.) $x^2+102x+2597=0.$

(16.) $x^2-4x-2597=0.$

(17.) $x^2+6\sqrt{7}x+55=0.$

(18.) $x^2-2(1+\sqrt{2})x+2\sqrt{2}=0.$

(19.) $x^2+(23+12i)x+97+137i=0.$

(20.) $x^2-(8-2i)x=38i-31.$

(21.) $(x-1)(x-2)+(x-1)(x-3)+(x-2)(x-3)=0$.
(22.) $(x-1)^3+(x-1)^2(x-2)-2(x+1)^3=0$.
(23.) $(x-\frac{3}{4})(x-\frac{6}{8})+(x-\frac{3}{4})(x-\frac{1}{2})=0$.
(24.) $(x-a)^2+(x-b)^2=a^2+b^2$. (25.) $x^2+4ax=(b-c)^2+4(bc-a^2)$.
(26.) $x^2+(b-c)x=a^2+bc+ca+ab$.
(27.) $x^2+1=x\left(\sqrt{\frac{m}{n}}+\sqrt{\frac{n}{m}}\right)$.
(28.) $(a+b)(abx^2-2)=(a^2+b^2)x$.
(29.) $(a-b)x^2-(a^2+ab+b^2)x+ab(2a+b)=0$.
(30.) $(c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0$.
(31.) $(a^2-ax+c^2)(a^2+ax+c^2)=a^4+a^2c^2+c^4$.
(32.) $x^2-2(a^2+b^2+c^2)x+a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2=2abc(a+b+c)$
(33.) $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3=0$.
(34.) Evaluate $\sqrt{(7+\sqrt{(7+\sqrt{(7+\sqrt{(7\ldots \text{ad } \infty\ldots))))}}}}$.

EQUATIONS WHOSE SOLUTION CAN BE EFFECTED BY MEANS OF QUADRATIC EQUATIONS.

§ 5.] *Reduction by Factorisation.*—If we know one root of an integral equation

$$f(x)=0 \qquad (1),$$

say $x=a$, then, by the remainder theorem, we know that $f(x)=(x-a)\phi(x)$, where $\phi(x)$ is lower in degree by one than $f(x)$. Hence (1) is equivalent to

$$\left\{\begin{array}{c} x-a=0 \\ \phi(x)=0 \end{array}\right\} \qquad (2).$$

The solution of (1) now depends on the solution of $\phi(x)=0$. It may happen that $\phi(x)=0$ is a quadratic equation, in which case it may be solved as usual; or, if not, we may be able to reduce the equation $\phi(x)=0$ by guessing another root; and so on.

Example 1.

To find the cube roots of -1.

Let x be any cube root of -1, then, by the definition of a cube root, we must have $x^3=-1$. We have therefore to solve the equation

$$x^3+1=0.$$

We know one root of this equation, namely, $x=-1$; the equation, in fact, is equivalent to

$$(x+1)(x^2-x+1)=0,$$

that is, to

$$\left\{\begin{array}{c} x+1=0 \\ x^2-x+1=0 \end{array}\right\}.$$

The quadratic $x^2-x+1=0$, solved as usual, gives $x=(1\pm i\sqrt{3})/2$.

Hence the three cube roots of -1 are -1, $(1+i\sqrt{3})/2$, $(1-i\sqrt{3})/2$, which agrees with the result already obtained in chap. xii. by means of Demoivre's Theorem.

Example 2.

$$7x^3-13x^2+3x+3=0.$$

This equation is obviously satisfied by $x=1$. Hence it is equivalent to

$$(7x^2-6x-3)(x-1)=0.$$

The roots of the quadratic $7x^2-6x-3=0$ are $(3\pm\sqrt{30})/7$. Hence the three roots of the original cubic are 1, $(3+\sqrt{30})/7$, $(3-\sqrt{30})/7$.

It may happen that we are able by some artifice to throw an integral equation into the form

$$PQR\ .\ .\ .=0,$$

where P, Q, R, . . . are all integral functions of x of the 2nd degree. The roots of the equation in question are then found by solving the quadratics

$$P=0,\quad Q=0,\quad R=0,\quad .\ .\ .$$

Example 3.

$$p(ax^2+bx+c)^2-q(dx^2+ex+f)^2=0.$$

This equation is obviously equivalent to

$$\{\sqrt{p}(ax^2+bx+c)+\sqrt{q}(dx^2+ex+f)\}\{\sqrt{p}(ax^2+bx+c)-\sqrt{q}(dx^2+ex+f)\}=0.$$

Hence its roots are the four roots of the two quadratics

$$(a\sqrt{p}+d\sqrt{q})x^2+(b\sqrt{p}+e\sqrt{q})x+(c\sqrt{p}+f\sqrt{q})=0,$$
$$(a\sqrt{p}-d\sqrt{q})x^2+(b\sqrt{p}-e\sqrt{q})x+(c\sqrt{p}-f\sqrt{q})=0,$$

which can be solved in the usual way.

§ 6.] *Integralisation and Rationalisation.*—We have seen in chap. xiv. that every algebraical equation can be reduced to an integral equation, which will be satisfied by all the finite roots of the given equation, but some of whose roots may happen to be extraneous to the given equation. The student should recur to the principles of chap. xiv., and work out the full solutions of as many of the exercises of that chapter as he can. In the exercises that follow in the present chapter particular attention should be paid to the distinction between solutions which are and solutions which are not extraneous to the given equation.

The following additional examples will serve to illustrate the point just alluded to, and to exemplify some of the artifices that are used in the reduction of equations having special peculiarities.

Example 1.

$$\frac{1}{x+a+b}+\frac{1}{x-a+b}+\frac{1}{x+a-b}+\frac{1}{x-a-b}=0.$$

If we combine the first and last terms, and also the two middle terms, we derive the equivalent equation

$$\frac{2x}{x^2-(a+b)^2}+\frac{2x}{x^2-(a-b)^2}=0.$$

If we now multiply by $\{x^2-(a+b)^2\}\{x^2-(a-b)^2\}$ we deduce the equation

$$2x\{2x^2-2(a^2+b^2)\}=0\ ;$$

and it may be that we introduce extraneous solutions, since the multiplier used is a function of x.

The equation last derived is equivalent to

$$\left\{\begin{array}{r} x=0 \\ x^2-(a^2+b^2)=0 \end{array}\right\}.$$

Hence the roots of the last derived equation are 0, $+\sqrt{(a^2+b^2)}$, $-\sqrt{(a^2+b^2)}$.

Now, the roots, if any, introduced by the factor $\{x^2-(a+b)^2\}\{x^2-(a-b)^2\}$ must be $\pm(a+b)$ or $\pm(a-b)$. Hence none of the three roots obtained from the last derived equation are, in the present case, extraneous.

Example 2.

$$\frac{a-x}{\sqrt{a}+\sqrt{(a-x)}}+\frac{a+x}{\sqrt{a}+\sqrt{(a+x)}}=\sqrt{a} \qquad (\alpha).$$

If we rationalise the denominators on the left, we have

$$\frac{(a-x)\{\sqrt{a}-\sqrt{(a-x)}\}}{x}+\frac{(a+x)\{\sqrt{a}-\sqrt{(a+x)}\}}{-x}=\sqrt{a} \qquad (\beta).$$

From (β), after multiplying both sides by x, and transposing all the terms that are rational in x, we obtain

$$(a+x)^{\frac{3}{2}}-(a-x)^{\frac{3}{2}}=3x\sqrt{a} \qquad (\gamma).$$

From (γ), by squaring and transposing, we deduce

$$2a^3-3ax^2=2(a^2-x^2)^{\frac{3}{2}} \qquad (\delta).$$

From (δ), by squaring and transposing, we have finally the integral equation

$$(4x^2-3a^2)x^4=0 \qquad (\epsilon).$$

The roots of (ϵ) are 0 (*repeated* four times, but that does not concern us so far as the original irrational equation * (α) is concerned) and $\pm a\sqrt{3}/2$.

It is at once obvious that $x=0$ is a root of (α).

If we observe that $\sqrt{(1\pm\sqrt{3}/2)}=(\sqrt{3}\pm1)/2$, we see that $\pm a\sqrt{3}/2$ are roots of (α), provided

$$\frac{2\mp\sqrt{3}}{2\mp1+\sqrt{3}}+\frac{2\pm\sqrt{3}}{2\pm1+\sqrt{3}}=1,$$

that is, provided

$$\frac{2-\sqrt{3}}{1+\sqrt{3}}+\frac{2+\sqrt{3}}{3+\sqrt{3}}=1,$$

which is not true.

Hence the only root of (α) is $x=0$.

* For we have established no theory regarding the number of the roots of an *irrational* equation as such.

Example 3.

$$\frac{\sqrt{(a+x)}}{\sqrt{a}+\sqrt{(a+x)}}=\frac{\sqrt{(a-x)}}{\sqrt{a}-\sqrt{(a-x)}} \quad (\alpha).$$

By a process almost identical with that followed in last example, we deduce from (α) the equation

$$4x^4-3a^2x^2=0 \quad (\beta).$$

The roots of (β) are 0, and $\pm a\sqrt{3}/2$; but it will be found that none of these satisfy the original equation (α).

Example 4.

$$\sqrt{(2x^2-4x+1)}+\sqrt{(x^2-5x+2)}=\sqrt{(2x^2-2x+3)}+\sqrt{(x^2-3x+4)} \quad (\alpha).$$

The given equation is equivalent to

$$\sqrt{(2x^2-4x+1)}-\sqrt{(x^2-3x+4)}=\sqrt{(2x^2-2x+3)}-\sqrt{(x^2-5x+2)}.$$

From this last, by squaring, we deduce

$$3x^2-7x+5-2\sqrt{(2x^2-4x+1)(x^2-3x+4)}$$
$$=3x^2-7x+5-2\sqrt{(2x^2-2x+3)(x^2-5x+2)},$$

which is equivalent to

$$\sqrt{(2x^4-10x^3+21x^2-19x+4)}=\sqrt{(2x^4-12x^3+17x^2-19x+6)} \quad (\beta).$$

From (β), by squaring and transposing and rejecting the factor 2, we deduce

$$x^3+2x^2-1=0 \quad (\gamma).$$

One root of (γ) is $x=-1$, and (γ) is equivalent to

$$(x+1)(x^2+x-1)=0.$$

Hence the roots of (γ) are -1 and $(-1\pm\sqrt{5})/2$.

Now $x=-1$ obviously satisfies (α). We can show that the other two roots of (γ) are extraneous to (α); for, if x have either of the values $(-1\pm\sqrt{5})/2$, then $x^2+x-1=0$, therefore $x^2=-x+1$. Using this value of x^2, we reduce (α) to $\sqrt{(-6x+3)}=\sqrt{(-4x+5)}$. This last equation involves the truth of the equation $-6x+3=-4x+5$, which is satisfied by $x=-1$, and not by either of the values $x=(-1\pm\sqrt{5})/2$.

N.B.—An interesting point in this example is the way the terms of (α) are disposed before we square for the first time.

Example 5.

$$\frac{1-\sqrt{(1-x^2)}}{1+\sqrt{(1-x^2)}}=27\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}} \quad (\alpha).$$

Multiply the numerator and denominator on the left by $1-\sqrt{(1-x^2)}$, and the numerator and denominator on the right by $\sqrt{(1+x)}-\sqrt{(1-x)}$, and we obtain the equivalent equation

$$\frac{(1-\sqrt{1-x^2})^2}{x^2}=27\frac{x}{1-\sqrt{(1-x^2)}}.$$

Multiply both sides of the last equation by $x^2(1-\sqrt{1-x^2})$, and we deduce

$$\{1-\sqrt{(1-x^2)}\}^3=27x^3 \quad (\beta).$$

If 1, ω, ω^2 (see chap. xii., § 20) be the three cube roots of $+1$, then (β) is equivalent to

$$\left.\begin{cases} 1-\sqrt{(1-x^2)}=3x \\ 1-\sqrt{(1-x^2)}=3\omega x \\ 1-\sqrt{(1-x^2)}=3\omega^2 x \end{cases}\right\} \qquad (\gamma).$$

By rationalisation we deduce from (γ) the three integral equations

$$\left.\begin{cases} 10x^2-6x=0 \\ (1+9\omega^2)x^2-6\omega x=0 \\ (1+9\omega)x^2-6\omega^2 x=0 \end{cases}\right\} \qquad (\delta).$$

The roots of these equations (δ) are 0, 3/5 ; 0, $6\omega/(1+9\omega^2)$; 0, $6\omega^2/(1+9\omega)$.

The student will have no difficulty in settling which of them satisfy the original equation (α).

Exercises XXIX.

(1.) $x^6-1=(x^2+\frac{3}{2})^2(x^2-1)$.

(2.) $x^3-(a+b+c)x^2-(a^2+b^2+c^2-bc-ca-ab)x+a^3+b^3+c^3-3abc=0$.

(3.) $x^4-40x+39=0$.

(4.) $x^4+2(a-2)x^3+(a-2)^2x^2+2a^2(a-2)x+a^4=0$.

(5.) $2x^3-x^2-2x-8=0$. (6.) $ax^3+x+a+1=0$.

(7.) $x^4-3x^3+4x^2-3x+1=0$.

(8.) $\dfrac{x^2}{p}+\dfrac{x}{p^2}+\dfrac{1}{p^3}=\dfrac{p}{x^2}+\dfrac{p^2}{x}+p^3$.

(9.) $x^4-6x^3+10x^2-8x+16=0$. (10.) $x^4-6=5x(x^2-x-1)$.

(11.) $(x^2+6x+9)(x^2+8x+16)=(x^2+4x+4)(x^2-12x+36)$.

Exercises XXX.

(1.) $ax=2\left(\dfrac{1}{a}+\dfrac{1}{b}\right)\Big/\left(x+\dfrac{1}{a}-\dfrac{1}{b}\right)$.

(2.) $\dfrac{a}{b+x}+\dfrac{b}{a+x}=1$.

(3.) $\dfrac{2x^2-x-1}{x-2}+\dfrac{2x^2-3x-8}{x-3}=\dfrac{8x^2-8}{2x-3}$.

(4.) $\dfrac{9x+5}{12}-\dfrac{4x-2}{7x-1}=\dfrac{12x+3}{16}-\dfrac{4x+3}{7x+9}+\dfrac{11}{48}$.

(5.) $x-3=\dfrac{x^3-27}{x^2+8}$.

(6.) $\dfrac{ax+b}{c}+\dfrac{ax+b}{cx+b}=\dfrac{2ax+d}{2c}+\dfrac{b}{c}$.

(7.) $\dfrac{ax+b}{cx+b}+\dfrac{bx+a}{cx+a}=\dfrac{(a+b)(x+2)}{cx+a+b}$.

(8.) $\dfrac{1}{2}\dfrac{x+a}{x-a}+\dfrac{1}{2}\dfrac{x+b}{x-b}-\dfrac{x+c}{x-c}=\dfrac{ax}{x^2-a^2}+\dfrac{bx}{x^2-b^2}-\dfrac{2cx}{x^2-c^2}$.

(9.) $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{b}{x-a}+\dfrac{a}{x-b}$.

(10.) $\frac{a-c}{2b+x}+\frac{b-c}{2a+x}=\frac{a+b-2c}{a+b+x}$.

(11.) $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{x^2+a^2}{x^2-a^2}+\frac{x^2-a^2}{x^2+a^2}$.

(12.) $\frac{(x-a)(x-b)}{x-a-b}=\frac{(x-c)(x-d)}{x-c-d}$.

(13.) $\left(\frac{a^2+ax+x^2}{a^2-ax+x^2}\right)^2=\frac{a+2x}{a-2x}$.

(14.) $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3}=\frac{2x^2+2x+1}{x+1}$.

(15.) $\frac{x}{x^2-2x-15}-\frac{7\frac{1}{2}}{x^2+2x-35}=\frac{1}{x^2+10x+21}$.

(16.) $\frac{2x+3a}{2x-3a}+\frac{2x-3a}{2x+3a}=\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

(17.) $\frac{(x-a)^2}{(x-b)(x-c)}+\frac{(x-b)^2}{(x-c)(x-a)}+\frac{(x-c)^2}{(x-a)(x-b)}=3$.

Exercises XXXI.

(1.) $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x(x-1)}{4}$.

(2.) $\frac{4}{\sqrt{(x+2)}}=\sqrt{(x+2)}+2\sqrt{x}$.

(3.) $x+\frac{x^2}{(\sqrt{(x+a)}-\sqrt{a})^2}=2a$.

(4.) $(a^2+bx)\sqrt{(a^2+c^2)}=(a^2+bc)\sqrt{(a^2+x^2)}$.

(5.) $6x(x-1)-3\sqrt{\{3(x-2)(x+1)-2(x-5)\}}=4(x+3)$.

(6.) $\sqrt{(x+\sqrt{x})}+\sqrt{(x-\sqrt{x})}=a\sqrt{x}/\sqrt{(x+\sqrt{x})}$.

(7.) $(1+x)\sqrt{(1-x^2)}+(x-1)=0$.

(8.) $(x-3)/\sqrt{(x^2-6x+36)}=(x-4)/\sqrt{(x^2-8x+64)}$.

(9.) $(2x-a)/\sqrt{(x^2-ax+a^2)}=(2x-b)/\sqrt{(x^2-bx+b^2)}$.

(10.) $\sqrt{\left(\frac{3}{x+1}\right)}=3\left\{\sqrt{\left(\frac{2x+1}{2x-1}\right)}-\sqrt{\left(\frac{2x-1}{2x+1}\right)}\right\}$.

(11.) $\sqrt{(x^2+6x+1)}-\sqrt{(x^2+6x+4)}+\sqrt{(x^2+6x-3)}=0$.

(12.) $\sqrt{(a^2+bx)}+\sqrt{(b^2+ax)}=3(a+b)$.

(13.) $\sqrt{\{a(bx-a^2)/b\}}+\sqrt{\{b(ax-b^2)/a\}}=a-b$.

(14.) $\sqrt{(a+x)}+\sqrt{(b+x)}=2\sqrt{(a+b+x)}$.

Consider more especially the case where $a=b$.

(15.) $\sqrt{(x+4)}-\sqrt{(x-4)}=\sqrt{(x-1)}$.

(16.) $2x\sqrt{(x^2+a^2)}+2x\sqrt{(x^2+b^2)}=a^2-b^2$.

(17.) $\sqrt{(x^2+4x+3)}-\sqrt{(x^2+3x+2)}=2(x+1)$.

Two solutions, $x=-1$ and another.

(18.) $x^2+a^2+\sqrt{(x^4+a^4)}=2x\sqrt{\{x^2+\sqrt{(x^4+a^4)}\}}$.

(19.) $x=\sqrt{\{ax+x^2-a\sqrt{(ax+x^2)}\}}$.

(20.) $\dfrac{7}{\sqrt{(x-6)}+4}+\dfrac{12}{\sqrt{(x-6)}+9}+\dfrac{1}{\sqrt{(x-6)}-4}+\dfrac{6}{\sqrt{(x-6)}-9}=0$.

(21.) $\sqrt{(a^2+x^2)}+\sqrt{(2ax)}=\sqrt{(a^2+3ax)}+\sqrt{(x^2+3ax)}$.

(22.) $\dfrac{1}{\sqrt{(a+x)}-\sqrt{a}}+\dfrac{1}{\sqrt{a}+\sqrt{(a+x)}}=\dfrac{m}{\sqrt{(a+x)}-\sqrt{(a-x)}}$.

(23.) $\sqrt{a}+\sqrt{(a+x)}-\sqrt{(a-x)}=\sqrt[4]{(a^2-x^2)}$.

(24.) $m\sqrt{(a+x)}+n\sqrt{(a-x)}=\sqrt{(m^2+n^2)}\sqrt[4]{(a^2-x^2)}$.

(25.) Rationalise and solve $\Sigma\sqrt{(x-b-c)}=\sqrt{x}$.

(26.) $\sqrt{\{(x^2+a^2)(x^2+b^2)\}}+x\{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}\}=nb^2+x^2$.

(27.) $a+(x+b)\sqrt{\{(x^2+a^2)/(x^2+b^2)\}}=b+(x+a)\sqrt{\{(x^2+b^2)/(x^2+a^2)\}}$.

§ 7.] *Reduction of Equations by change of Variable.* If we have an equation which is reducible to the form

$$\{f(x)\}^2+p\{f(x)\}+q=0 \qquad (\alpha),$$

then, if we put $\xi=f(x)$, we have the quadratic equation

$$\xi^2+p\xi+q=0 \qquad (\beta)$$

to determine ξ. Solving (β), we obtain for ξ the values $\{-p\pm\sqrt{(p^2-4q)}\}/2$. Hence (α) is equivalent to

$$\left.\begin{aligned} f(x)&=\frac{-p+\sqrt{(p^2-4q)}}{2} \\ f(x)&=\frac{-p-\sqrt{(p^2-4q)}}{2} \end{aligned}\right\} \qquad (\gamma).$$

If the function $f(x)$ be of the 1st or 2nd degree in x, the equations (γ) can be solved at once; and all the roots obtained will be roots of (α).

Even when the equations (γ) are not, as they stand, linear or quadratic equations, it may happen that they are reducible to such, or that solutions can in some way be obtained, and thus one or more solutions will be found for the original equation (α).

In practice it is unnecessary to actually introduce the auxiliary variable ξ. We should simply speak of (α) as a quadratic in $f(x)$, and proceed to solve for $f(x)$ accordingly.

Example 1.

$$x^{2p/q}+4x^{p/q}-12=0.$$

We may write this equation in the form

$$(x^{p/q})^2+4(x^{p/q})-12=0.$$

It may therefore be regarded as a quadratic equation in $x^{p/q}$. Solving, we find

$$x^{p/q} = +2, \quad x^{p/q} = -6.$$

From the first of these we have

$$x^p = 2^q.$$

Hence, if $1, \omega, \omega^2, \ldots, \omega^{p-1}$ be the pth roots of $+1$, we find the following p values for x:—

$$2^{q/p}, \quad \omega 2^{q/p}, \quad \omega^2 2^{q/p}, \quad \ldots, \quad \omega^{p-1} 2^{q/p}.$$

In like manner, from $x^{p/q} = -6$, we obtain, if q be even, the p values

$$6^{q/p}, \quad \omega 6^{q/p}, \quad \omega^2 6^{q/p}, \quad \ldots, \quad \omega^{p-1} 6^{p/q};$$

and, if q be odd, the p values

$$\omega' 6^{q/p}, \quad \omega'^3 6^{q/p}, \quad \omega'^5 6^{q/p}, \quad \ldots, \quad \omega'^{2p-1} 6^{q/p},$$

where $\omega', \omega'^3, \ldots, \omega'^{2p-1}$ are the pth roots of -1.

Example 2.

$$x^2 + 3 = 2\sqrt{(x^2 - 2x + 2)} + 2x.$$

This equation may be written

$$x^2 - 2x + 2 - 2\sqrt{(x^2 - 2x + 2)} + 1 = 0;$$

that is,

$$\{\sqrt{(x^2 - 2x + 2)}\}^2 - 2\{\sqrt{(x^2 - 2x + 2)}\} + 1 = 0,$$

which is a quadratic in $\sqrt{(x^2 - 2x + 2)}$.

Solving this quadratic we have

$$\sqrt{(x^2 - 2x + 2)} = 1.$$

Whence

$$x^2 - 2x + 2 = 1,$$

that is,

$$(x - 1)^2 = 0.$$

The roots of this last equation are 1, 1, and $x = 1$ satisfies the original equation.

Example 3.

$$2^{2x} - 3 \cdot 2^{x+2} + 32 = 0.$$

We may write this equation as follows,

$$(2^x)^2 - 12(2^x) + 32 = 0;$$

that is,

$$(2^x - 4)(2^x - 8) = 0.$$

Hence the given equation is equivalent to

$$\left\{\begin{array}{l} 2^x = 4 \\ 2^x = 8 \end{array}\right\}.$$

The first of these has for one real solution $x = 2$; the second has the real solution $x = 3$.

Example 4.

$$(x + a)(x + a + b)(x + a + 2b)(x + a + 3b) = c^4.$$

Associating the two extreme and the two intermediate factors on the left, we may write this equation as follows,

$$\{x^2 + (2a + 3b)x + a(a + 3b)\}\{x^2 + (2a + 3b)x + (a + b)(a + 2b)\} = c^4.$$

If $\xi=x^2+(2a+3b)x+(a^2+3ab)$, the last equation may be written

$$\xi(\xi+2b^2)=c^4;$$

that is,
$$\xi^2+2b^2\xi+b^4=b^4+c^4.$$
Hence
$$\xi=-b^2\pm\sqrt{(b^4+c^4)}.$$

The original equation is therefore equivalent to the two quadratics

$$x^2+(2a+3b)x+a^2+3ab+b^2=\pm\sqrt{(b^4+c^4)}.$$

§ 8.] *Reciprocal Equations.*—A very important class of equations of the 4th degree (biquadratics) can be reduced to quadratics by the method we are now illustrating.

Consider the equations

$$ax^4+bx^3+cx^2+bx+a=0 \qquad (1),$$
$$ax^4+bx^3+cx^2-bx+a=0 \qquad (\text{I.}),$$

where the coefficients equidistant from the ends are either equal, or, in the case of the second and fourth coefficients, equal or numerically equal with opposite signs. Such equations are called *reciprocal.**

If we divide by x^2, we reduce (1) and (I.) to the forms

$$a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0 \qquad (2),$$
$$a\left(x^2+\frac{1}{x^2}\right)+b\left(x-\frac{1}{x}\right)+c=0 \qquad (\text{II.})$$

These are equivalent to

$$a\left(x+\frac{1}{x}\right)^2+b\left(x+\frac{1}{x}\right)+c-2a=0 \qquad (3),$$
$$a\left(x-\frac{1}{x}\right)^2+b\left(x-\frac{1}{x}\right)+c+2a=0 \qquad (\text{III.})$$

3 and III. are quadratics in $x+1/x$ and $x-1/x$ respectively. If their roots be α, β, and γ, δ respectively, then (3) is equivalent to

$$\left\{\begin{array}{l} x+\dfrac{1}{x}=\alpha \\ x+\dfrac{1}{x}=\beta \end{array}\right\};$$

* If in equation (1) we write $1/\xi$ for x, we get an equation which is equivalent to $a\xi^4+b\xi^3+c\xi^2+b\xi+a=0$. Hence, if ξ be any root of (1), $1/\xi$ is also a root. In other words, two of the four roots of (1) are the reciprocals of the remaining two. In like manner it may be shown that two of the roots of (I.) are the reciprocals of the remaining two with the sign changed.

that is, to

$$\left.\begin{array}{l} x^2 - \alpha x + 1 = 0 \\ x^2 - \beta x + 1 = 0 \end{array}\right\} \quad (4).$$

Similarly, III. is equivalent to

$$\left.\begin{array}{l} x^2 - \gamma x - 1 = 0 \\ x^2 - \delta x - 1 = 0 \end{array}\right\} \quad (IV.)$$

The four roots of the two quadratics (4) or (IV.) are the roots of the biquadratic (1) or (I.)

Generalisation of the Reciprocal Equation.—If we treat the general biquadratic

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

in the same way as we treated equations (1) and (I.), we reduce it to the form

$$a\left(x^2 + \frac{e}{ax^2}\right) + b\left(x + \frac{d}{bx}\right) + c = 0.$$

Now, if $e/a = d^2/b^2$, this last equation may be written

$$a\left(x + \frac{d}{bx}\right)^2 + b\left(x + \frac{d}{bx}\right) + c - 2\frac{ad}{b} = 0,$$

which is a quadratic in $x + d/bx$.

Cor. *It should be noticed that the following reciprocal equations of the 5th degree can be reduced to reciprocal biquadratics, and can therefore be solved by means of quadratics, namely,*

$$ax^5 + bx^4 + cx^3 \pm cx^2 \pm bx \pm a = 0,$$

where, in the ambiguities, the upper signs go together and the lower signs together.

For the above may be written

$$a(x^5 \pm 1) + bx(x^3 \pm 1) + cx^2(x \pm 1) = 0,$$

from which it appears that either $x + 1$ or $x - 1$ is a factor on the left-hand side. After this factor is removed, the equation becomes a reciprocal biquadratic, which may be solved in the manner already explained. The roots of the quintic are either $+1$ or -1, and the four roots of this biquadratic.

In an appendix to this volume is given a discussion of the general solution of the cubic and biquadratic, and of the cases where they can be solved by means of quadratics.

Example 1.

To find the fifth roots of $+1$. Let x be any fifth root of $+1$; then $x^5=1$. Hence we have to solve the equation

$$x^5-1=0.$$

This is equivalent to

$$\left\{\begin{array}{r} x-1=0 \\ x^4+x^3+x^2+x+1=0 \end{array}\right\}.$$

The latter equation is a reciprocal biquadratic, and may be written

$$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-1=0.$$

After solving this equation for $x+1/x$, we find

$$x+\frac{1}{x}=-\frac{1+\sqrt{5}}{2}, \qquad x+\frac{1}{x}=-\frac{1-\sqrt{5}}{2}.$$

These give the two quadratics

$$x^2+\frac{1+\sqrt{5}}{2}x+1=0, \qquad x^2+\frac{1-\sqrt{5}}{2}x+1=0.$$

These again give the following four values for x:—

$$-(1+\sqrt{5})/4\pm i\sqrt{(10-2\sqrt{5})/4}, \quad -(1-\sqrt{5})/4\pm i\sqrt{(10+2\sqrt{5})/4},$$

these, together with 1, are the five fifth roots of $+1$. This will be found to agree with the result obtained by using chap. xii., § 19.

Example 2.

$$(x+a)^4+(x+b)^4=17(a-b)^4.$$

This equation may be written

$$(x+a)^4+(x+b)^4=17\{(x+a)-(x+b)\}^4,$$

from which, by dividing by $(x+b)^4$, we deduce

$$\left(\frac{x+a}{x+b}\right)^4+1=17\left\{\frac{x+a}{x+b}-1\right\}^4,$$

or

$$\xi^4+1=17(\xi-1)^4,$$

where

$$\xi=(x+a)/(x+b).$$

This equation in ξ is reciprocal, and may be written thus—

$$\left(\xi+\frac{1}{\xi}\right)^2-\frac{17}{4}\left(\xi+\frac{1}{\xi}\right)+\frac{35}{8}=0.$$

Hence

$$\xi+\frac{1}{\xi}=\frac{5}{2},$$

or

$$\xi+\frac{1}{\xi}=\frac{7}{4}.$$

From this last pair we deduce

$$\xi=2, \text{ or } \tfrac{1}{2}\text{; and } \xi=\frac{7\pm i\sqrt{(15)}}{8}.$$

Hence we have the four equations

$$\frac{x+a}{x+b}=2, \quad \frac{x+a}{x+b}=\frac{1}{2}, \quad \frac{x+a}{x+b}=\frac{7\pm i\sqrt{(15)}}{8}.$$

From these, four values of x can at once be deduced. The real values are $x=a-2b$ and $x=b-2a$.

§ 9.] *By introducing auxiliary variables, we can always make any* IRRATIONAL *equation in one variable depend on a system of* RATIONAL *equations in one or more variables.* For example, if we have

$$\sqrt{(x+a)}+\sqrt{(x+b)}+\sqrt{(x+c)}=d,$$

and we put $u=\sqrt{(x+a)}$, $v=\sqrt{(x+b)}$, $w=\sqrt{(x+c)}$, then we deduce the rational system

$$u+v+w=d, \quad u^2=x+a, \quad v^2=x+b, \quad w^2=x+c.$$

Whether such a transformation will facilitate the solution depends on the special circumstances of any particular case. The following is an example of the success of the artifice in question.

Example.

$$(a+x)^{\frac{1}{4}}+(a-x)^{\frac{1}{4}}=b.$$

We may write the given equation thus—

$$(a+x)^{\frac{1}{4}}+(a-x)^{\frac{1}{4}}=\frac{b}{(2a)^{\frac{1}{4}}}\{(a+x)+(a-x)\}^{\frac{1}{4}}.$$

Hence we deduce

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{4}}+1=\frac{b}{(2a)^{\frac{1}{4}}}\left\{\frac{a+x}{a-x}+1\right\}^{\frac{1}{4}}.$$

Let now

$$y=\{(a+x)/(a-x)\}^{\frac{1}{4}},$$

we then have

$$y+1=\frac{b}{(2a)^{\frac{1}{4}}}(y^4+1)^{\frac{1}{4}}.$$

From the last equation we deduce

$$2a(y+1)^4=b^4(y^4+1),$$

which is a reciprocal biquadratic, and can therefore be solved by means of quadratics. Having thus determined y, we deduce the value of x by means of the equation $(a+x)/(a-x)=y^4$.

Exercises XXXII.

(1.) $x^{2m}-x^m(b^m+c^m)+b^mc^m=0$.

(2.) $e^{3x/2}+qe^{-3x/2}=p$; show that the sum of the two real values of x is $\log_e q^{\frac{2}{3}}$.

(3.) $2x^{(p+q)/2pq}=\frac{a^2-b^2}{a^2+b^2}(x^{1/p}+x^{1/q})$.

(4.) $(9^x)^x - 2(3^x)^x 3^{x+1} = 3^{2x+3}$.

(5.) $3^{x+1} - \dfrac{15}{3^{x-1}} + 3^{x-2} - \dfrac{47}{3^{x-2}} = 0$.

(6.) $(x+\lambda) + 1/(x+\lambda) = \mu$.

(7.) $(1 - x + x^2)/(1 + a^2 - x) = (1 + a^2 + x)/(1 + x + x^2)$.

(8.) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$. (9.) $6x^4 - 31x^3 + 51x^2 - 31x + 6 = 0$.

(10.) $2x^4 - 7x^3 + 7x^2 - 7x + 2 = 0$. (11.) $8x^4 - 42x^3 + 29x^2 + 42x + 8 = 0$.

(12.) $ax^3 + bx^2 + bx + a = 0$. (13.) $ax^3 + bx^2 - bx - a = 0$.

(14.) $ax^4 + bx^2 + c = 0$. (15.) $ax^4 + bx^3 - bx - a = 0$.

(16.) $a^2x^4 + 2abx^3 + b^2x^2 - c^2 = 0$. (17.) $x^5 + 1 = 0$.

(18.) $x^5 + 7x^4 + 9x^3 - 9x^2 - 7x - 1 = 0$.

(19.) $12x^5 + x^4 + 13x^3 - 13x^2 - x - 12 = 0$.

(20.) Show that the biquadratic $ax^4 + bx^3 + cx^2 + dx + e = 0$ can be solved by means of quadratics, provided $b/2a = 4ad/(4ac - b^2)$.

(21.) $x^4 + 10x^3 + 22x^2 - 15x + 2 = 0$.

(22.) $x^4 + 2(p-q)x^3 + (p^2+q^2)x^2 + 2pq(p-q)x + pq(p^2+pq+q^2) = 0$.

(23.) $3/(x^2 - 7x + 3) - 2/(x^2 + 7x + 2) = 5$.

(24.) $x^4\left(1 + \dfrac{1}{3x}\right)^2 - (3x^2 + x) = 70$. (25.) $\sqrt{(1 - x^2)} = 1 + \sqrt{(1 + x^2)}/x$.

(26.) $\sqrt{(x^2+1)} + 4 = 5/\sqrt{(x^2+1)}$. (27.) $(x+5)^{\frac{1}{2}} + (x+5)^{-\frac{1}{2}} = 2$.

(28.) $\left\{\left(\dfrac{x-a}{x+a}\right)^{\frac{1}{2}} - b\right\}(x-a)^{\frac{1}{2}} = \left\{\left(\dfrac{x+a}{x-a}\right)^{\frac{1}{2}} - b\right\}(x+a)^{\frac{1}{2}}$.

(29.) $2x^2 + 2\sqrt{(x^2 + 4x - 5)} = 4x^2 + 8x + 5$.

(30.) $x^2 + 7x - 3 = \sqrt{(2x^2 + 14x + 2)}$.

(31.) $(x-7)^{\frac{1}{2}} + (x+9)^{\frac{1}{2}} + 2(x^2 + 2x - 63)^{\frac{1}{2}} = 70 - 2x$.

(32.) $\sqrt{(x^2 + px + a)} + \sqrt{(x^2 + px + b)} + \sqrt{(x^2 + px + c)} = 0$.

(33.) Show that the imaginary 7th roots of $+1$ are the roots of $x^2 - \alpha x + 1 = 0$, $x^2 - \beta x + 1 = 0$, $x^2 - \gamma x + 1 = 0$, where α, β, γ are the roots of the cubic $x^3 + x^2 - 2x - 1 = 0$.

(34.) $x^4 + \frac{1}{4} = x\sqrt{2}\sqrt{(x^4 - \frac{1}{4})}$. (35.) $5(1 + x^5)/(1 - x^5) = \{(1+x)/(1-x)\}^3$.

(36.) $(a-x)^5 + (x-b)^5 = (a-b)^5$. (37.) $\sqrt[4]{x} + \sqrt[4]{(x-1)} = \sqrt[4]{(x+1)}$.

(38.) $(x+3)(x+8)(x+13)(x+18) = 51$.

SYSTEMS WITH MORE THAN ONE VARIABLE WHICH CAN BE SOLVED BY MEANS OF QUADRATICS.

§ 10.] According to the rule stated without proof in chap. xiv., § 6, if we have a system of two equations of the lth and mth degrees respectively in two variables, x and y, that system has in general lm solutions. Hence, if we eliminate y and deduce from the given system an equation in x alone, that equation will in general be of the lmth degree, since there must in

general be as many different values of x as there are solutions of the original systems. We shall speak of this equation as the *Resultant Equation* in x.

In like manner, if we have a system of three equations of the lth, mth, and nth degrees respectively, in three variables x, y, z, the system has in general lmn solutions; and the resultant equation in x obtained by eliminating y and z will be of the lmnth degree; and so on.

From this it appears that *the only perfectly general case in which the solution of a system of equations will depend on a quadratic equation is that in which all the equations but one are of the 1st degree, and that one is of the 2nd.*

It is quite easy to obtain the solution in this case, and thus verify in a particular instance the general rule from which we have been arguing. All we have to do is to solve the $n-1$ linear equations, and thereby determine $n-1$ of the variables as linear functions of the nth variable. On substituting these values in the nth equation, which we suppose of the 2nd degree in all the n variables, it becomes an equation of the 2nd degree in the nth variable. We thus obtain two values of the nth variable, and hence two corresponding values for each of the other $n-1$ variables; that is to say, we obtain two solutions of the system.

Example 1.

$$lx+my+n=0 \qquad (1),$$

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad (2).$$

(1) is equivalent to

$$y=-\frac{lx+n}{m} \qquad (3);$$

and this value of y reduces (2) to

$$am^2x^2-2hmx(lx+n)+b(lx+n)^2+2gm^2x-2fm(lx+n)+cm^2=0,$$

that is,

$$(am^2-2hlm+bl^2)x^2+2(gm^2-hmn+bnl-flm)x+(bn^2-2fmn+cm^2)=0 \quad (4).$$

The original system (1), (2) is therefore equivalent to (3), (4). Now (4) gives two values for x, and for each of these (3) gives a corresponding value of y.

For example, the two equations

$$3x+2y+1=0, \quad x^2+2xy+y^2-x+y+3=0,$$

will be found to be equivalent to

$$y=-\tfrac{3}{2}x-\tfrac{1}{2}, \quad x^2-8x+11=0.$$

Hence the two solutions of the system are

$$x = 4 + \sqrt{5}, \qquad 4 - \sqrt{5};$$
$$y = -\tfrac{13}{2} - \tfrac{3}{2}\sqrt{5}, \qquad -\tfrac{13}{2} + \tfrac{3}{2}\sqrt{5}.$$

Example 2.

$$3x + 2y - z = 1, \quad x + y - 3z = 2, \quad x^2 + y^2 + z^2 = 1.$$

The system is equivalent to

$$x = -5z - 3, \quad y = 8z + 5, \quad 90z^2 + 110z + 33 = 0.$$

The solutions are

$$x = \tfrac{1}{18} \mp \tfrac{5}{18}\sqrt{\tfrac{11}{5}}, \quad y = \tfrac{1}{9} \pm \tfrac{4}{9}\sqrt{\tfrac{11}{5}}, \quad z = -\tfrac{11}{18} \pm \tfrac{1}{18}\sqrt{\tfrac{11}{5}},$$

the upper signs going together and the lower together.

§ 11.] For the sake of contrast with the case last considered, and as an illustration of an important method in elimination, let us consider the most general system of two equations of the 2nd degree in two variables, namely—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1),$$
$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \qquad (2).$$

We may write these equations in the forms—

$$by^2 + 2(hx + f)y + (ax^2 + 2gx + c) = 0,$$
$$b'y^2 + 2(h'x + f')y + (a'x^2 + 2g'x + c') = 0,$$

say

$$by^2 + py + q = 0 \qquad (1'),$$
$$b'y^2 + p'y + q' = 0 \qquad (2'),$$

where $p = 2(hx + f)$, $q = ax^2 + 2gx + c$, &c.

If we multiply $(1')$ and $(2')$ by b' and by b respectively, and subtract, and also multiply them by q' and by q respectively, and subtract, we deduce

$$(pb' - p'b)y + (b'q - bq') = 0 \qquad (3),$$
$$(b'q - bq')y^2 + (p'q - pq')y = 0 \qquad (4);$$

and provided $bq' - b'q \neq 0$, (3) and (4) will be equivalent to $(1')$ and $(2')$. In general, the values of x which make $bq' - b'q = 0$ will not belong to the solutions of (3) and (4), nor will the value $y = 0$ belong to those solutions. Hence we may say that, in general, the system

$$(pb' - p'b)y + (b'q - bq') = 0 \qquad (3'),$$
$$(b'q - bq')y + (p'q - pq') = 0 \qquad (4'),$$

is equivalent to $(1')$ and $(2')$.

Again, if we multiply $(3')$ and $(4')$ by $b'q - bq'$ and by $pb' - p'b$ respectively, and subtract, we deduce

$$(b'q - bq')^2 - (pb' - p'b)(p'q - pq') = 0 \qquad (5),$$

and, provided $bq' - b'q \neq 0$, (4′) and (5) will be equivalent to (3′) and (4′).

Hence we finally arrive at the conclusion that, in general, the system

$$\begin{aligned}\{b'(ax^2 + 2gx + c) - b(a'x^2 + 2g'x + c')\}^2 - 4\{b'(hx + f)\\ - b(h'x + f')\}\{(h'x + f')(ax^2 + 2gx + c) - (hx + f)(a'x^2 + 2g'x + c')\}\\ = 0 \quad (6),\end{aligned}$$

$$\begin{aligned}\{b'(ax^2 + 2gx + c) - b(a'x^2 + 2g'x + c')\}y\\ + 2\{(h'x + f')(ax^2 + 2gx + c) - (hx + f)(a'x^2 + 2g'x + c')\}\\ = 0 \quad (7),\end{aligned}$$

is equivalent to (1) and (2).

The first of these is a biquadratic giving four values for x, and, since (7) is of the 1st degree in y, for each value of x we obtain one and only one value of y. We have therefore four solutions, as the general rule requires.

In general, the resultant biquadratic (6) will not be reducible to quadratics. It may, however, happen to be so reducible in particular cases. The following are a few of the more important:—

I. If, for example, $b'/b = f'/f = c'/c$, then (6) reduces to

$$\begin{aligned}x^2[\{b'(ax + 2g) - b(a'x + 2g')\}^2 - 4(b'h - bh')\{(h'x + f')(ax + 2g)\\ - (hx + f)(a'x + 2g') + (h'c - hc')\}] = 0,\end{aligned}$$

two of whose roots are zero, the other two being determinable by means of a quadratic equation.

II. Again, if $a'/a = b'/b = h'/h$, it will be found that the two highest terms disappear from (6). Hence in this case two of its roots become infinite (see chap. xviii., § 6), and the remaining two can be found by means of a quadratic equation.

III. If $f = 0$, $g = 0$, $f' = 0$, $g' = 0$, it will be found that only even powers of x occur in (6). The resultant then becomes a quadratic in x^2.

IV. The resultant biquadratic may come under the reciprocal class discussed in § 8 above.

Most of these exceptional cases are of interest in the theory of conics, because they relate to cases where the intersection of two conics can be constructed by means of the ruler and compasses alone. The general theory is given in the Appendix to this vol.

Example 1.

The system $3x^2+2xy+y^2=17, \quad x^2-2xy+5y^2=5,$

is equivalent to $12xy+14x^2-80=0, \quad 73x^4-692x^2+1600=0.$

The solutions of which are

$$x=+2, \quad -2, \quad +20/\sqrt{(73)}, \quad -20/\sqrt{(73)};$$
$$y=+1, \quad -1, \quad +1/\sqrt{(73)}, \quad -1/\sqrt{(73)}.$$

Example 2.

$$n(x^2+2y)=l(1+2xy), \quad n(y^2+2x)=m(1+2xy).$$

Here the elimination is easy, because the first equation is of the 1st degree in y. We deduce from it

$$y=\frac{nx^2-l}{2(lx-n)}.$$

This reduces the second equation to

$$n(nx^2-l)^2+8nx(lx-n)^2=4m(lx-n)^2+4mx(lx-n)(nx^2-l),$$

which is equivalent to

$$(n^2-4lm)x^4+4(2l^2+mn)x^3-18nlx^2+4(2n^2+lm)x+(l^2-4mn)=0.$$

If $n=l$, this biquadratic is reciprocal, and its solution depends upon

$$(l-4m)\xi^2+4(2l+m)\xi+(8m-20l)=0,$$

where $\xi=x+1/x$.

In general, if we have an equation of the 1st degree in x and y together with an equation of the nth degree in x and y, the resultant equation in x will be of the nth degree. In particular cases, owing to the existence of zero or infinite roots, or for other special reasons, this equation may be reducible to quadratics.

Example.

$$x+y=18, \quad x^3+y^3=4914,$$

is equivalent to

$$y=18-x, \quad x^3+(18-x)^3=4914.$$

The second of these two last equations reduces, as it happens, to

$$x^2-18x+17=0.$$

Hence the finite solutions of the given system are

$$x=17, \ 1;$$
$$y=1, \ 17.$$

§ 12.] A very important class of equations are the so-called *Homogeneous Systems.* The kind that most commonly occurs is that in which each equation consists of a homogeneous function of the variables equated to a constant. The artifice usually employed for solving such equations is to introduce as auxiliary variables the ratios of all but one of the variables to that one. Thus, for example, if the variables were x and y, we should put $y=vx$, and then treat v and x as the new variables.

Example 1.

$$x^2+xy=12, \quad xy-2y^2=1.$$

Put $y=vx$, and the two equations become

$$x^2(1+v)=12, \quad x^2(v-2v^2)=1.$$

From these two we derive

$$x^2(1+v)-12x^2(v-2v^2)=0,$$

that is,

$$x^2\{24v^2-11v+1\}=0.$$

Since $x=0$ evidently affords no solution of the given system, we see that the original system is equivalent to

$$x^2(1+v)=12, \quad 24v^2-11v+1=0.$$

Solving the quadratic for v, we find $v=1/3$ or $1/8$.

Corresponding to $v=1/3$, the first of the last pair of equations gives $x^2=9$, that is, $x=\pm 3$.

Corresponding to $v=1/8$, we find in like manner $x=\pm 4\sqrt{(2/3)}$.

Hence, bearing in mind that y is derived from the corresponding value of x by using the corresponding value of v in the equation $y=vx$, we have, for the complete set of solutions,

$$x=+3, \quad -3, \quad +4\sqrt{(2/3)}, \quad -4\sqrt{(2/3)};$$
$$y=+1, \quad -1, \quad +1/\sqrt{6}, \quad -1/\sqrt{6}.$$

Example 2.

$$x^2+2yz=l, \quad y^2+2zx=m, \quad z^2+2xy=n.$$

Let $x=uz$, $y=vz$, then the equations become

$$(u^2+2v)z^2=l, \quad (v^2+2u)z^2=m, \quad (1+2uv)z^2=n.$$

Eliminating z, we have, since $z=0$ forms in general no part of any solution,

$$n(u^2+2v)=l(1+2uv), \quad n(v^2+2u)=m(1+2uv).$$

We have already seen how to treat this pair of equations (see § 11, Example 2). The system has in general four different solutions, which can be obtained by solving a biquadratic equation (reducible to quadratics when $n=l$).

If we take any one of these solutions, the equation $(1+2uv)z^2=n$ gives two values of z. The relations $x=uz$, $y=vz$, then give one value of x and one value of y corresponding to each of the two values of z.

We thus obtain all the eight solutions of the given system.

There is another class of equations in the solution of which the artifice just exemplified is sometimes successful, namely, that in which each equation consists of a homogeneous function of the variables equated to another homogeneous function of the variables of the same or of different degree.

Example 3.

The system

$$ax^2+bxy+cy^2=dx+ey, \quad a'x^2+b'xy+c'y^2=d'x+e'y \qquad (1)$$

is equivalent to

$$(a+bv+cv^2)x^2=(d+ev)x, \quad (a'+b'v+c'v^2)x^2=(d'+e'v)x \qquad (2)$$

where $y=vx$.

From this last system we derive the system

$$\left.\begin{aligned} x^2\{(a+bv+cv^2)(d'+e'v)-(a'+b'v+c'v^2)(d+ev)\}&=0 \\ x\{(a+bv+cv^2)x-(d+ev)\}&=0 \end{aligned}\right\} \quad (3),$$

which is equivalent (see chap. xiv., § 11) to (2), along with

$$(a+bv+cv^2)x^2=0 \quad (4),$$

$$(d+ev)x=0 \quad (5).$$

If we observe that $x=0$, $y=0$ is a solution of the system (1), and keep account of it separately, and observe further that values of v which satisfy both (4) and (5) do not in general exist, we see that the system (1) is equivalent to

$$(a+bv+cv^2)(d'+e'v)-(a'+b'v+c'v^2)(d+ev)=0 \quad (6)$$

along with $$(a+bv+cv^2)x-(d+ev)=0 \quad (7)$$

and $$x=0, \quad y=0.$$

The solution of the given system now depends on the cubic (6). The three roots of this cubic substituted in (7) give us three values of x, and $y=vx$ gives three corresponding solutions of (1). Thus, counting $x=0$, $y=0$, we have obtained all the four solutions of (1).

The cubic (6) will not be reducible to quadratics except in particular cases, as, for example, when $ad'-a'd=0$ or $ce'-c'e=0$.

For example, the system

$$3x^2-2xy+3y^2=x+12y, \quad 6x^2+3xy-2y^2=2x+29y,$$

is equivalent to $x=0$, $y=0$, together with

$$v(111v^2-86v+8)=0, \quad (3-2v+3v^2)x=1+12v.$$

The values of v are 2/3, 4/37, and 0. Hence the solutions of the system are

$$x=0, \quad 3, \quad 185/227, \quad 1/3;$$
$$y=0, \quad 2, \quad 20/227, \quad 0.$$

§ 13.] *Symmetrical Systems.*—A system of equations is said to be symmetrical when the interchange of any pair of the variables derives from the given system an identical system. For example,

$$x+y=a, \quad x^2+y^2=b; \qquad x^3+y=a, \quad y^3+x=a;$$
$$x+y+z=a, \quad x^2+y^2+z^2=b, \quad yz+zx+xy=c,$$

are all symmetrical systems.

There is a peculiarity in the solutions of such systems, which can be foreseen from their nature. Let us suppose in the first place that the system is such that it would in general have an even number of solutions, four say. If we take half the solutions, say

$$x=\alpha_1, \ \alpha_2,$$
$$y=\beta_1, \ \beta_2,$$

then, since the equations are still satisfied when the values of x and y are interchanged, the remaining half of the solutions are

$$x=\beta_1, \ \beta_2,$$
$$y=\alpha_1, \ \alpha_2.$$

If the whole number of solutions were odd, five say, then four of the solutions would be arranged as above, and the fifth (if finite, which in many cases it would not be) must be such that the values of x and y are equal; otherwise the interchanges of the two would produce a sixth solution, which is inadmissible, if the system have only five solutions.*

These considerations suggest two methods of solving such equations.

1*st Method.*—Replace the variables by a new system of variables, consisting of one, say x, of the former, and the ratios to it of the others, u, v, . . . say. Eliminate x, v, . . . and obtain an equation in u alone; then this equation will be a reciprocal equation; for the values of u are

$$u = \frac{\alpha_1}{\beta_1}, \quad \frac{\beta_1}{\alpha_1}, \quad \frac{\alpha_2}{\beta_2}, \quad \frac{\beta_2}{\alpha_2}, \text{ \&c. (and, it may be, } u = 1),$$

that is to say, along with each root there is another, which is its reciprocal. The degree of this resultant equation can therefore in all cases be reduced by adjoining a certain quadratic, just as in the case of a reciprocal biquadratic.

2*nd Method.*—Replace the variables x, y, z, . . . by an equal number of symmetric functions of x, y, z, . . ., say by Σx, Σxy, Σxyz, . . ., &c., and solve for these.

The nature of the method, its details, and the reason of its success, will be best understood by taking the case of two variables, x and y.

Let us put $u = x + y$, $v = xy$. After separating the solutions, if any, for which $x = y$, we may replace the given system by a system each equation of which is symmetrical. We know, by the general theory of symmetric functions (see chap. xviii., § 4), that every integral symmetric function can be expressed as an

* We have supposed that for all the solutions (except one in the case of an odd system) $x \neq y$. It may, however, happen that $x = y$ for one or more solutions. Such solutions cannot be paired with others, since an interchange of values does not produce a new solution. This peculiarity must always arise in systems which are symmetrical as a whole, but not symmetrical in the individual equations. As an example, we may take the symmetrical system $x^3 + y = a$, $y^3 + x = a$, three of whose solutions are such that $x = y$.

integral function of u and v. Hence it will always be possible to transform the given system into an equivalent system in u and v.

We observe further that, in general, u and v will each have as many values as there are solutions of the given system, and no more; but that the values of u and v corresponding to two solutions, such as $x=\alpha_1$, $y=\beta_1$, and $x=\beta_1$, $y=\alpha_1$, are equal. Hence in the case of symmetrical equations the number of solutions of the system in u and v must in general be less than usual.

Corresponding to any particular values of u and v, say $u=\alpha$, $v=\beta$, we have the quadratic system $x+y=\alpha$, $xy=\beta$, which gives the two solutions

$$x=\{\alpha \pm \sqrt{(\alpha^2-4\beta)}\}/2, \quad y=\{\alpha \mp \sqrt{(\alpha^2-4\beta)}\}/2.$$

If we had a system in three variables, x, y, z, then we should assume $u=x+y+z$, $v=yz+zx+xy$, $w=xyz$, and attempt to solve the system in u, v, w. Let $u=\alpha$, $v=\beta$, $w=\gamma$, be any solution of this system; then, since

$$(\xi-x)(\xi-y)(\xi-z) \equiv \xi^3-u\xi^2+v\xi-w,$$

we see that the three roots of

$$\xi^3-\alpha\xi^2+\beta\xi-\gamma=0$$

constitute a solution of the original system, and, since the equations are symmetrical, any one of the six permutations of these roots is also a solution. In this case, therefore, the number of solutions of the system in u, v, w would, in general, be less than the corresponding number for the system in x, y, z.

The student should study the following examples in the light of these general remarks:—

Example 1.
$$\left.\begin{aligned} A(x^2+y^2)+Bxy+C(x+y)+D&=0 \\ A'(x^2+y^2)+B'xy+C'(x+y)+D'&=0 \end{aligned}\right\} \qquad (1).$$

If we put $y=vx$, and then eliminate x by the method employed in § 11, the resultant equation in v is

$$\{(D'A)+(D'B)v+(D'A)v^2\}^2=(D'C)(1+v)^2\{(C'A)+(C'B)v+(C'A)v^2\} \qquad (2),$$

where $(D'A)$ stands for $D'A-DA'$, $(D'B)$ for $D'B-DB'$, and so on.

The biquadratic (2) is obviously reciprocal, and can therefore be solved by means of quadratics.

The solution can then be completed by means of the equation

$$\{(D'A)+(D'B)v+(D'A)v^2\}x+(D'C)(1+v)=0 \qquad (3).$$

As an instance of this method the student should work out in full the solution of the system

$$2(x^2+y^2)-3xy+2(x+y)-39=0,$$
$$3(x^2+y^2)-4xy+\ \ (x+y)-50=0.$$

We may treat the above example by the second method of the present paragraph as follows. The system (1) may be written

$$A\,(x+y)^2+(B\ -2A\ )xy+C\,(x+y)+D\ =0,$$
$$A'(x+y)^2+(B'-2A')xy+C'(x+y)+D'=0\ ;$$

that is,

$$\left.\begin{aligned}Au^2\ +(B\ -2A\ )v+Cu\ +D\ =0\\ A'u^2+(B'-2A')v+C'u+D'=0\end{aligned}\right\} \qquad (4).$$

Eliminating first u^2 and then v, we deduce the equivalent system

$$\left.\begin{aligned}(A'B)v+(A'C)u+(A'D)=0\\ (A'B)u^2+\{(C'B)-2(C'A)\}u+\{(D'B)-2(D'A)\}=0\end{aligned}\right\} \qquad (5),$$

where (A'B), &c., have the same meaning as above.

The system (5) has two solutions,

$$u=\alpha,\ \alpha',$$
$$v=\beta,\ \beta',$$

say, corresponding to which we find for the original system

$$x=\{\alpha\pm\sqrt{(\alpha^2-4\beta)}\}/2,\quad \{\alpha'\pm\sqrt{(\alpha'^2-4\beta')}\}/2,$$
$$y=\{\alpha\mp\sqrt{(\alpha^2-4\beta)}\}/2,\quad \{\alpha'\mp\sqrt{(\alpha'^2-4\beta')}\}/2,$$

in all four solutions.

This method should be tested on the numerical example given above.

Example 2. $\qquad x^4+y^4=82,\quad x+y=4.$

We have

$$\begin{aligned}x^4+y^4&\equiv(x+y)^4-4xy(x^2+y^2)-6x^2y^2,\\ &\equiv(x+y)^4-4xy\{(x+y)^2-2xy\}-6x^2y^2,\\ &\equiv u^4-4u^2v+2v^2.\end{aligned}$$

Hence the given system is equivalent to

$$u^4-4u^2v+2v^2=82,\quad u=4.$$

Using the value of u given by the second equation, we reduce the first to

$$v^2-32v+87=0.$$

The roots of this quadratic are 3 and 29. Hence the solution of the u, v system is

$$u=4,\ 4,$$
$$v=3,\ 29.$$

From $x+y=4$; $xy=29$, we derive $(x-y)^2=-100$, that is, $x-y=\pm10i$; combining this with $x+y=4$, we have $x=2\pm5i$, $y=2\mp5i$.

From $x+y=4$, $xy=3$, we find $x=3$, $y=1$; $x=1$, $y=3$.

All the four solutions have thus been found.

Example 3. $\qquad x^4=mx+ny,\quad y^4=nx+my \qquad (1).$

Let us put $y=vx$; then, removing the factor x in both equations, and noting the corresponding solution, $x=0$, $y=0$, we have

$$x^3=m+nv,\quad v^4x^3=n+mv.$$

These are equivalent to

$$x^3=m+nv,\quad v^4(m+nv)=mv+n \qquad (2).$$

The second of these may be written

$$n(v^5-1)+mv(v^3-1)=0 \qquad (3),$$

and is therefore equivalent to

$$\left.\begin{aligned} v-1=0 \\ v^4+\left(\frac{m}{n}+1\right)v^3+\left(\frac{m}{n}+1\right)v^2+\left(\frac{m}{n}+1\right)v+1=0 \end{aligned}\right\}.$$

The second of these is a reciprocal biquadratic. Hence all the five roots of (3) can be found without solving any equation of higher degree than the 2nd.

To the root $v=1$ correspond the three solutions,

$$x=y=(m+n)^{1/3}, \quad \omega(m+n)^{1/3}, \quad \omega^2(m+n)^{1/3},$$

of the original system, where $(m+n)^{1/3}$ is the real value of the cube root, and ω, ω^2 are the imaginary cube roots of unity.

In like manner three solutions of (1) are obtained for each of the remaining four roots of (3). Hence, counting $x=0$, $y=0$, we obtain all the sixteen solutions of (1).

The reader should work out the details of the numerical case

$$x^4=2x+3y, \quad y^4=3x+2y,$$

and calculate all the real roots, and all the coefficients in the complex roots, to one or two places of decimals.

Example 4.

$$yz+zx+xy=26,$$
$$yz(y+z)+zx(z+x)+xy(x+y)=162,$$
$$yz(y^2+z^2)+zx(z^2+x^2)+xy(x^2+y^2)=538.$$

If we put $u=x+y+z$, $v=yz+zx+xy$, $w=xyz$, the above system reduces to

$$v=26, \quad uv-3w=162, \quad (u^2-2v)v-uw=538.$$

Hence $$26u-3w=162, \quad 26u^2-uw=1890.$$

Hence $$26u^2+81u-2835=0.$$

The roots of this quadratic are $u=9$ and $u=-315/26$.

We thus obtain for the values of u, v, w, 9, 26, 24, and $-315/26$, 26, -159. Hence we have the two cubics

$$\xi^3-9\xi^2+26\xi-24=0,$$
$$\xi^3+\frac{315}{26}\xi^2+26\xi+159=0.$$

Twelve of the roots of the original system consist of the six permutations of the three roots of the first cubic, together with the six permutations of the roots of the second cubic.

The first cubic evidently has the root $\xi=2$; and the other two are easily found to be 3 and 4. Hence we have the following six solutions :—

$$x=2,\ 2,\ 3,\ 3,\ 4,\ 4;$$
$$y=3,\ 4,\ 4,\ 2,\ 2,\ 3;$$
$$z=4,\ 3,\ 2,\ 4,\ 3,\ 2.$$

Other six are to be found by solving the second cubic.

§ 14.] We conclude this chapter with a few miscellaneous examples of artifices that are suggested merely by the peculi-

arities of the particular case. Some of them have a somewhat more general character, as the student will find in working the exercises in set XXXIV. A moderate amount of practice in solving puzzles of this description is useful as a means of cultivating manipulative skill; but he should beware of wasting his time over what is after all merely a chapter of accidents.

Example 1.

$$\frac{ax}{a+x}+\frac{by}{b+y}=\frac{(a+b)c}{a+b+c}, \quad x+y=c.$$

Let $$a+x=(a+b+c)\xi, \quad b+y=(a+b+c)\eta;$$
the system then reduces to

$$a^2/\xi+b^2/\eta=(a+b)^2, \quad \xi+\eta=1.$$

This again is equivalent to

$$\{(a+b)\xi-a\}^2=0, \quad \xi+\eta=1.$$

Hence we have the solution $\xi=a/(a+b)$, $\eta=b/(a+b)$ twice over.

The solutions of the original system are therefore $x=ac/(a+b)$, $y=bc/(a+b)$ twice over.

Example 2.

$$ax^2+bxy+cy^2=bx^2+cxy+ay^2=d \qquad (1).$$

This system is equivalent to

$$(a-b)x^2+(b-c)xy+(c-a)y^2=0 \qquad (2),$$
$$ax^2+bxy+cy^2=d \qquad (3).$$

The equation (2) (see chap. XVI., § 9) is equivalent to

$$x^2=(c\sigma+1)\rho, \quad xy=(a\sigma+1)\rho, \quad y^2=(b\sigma+1)\rho \qquad (4),$$

where ρ and σ are undetermined.

Since $x^2y^2=(xy)^2$, we must have

$$(c\sigma+1)(b\sigma+1)=(a\sigma+1)^2.$$

Hence we deduce $$\sigma=0, \quad \sigma=\frac{b+c-2a}{a^2-bc} \qquad (5).$$

The first of these, taken in conjunction with (4), gives $x=y$; and hence

$$x=y=\pm\sqrt{\frac{d}{a+b+c}};$$

that is to say, two solutions of (1). If we take the second value of σ we find

$$x^2=\frac{\rho(c-a)^2}{a^2-bc}, \quad xy=\frac{\rho(c-a)(a-b)}{a^2-bc}, \quad y^2=\frac{\rho(a-b)^2}{a^2-bc} \qquad (6),$$

where it remains to determine $\rho/(a^2-bc)$. This can be done by substituting in (3). We thus find

$$\rho/(a^2-bc)=d/(a^3+ac^2-ca^2+ab^2-a^2b-abc).$$

We now deduce from (6)

$$x=\frac{\pm(c-a)d^{1/2}}{(a^3+ac^2-ca^2+ab^2-a^2b-abc)^{1/2}}, \quad y=\pm, \text{ \&c.};$$

two more solutions of the original system.

The system (2), (3) could also be solved very simply by putting $y=vx$, as in § 12.

Example 3.

$$yz=a^2, \quad zx=b^2, \quad xy=c^2.$$

These equations give

$$\frac{zx\times xy}{yz}=\frac{b^2\times c^2}{a^2},$$

that is,

$$x^2=\frac{b^2c^2}{a^2}.$$

Hence $x=\pm bc/a$; the two last equations of the original system then give $y=\pm ca/b$, $z=\pm ab/c$. The upper signs go together and the lower together; so that we have only obtained two out of the possible eight solutions.

Example 4.

$$x(y+z)=a^2, \quad y(z+x)=b^2, \quad z(x+y)=c^2.$$

This can be reduced to last by solving for yz, zx, xy.

Example 5.

$$x(x+y+z)=a^2, \quad y(x+y+z)=b^2, \quad z(x+y+z)=c^2.$$

Let $x+y+z=\rho$. Then, if we add the three equations, we have

$$\rho^2=a^2+b^2+c^2.$$

Hence $\rho=\pm\sqrt{(a^2+b^2+c^2)}$; and we have

$$x=\frac{\pm a^2}{\sqrt{(a^2+b^2+c^2)}}, \quad y=\frac{\pm b^2}{\sqrt{(a^2+b^2+c^2)}}, \quad z=\frac{\pm c^2}{\sqrt{(a^2+b^2+c^2)}}.$$

Example 6.

To find the real solutions of

$$\begin{array}{llll} x^2+\eta^2+\zeta^2=a^2 & (1), & \xi(y+z)+\eta\zeta=bc & (4), \\ y^2+\zeta^2+\xi^2=b^2 & (2), & \eta(z+x)+\zeta\xi=ca & (5), \\ z^2+\xi^2+\eta^2=c^2 & (3), & \zeta(x+y)+\xi\eta=ab & (6). \end{array}$$

From (2), (3), and (4) we deduce

$$\{\xi(y+z)+\eta\zeta\}^2-\{y^2+\zeta^2+\xi^2\}\{z^2+\xi^2+\eta^2\}=0;$$

that is,

$$(\xi^2-yz)^2+(\xi\eta-\zeta z)^2+(\zeta\xi-\eta y)^2=0 \qquad (7).$$

Every solution of the given system must satisfy (7). Now, since $(\xi^2-yz)^2$, $(\xi\eta-z\zeta)^2$, $(\zeta\xi-\eta y)^2$ are all positive, provided x, y, z, ξ, η, ζ be all real, it follows that for all real solutions we must have $\xi^2=yz$, $\xi\eta=\zeta z$, $\zeta\xi=\eta y$.

Hence, from the symmetry of the system, we must have

$$\xi^2=yz, \quad \eta^2=zx, \quad \zeta^2=xy, \qquad (8),$$

$$x=\frac{\eta\zeta}{\xi}, \quad y=\frac{\zeta\xi}{\eta}, \quad z=\frac{\xi\eta}{\zeta} \qquad (9).$$

By means of (8) we reduce (1), (2), (3) to

$$x(x+y+z)=a^2, \quad y(x+y+z)=b^2, \quad z(x+y+z)=c^2.$$

Hence, by Example 5, we have

$$x=\frac{\pm a^2}{\sqrt{(a^2+b^2+c^2)}}, \quad y=\frac{\pm b^2}{\sqrt{(a^2+b^2+c^2)}}, \quad z=\frac{\pm c^2}{\sqrt{(a^2+b^2+c^2)}}.$$

From (8) we now derive

$$\xi=\frac{\pm bc}{\sqrt{(a^2+b^2+c^2)}}, \quad \eta=\frac{\pm ca}{\sqrt{(a^2+b^2+c^2)}}, \quad \zeta=\frac{\pm ab}{\sqrt{(a^2+b^2+c^2)}}.$$

If we take account of (4), (5), (6) we see that the upper signs must go together throughout, and the lower together throughout; so that we find only two real solutions.

Example 7.

$$x(x-a)=yz, \quad y(y-b)=zx, \quad z(z-c)=xy \qquad (1).$$

From the first two equations we derive $(x-y)(x+y+z)=ax-by$, which, if we put $\rho=x+y+z$, may be written $(\rho-a)x=(\rho-b)y$. Hence, bearing in mind the symmetry of the system, we have

$$x=\frac{\sigma}{\rho-a}, \quad y=\frac{\sigma}{\rho-b}, \quad z=\frac{\sigma}{\rho-c} \qquad (2),$$

where ρ and σ have to be determined.

From the first equation of (1) we have

$$\frac{\sigma}{\rho-a}\left(\frac{\sigma}{\rho-a}-a\right)=\frac{\sigma^2}{(\rho-b)(\rho-c)}.$$

Removing the factor σ, to which will correspond the solution $x=y=z=0$, we find

$$\sigma\{(2a-b-c)\rho+(bc-a^2)\}=a(\rho-a)(\rho-b)(\rho-c) \qquad (3).$$

Similarly we find

$$\sigma\{(2b-c-a)\rho+(ca-b^2)\}=b(\rho-a)(\rho-b)(\rho-c) \qquad (4).$$

From (3) and (4) we now eliminate σ, observing that in the process we reject the factors σ, $\rho-a$, $\rho-b$, $\rho-c$, which correspond to three solutions, namely,

$$\begin{array}{lll} x=a, & 0, & 0; \\ y=0, & b, & 0; \\ z=0, & 0, & c. \end{array}$$

We thus deduce

$$\rho=\frac{bc+ca+ab}{a+b+c},$$

which gives one more solution. We have in fact $\rho-a=(bc-a^2)/\Sigma a$, $\rho-b=(ca-b^2)/\Sigma a$, $\rho-c=(ab-c^2)/\Sigma a$.

Hence (2) gives

$$x=\frac{\sigma}{\rho-a}=\frac{(ca-b^2)(ab-c^2)}{3abc-\Sigma a^3},$$

and, by symmetry, we have two corresponding values for y and z.

This example is worthy of notice on account of the symmetrical method which is used for treating the given system of equations. The solution might be obtained fully as readily by putting $x=uz$, $y=vz$, and proceeding as in § 13, Example 3.

EXERCISES XXXIII.

(1.) $x+y=30, \quad xy=216.$

(2.) $x-y=3, \quad x^2+y^2=65.$

(3.) $x^2+y^2=58, \quad xy=21.$

(4.) $x+y=8, \quad 3x^2-2xy+y^2=54.$

(5.) $x+2y=x^2, \quad 2x+y=y^2.$

(6.) $x^2+y^2+2(x+y)=11, \quad 3xy=2(x+y).$

(7.) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}, \quad x+y=b.$

(8.) $a(x^2+y^2)=px-qy, \quad b(x^2+y^2)=qx-py.$

(9.) $(x+y)/(1+xy)=a, \quad (x-y)/(1-xy)=b.$

(10.) $ax+by=c, \quad b/x+a/y=d.$

(11.) If $ax+by=1$, $cx^2+dy^2=1$, have only one solution, then $a^2/c+b^2/d=1$, and the solution in question is $x=a/c$, $y=b/d$.

(12.) $2x^2-3xy=1, \quad y^2+5xy=34.$ (13.) $x^2+xy=84, \quad xy+y^2=60.$

(14.) $x^3+4xy+y^3=38, \quad x+y=2.$

(15.) $1/x^2+1/xy=1/a^2, \quad 1/y^2+1/xy=1/b^2.$

(16.) $(px+qy)(x/p+y/q)=x^2+y^2+p^2+q^2, \quad x/p+y/q=\sqrt{5}.$

(17.) $x^2+a^2=y^2+b^2=(x+y)^2+(a-b)^2.$

(18.) $(x-y)^2=a^3(x+y), \quad (x+y)^2=b^3(x-y).$

(19.) $(a^2-b^2)/(x^2+y^2)+(a^2+b^2)/(x^2-y^2)=1, \quad x^2/p^2-y^2/q^2=0.$

(20.) $\dfrac{x+3}{x-3}+\dfrac{y-3}{y+3}=2, \quad \dfrac{x-3}{2x+3}+\dfrac{y-3}{2y+3}=1.$

(21.) $2(x-y)+xy=3xy-(x-y)=7.$

(22.) $(x+y)/7=8/(x+y+1), \quad xy=12.$

(23.) $x+1/y=10/x, \quad y+1/x=10x.$

(24.) $3(x^2+y^2)-2xy=27, \quad 4(x^2+y^2)-6xy=16.$

(25.) $x^3-y^3=208, \quad x-y=4.$

(26.) $x^2y+xy^2=162, \quad x^3+y^3=243.$

(27.) $x^2y+xy^2=30, \quad x^4y^2+x^2y^4=468.$

(28.) $x^3+y^3=(a+b)(x-y), \quad x^2+xy+y^2=a-b.$

(29.) $x^4+x^2y^2+y^4=741, \quad x^2-xy+y^2=19.$

(30.) $xy(x+y)=48, \quad x^3+y^3=72.$ (31.) $x^4+y^4=a^4, \quad x+y=b.$

(32.) $x^4+y^4=97, \quad x+y=5.$

(33.) $x^4+y^4=(p^2+2)x^2y^2, \quad x+y=a.$ (34.) $x^5+y^5=33, \quad x+y=3.$

(35.) $x^2-y^2=2xy+x+y, \quad x^3-y^3=3xy(x+y).$

(36.) $(x+y)(x^3-y^3)=819, \quad (x-y)(x^3+y^3)=399.$

(37.) $x^2/y+y^2/x=2, \quad x+y=5.$

(38.) $x^2y^2(x^4-y^4)=a^4, \quad xy(x^4+y^4)(x^2-y^2)=a.$

(39.) $x^4-x^2+y^4-y^2=84, \quad x^2+x^2y^2+y^2=49.$

(40.) $x^3/y=a^2-xy, \quad y^3/x=b^2-xy.$

(41.) $x+y+\sqrt{(xy)}=14, \quad x^2+y^2+xy=84.$

(42.) $\sqrt{(1-a/x)}+\sqrt{(1-a/y)}=\sqrt{(1+a/b)}, \quad x+y=b.$

(43.) $x+y+\sqrt{(x^2-y^2)}=a, \quad 2y\sqrt{(x^2-y^2)}=b^2.$

(44.) $\sqrt{(x^2+12y)}+\sqrt{(y^2+12x)}=33, \quad x+y=23.$

(45.) $\sqrt{(x/y)}+\sqrt{(y/x)}=5/2, \quad \sqrt{(x^2/y)}+\sqrt{(y^2/x)}=9\sqrt{2}/2.$

(46.) $\sqrt{(x+a)}+\sqrt{(y-a)}=\frac{5}{2}\sqrt{a}, \quad \sqrt{(x-a)}+\sqrt{(y-a)}=\frac{3}{2}\sqrt{a}.$

(47.) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}, \quad (x^2+y^2)^{\frac{1}{2}}+(2xy)^{\frac{1}{2}}=b.$

(48.) $x^2+a^2+y^2+b^2=\sqrt{2}\{x(a+y)-b(a-y)\},$
$x^2-a^2-y^2+b^2=\sqrt{2}\{x(a-y)+b(a+y)\}.$

(49.) $(x^2+a^2)(y^2+b^2)=m(xy+ab)^2, \quad (x^2-a^2)(y^2-b^2)=n(bx-ay)^2.$

(50.) $x=y+\dfrac{a}{y+}\,\dfrac{a}{y+}\,\dfrac{a}{y+}\ \ldots, \quad y=x+\dfrac{b}{x+}\,\dfrac{b}{x+}\,\dfrac{b}{x+}\ \ldots$

(51.) $x^my^n=(\frac{3}{2})^{m-n}, \quad x^ny^m=(\frac{2}{3})^{m-n}.$

(52.) $x^{x+y}=y^{4a}, \quad y^{x+y}=x^a.$

Exercises XXXIV.

*(1.) $\Sigma x=0,\quad \Sigma ax=0,\quad \Sigma a^3x^3=3\Pi(b-c).$

(2.) $(y-a)(z-a)=bc,\quad (z-b)(x-b)=ca,\quad (x-c)(y-c)=ab.$

(3.) $yz+2(y+z)=11,\quad zx+2(z+x)=8,\quad xy+2(x+y)=16.$

(4.) $\dfrac{3}{x}+\dfrac{4}{y}+\dfrac{1}{z}=4,\quad yz+zx+xy=\frac{11}{6}xyz,\quad 2zx+3yz=2xy.$

(5.) $x(y+z)=24,\quad y(z+x)=18,\quad z(x+y)=20.$

(6.) $x(y+z)=y(z+x)=z(x+y)=1.$

(7.) $(z+x)(x+y)=a^2,\quad (x+y)(y+z)=b^2,\quad (y+z)(z+x)=c^2.$

(8.) $ax+yz=ay+zx=az+xy=p^2.$

(9.) $x^2+2yz=128,\quad y^2+2zx=153,\quad z^2+2xy=128.$

(10.) $a^2(y+z)^2=a^2x^2+1,\quad b^2(z+x)^2=b^2y^2+1,\quad c^2(x+y)^2=c^2z^2+1.$

(11.) $a(y+z-x)=(x+y+z)^2-2by,\quad b(z+x-y)=(x+y+z)^2-2cz,$
$c(x+y-z)=(x+y+z)^2-2ax.$

(12.) $\Sigma(x^2-yz)-2(x^2-yz)=a^2,\quad \Sigma(x^2-yz)-2(y^2-zx)=b^2,$
$\Sigma(x^2-yz)-2(z^2-xy)=c^2.$

(13.) $\Sigma bcx=0,\quad \Sigma ayz=0,\quad \Sigma x^2=1.$

(14.) $a(x+yz)=b(y+zx)=c(z+xy),\quad x^2+y^2+z^2+2xyz=1.$

(15.) $x(a+y+z)=y(a+z+x)=z(a+x+y)=3a(x+y+z).$

(16.) $x^2+y^2+z^2=a^2+2x(y+z)-x^2$, and the two equations derived from this one by interchanging $\begin{Bmatrix} xyz \\ abc \end{Bmatrix}$.

(17.) $ax^2=\dfrac{1}{y}+\dfrac{1}{z},\quad by^2=\dfrac{1}{z}-\dfrac{1}{x},\quad cz^2=\dfrac{1}{x}+\dfrac{1}{y}.$

(18.) $y^2z^2+z^2x^2+x^2y^2=49,\quad x^2+y^2+z^2=14,\quad x(y+z)=9.$

(19.) $(yz-x^2)/a^4x=(zx-y^2)/b^4y=(xy-z^2)/c^4z=1/xyz.$

(20.) $a^2x^2(y+z)^2=(a^2+x^2)y^2z^2$, and the two derived therefrom by interchanging $\begin{Bmatrix} xyz \\ abc \end{Bmatrix}$.

(21.) $\Sigma x^3=a(\Sigma x-2x)=b(\Sigma x-2y)=c(\Sigma x-2z).$

(22.) $(x-1)(y+z-5)=77,\quad (y-2)(z+x-4)=72,\quad (z-3)(x+y-3)=65.$

(23.) $u(y-x)/(z-u)=a,\quad z(y-x)/(z-u)=b,\quad y(u-z)/(x-y)=c,$
$x(u-z)/(x-y)=d.$

(24.) If $x^3+y^3+z^3+6xyz=a,\quad 3(y^2z+z^2x+x^2y)=b,\quad 3(yz^2+zx^2+xy^2)=c,$ show that

$$x+y+z=(a+b+c)^{\frac{1}{3}},\quad x+\omega y+\omega^2z=(a+\omega b+\omega^2c)^{\frac{1}{3}},$$
$$x+\omega^2y+\omega z=(a+\omega^2b+\omega c)^{\frac{1}{3}},$$

where $\omega^2+\omega+1=0$. Find all the real solutions when $a=72,\quad b=75,\quad c=69.$

(25.) $x^2-yz=a^2,\quad y^2-zx=b^2,\quad z^2-xy=c^2.$

---

* In this set of exercises Σ and Π refer to three letters only; and $\Pi(b-c)$ stands for $(b-c)(c-a)(a-b)$, and not for $(b-c)(c-a)(a-b)(c-b)(a-c)(b-a)$, as, strictly speaking, it ought to do.

Exercises XXXV.

Eliminate *

(1.) x from the system

$$\frac{ax+b}{cx+d}=\frac{a'x+b'}{c'x+d'}=\frac{a''x+b''}{c''x+d''}.$$

(2.) x and y from

$$\frac{lx}{a}-\frac{my}{b}=\frac{a+b}{a-b}, \quad \frac{al}{x}-\frac{bm}{y}=\frac{a-b}{a+b}, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(3.) x and y from

$$x^2+xy=a^2, \quad y^2+xy=b^2, \quad x^2+y^2=c^2.$$

(4.) x, y, z from

$$x(y+z)=a^2, \quad y(z+x)=b^2, \quad z(x+y)=c^2, \quad xyz=d^3.$$

(5.) x, y, z from

$$x+y+z=0, \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0, \quad \frac{a}{x}(x-p)=\frac{b}{y}(y-q)=\frac{c}{z}(z-r).$$

(6.) x, y, z from

$$\Sigma Ax^2=0, \quad \Sigma A'x^2=0, \quad \Sigma ax=0,$$

and show that the result is

$$\Sigma 1/\{b^2(CA')+c^2(AB')-a^2(BC')\}=0,$$

where
$$(CA')=CA'-C'A, \text{ \&c.}$$

(7.) Show that the following system of equations in x, y, z are inconsistent unless $r^3-p^3=3rq^2$, and that they have an infinite number of solutions if this condition be fulfilled.

$$\Sigma x^3-3xyz=p^3, \quad \Sigma yz=q^2, \quad \Sigma x=r.$$

Eliminate

(8.) x and y from

$$(a-x)(a-y)=p, \quad (b-x)(b-y)=q, \quad (a-x)(b-y)/(b-x)(a-y)=c.$$

(9.) x, y, z from

$$x+y-z=a, \quad x^2+y^2-z^2=b^2, \quad x^3+y^3-z^3=c^3, \quad xyz=a^3.$$

(10.) x, y, z from

$$ax+yz=bc, \quad by+zx=ca, \quad cz+xy=ab, \quad xyz=abc.$$

(11.) x, y, z from

$$\Sigma x^2=p^2, \quad \Sigma x^3=q^3, \quad \Sigma x^4=r^4, \quad xyz=s^3.$$

(12.) x, y, z from

$$(x+a)(y+b)(z+c)=abc, \quad (y-c)(z-b)=a^2, \quad (z-a)(x-c)=b^2, \quad (x-b)(y-a)=c^2.$$

(13.) The system

$$x_1x_2+y_1y_2=k_1^2, \quad x_2x_3+y_2y_3=k_2^2, \quad \ldots, \quad x_nx_1+y_ny_1=k_n^2,$$
$$x_1^2+y_1^2=x_2^2+y_2^2=\ldots=x_n^2+y_n^2=a^2,$$

either has no solution, or it has an infinite number of solutions.

---

* The eliminant is in all cases to be a rational integral equation.

CHAPTER XVIII.

General Theory of Integral Functions, more particularly of Quadratic Functions.

RELATIONS BETWEEN THE COEFFICIENTS OF A FUNCTION AND ITS ROOTS—SYMMETRICAL FUNCTIONS OF THE ROOTS.

§ 1.] By the remainder theorem (chap. v., § 15), it follows that if $a_1, a_2, \ldots, a_n$ be the n roots of the integral function

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n \qquad (1),$$

that is to say, the n values of x for which its value becomes 0, then we have the identity

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n$$
$$\equiv p_0(x - a_1)(x - a_2) \ldots (x - a_n) \qquad (2).$$

Now we have (see chap. iv., § 10)

$$(x - a_1)(x - a_2) \ldots (x - a_n) \equiv x^n - P_1x^{n-1} + P_2x^{n-2} - \ldots + (-1)^nP_n,$$

where $P_1, P_2, \ldots, P_n$ denote the sums of the products of the n quantities $a_1, a_2, \ldots, a_n$, taken 1, 2, . . ., n at a time respectively. Hence, if we divide both sides of (2) by p_0, we have the identity

$$x^n + \frac{p_1}{p_0}x^{n-1} + \frac{p_2}{p_0}x^{n-2} + \ldots + \frac{p_n}{p_0}$$
$$\equiv x^n - P_1x^{n-1} + P_2x^{n-2} - \ldots + (-1)^nP_n \qquad (3).$$

Since (3) is an identity, we must have

$$p_1/p_0 = -P_1, \quad p_2/p_0 = P_2, \quad \ldots, \quad p_n/p_0 = (-1)^nP_n \qquad (4).$$

In particular, if $p_0 = 1$, so that we have the function

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n \qquad (5),$$

then $\quad p_1 = -P_1, \quad p_2 = P_2, \quad \ldots, \quad p_n = (-1)^nP_n \qquad (6).$

Hence, if we consider the roots of the function

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n,$$

or, what comes to the same thing, the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0,$$

then $-p_1$ *is the sum of the* n *roots;* p_2 *the sum of all the products of the roots, taken two at a time;* $-p_3$ *the sum of all the products, taken three at a time, and so on.*

Thus, if α and β be the roots of the quadratic function $ax^2 + bx + c$, that is, the values of x which satisfy the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -b/a, \quad \alpha\beta = c/a \qquad (7).$$

Again, if α, β, γ be the roots of the cubic function $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = -b/a, \quad \beta\gamma + \gamma\alpha + \alpha\beta = c/a, \quad \alpha\beta\gamma = -d/a \qquad (8).$$

§ 2.] *If* $s_1, s_2, s_3, \ldots, s_r$ *stand for the sums of the 1st, 2nd, 3rd, . . ., rth powers of the roots* α *and* β *of the quadratic equation*

$$x^2 + p_1x + p_2 = 0 \qquad (1),$$

we can express $s_1, s_2, \ldots, s_r$ *as integral functions of* p_1 *and* p_2.

In the first place, we have, by § 1 (6),

$$s_1 = \alpha + \beta = -p_1 \qquad (2).$$

Again

$$\begin{aligned} s_2 &= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta, \\ &= p_1^2 - 2p_2 \qquad (3). \end{aligned}$$

To find s_3 we may proceed as follows. Since α and β are roots of (1), we have

$$\alpha^2 + p_1\alpha + p_2 = 0, \quad \beta^2 + p_1\beta + p_2 = 0 \qquad (4).$$

Multiplying these equations by α and β respectively, and adding, we obtain

$$s_3 + p_1s_2 + p_2s_1 = 0 \qquad (5).$$

Since s_1 and s_2 are integral functions of p_1 and p_2, (5) determines s_3 as an integral function of p_1 and p_2. We have, in fact,

$$\begin{aligned} s_3 &= -p_1(p_1^2 - 2p_2) + p_2p_1, \\ &= -p_1^3 + 3p_1p_2 \qquad (6). \end{aligned}$$

Similarly, multiplying the equations (4) by α^2 and β^2 respectively, and adding, we deduce

$$s_4 + p_1 s_3 + p_2 s_2 = 0 \quad (7).$$

Hence s_4 may be expressed as an integral function of p_1 and p_2, and so on.

We can now express any symmetric integral function whatever of the roots of the quadratic (1) *as an integral function of p_1 and p_2.*

Since any symmetric integral function is a sum of symmetrical integral homogeneous functions, it is sufficient to prove this proposition for a homogeneous symmetric integral function of the roots α and β. The most general such function of the rth degree may be written

$$A(\alpha^r + \beta^r) + B\alpha\beta(\alpha^{r-2} + \beta^{r-2}) + C\alpha^2\beta^2(\alpha^{r-4} + \beta^{r-4}) + \ldots,$$

that is to say,

$$As_r - Bp_2 s_{r-2} + Cp_2^2 s_{r-4} + \ldots \quad (8),$$

where A, B, C are coefficients independent of α and β.

Hence the proposition follows at once, for we have already shown that $s_r, s_{r-2}, s_{r-4}, \ldots$ can all be expressed as integral functions of p_1 and p_2.

It is important to notice that, since α and β may be any two quantities whatsoever, the result just arrived at is really a general proposition regarding any integral symmetric function of two variables, namely, that *any symmetric integral function of two variables α, β can be expressed as a rational integral function of the two elementary symmetrical functions $p_1 = -(\alpha + \beta)$ and $p_2 = \alpha\beta$.*

There are two important remarks to be made regarding this expression.

1st. *If all the coefficients of the given integral symmetric function be integers, then all the coefficients in the expression for it in terms of p_1 and p_2 will also be integers.*

This is at once obvious if we remark that at every step in the successive calculation of $s_1, s_2, s_3, \ldots$, &c., we substitute directly integral values previously obtained, so that the only possibility of introducing fractions would be through the coefficients A, B, C, . . . in (8).

2nd. Since all the equations above written become identities, homogeneous throughout, when for p_1 and p_2 we substitute their values $-(\alpha+\beta)$ and $\alpha\beta$ respectively; and since p_1 is of the 1st and p_2 of the 2nd degree in α and β, it follows that *in every term of any function of p_1 and p_2 which represents the value of a homogeneous symmetric function, the sum of the suffixes* of the p's must be equal to the degree of the symmetric function in α and β.* Thus, for example, in the expression (6) for s_3 the sum of the suffixes in the term $-p_1^3$, that is, $-p_1p_1p_1$, is 3; and in the term $3p_1p_2$ also 3.

This last remark is important, because it enables us to write down at once all the terms that can possibly occur in the expression for any given homogeneous symmetric function. All we have to do is to write down every product of p_1 and p_2, or of powers of these, in which the sum of the suffixes is equal to the degree of the given function.

Example 1.

To calculate $\alpha^4+\beta^4$ in terms of p_1 and p_2.

This is a homogeneous symmetric function of the 4th degree. Hence, by the rule just stated, we must have

$$\alpha^4+\beta^4\equiv Ap_1^4+Bp_1^2p_2+Cp_2^2,$$

where A, B, C are coefficients to be determined.

In the first place, let $\beta=0$, so that $p_1=-\alpha$, $p_2=0$. We must then have the identity $\alpha^4\equiv A\alpha^4$. Hence $A=1$.

We now have

$$\alpha^4+\beta^4\equiv(\alpha+\beta)^4+B(\alpha+\beta)^2\alpha\beta+C\alpha^2\beta^2.$$

Observing that the term $\alpha^3\beta$ does not occur on the left, we see that B must have the value -4.

Lastly, putting $\alpha=-\beta=1$, so that $p_1=0$, $p_2=-1$, we see that $C=2$. Hence

$$\alpha^4+\beta^4\equiv p_1^4-4p_1^2p_2+2p_2^2.$$

The same result might also be obtained as follows. We have

$$s_4+p_1s_3+p_2s_2=0.$$

Hence, using the values of s_2 and s_3 already calculated, we have

$$\begin{aligned} s_4&=-p_1(-p_1^3+3p_1p_2)-p_2(p_1^2-2p_2),\\ &=p_1^4-4p_1^2p_2+2p_2^2. \end{aligned}$$

Example 2.

Calculate $\alpha^5+\beta^5+\alpha^3\beta^2+\alpha^2\beta^3$ in terms of p_1 and p_2.

* This is called the *weight* of the symmetric function. See Salmon's *Higher Algebra*, § 56.

We have

$$\alpha^5+\beta^5+\alpha^3\beta^2+\alpha^2\beta^3=Ap_1^5+Bp_1^3p_2+Cp_1p_2^2.$$

Putting $\beta=0$, we find $A=-1$; considering the term $\alpha^4\beta$, we see that $B=5$; and, putting $\alpha=\beta=1$, we find $C=-6$. Hence

$$\alpha^5+\beta^5+\alpha^3\beta^2+\alpha^2\beta^3=-p_1^5+5p_1^3p_2-6p_1p_2^2.$$

Since any alternating integral function * of α, β, say $f(\alpha, \beta)$, merely changes its sign when α and β are interchanged, it follows that we have $f(\alpha, \beta)=-f(\beta, \alpha)$. Hence, if we put $\beta=\alpha$, we have $f(\alpha, \alpha)=-f(\alpha, \alpha)$; that is, $2f(\alpha, \alpha)=0$. Therefore $f(\alpha, \alpha)=0$. It follows from the remainder theorem that $f(\alpha, \beta)$ is exactly divisible by $\alpha-\beta$. Let the quotient be $g(\alpha, \beta)$. Then $g(\alpha, \beta)$ is a symmetric function of α, β. For $g(\alpha, \beta)=f(\alpha, \beta)/(\alpha-\beta)$, and $g(\beta, \alpha)=f(\beta, \alpha)/(\beta-\alpha)=-f(\alpha, \beta)/(\beta-\alpha)=f(\alpha,\beta)/(\alpha-\beta)$; that is, $g(\alpha, \beta)=g(\beta, \alpha)$. Hence any alternating integral function of α and β can be expressed as the product of $\alpha-\beta$ and some symmetric function of α and β. *Hence any alternating function of α and β can be expressed without difficulty as the product of $\pm\sqrt{(p_1^2-4p_2)}$, and an integral function of p_1 and p_2.*

Example 3.

To express $\alpha^5\beta-\alpha\beta^5$ in terms of p_1 and p_2.

We have

$$\begin{aligned}\alpha^5\beta-\alpha\beta^5&=\alpha\beta(\alpha^4-\beta^4),\\&=(\alpha-\beta)\alpha\beta(\alpha+\beta)(\alpha^2+\beta^2),\\&=\pm\sqrt{(p_1^2-4p_2)}\{p_1p_2(p_1^2-2p_2)\}.\end{aligned}$$

Every symmetric rational function of α and β can be expressed as the quotient of two integral symmetric functions of α and β, and can therefore be expressed as a rational function of p_1 and p_2.

Example 4.

$$\begin{aligned}\frac{\alpha^3+2\alpha^2\beta+2\alpha\beta^2+\beta^3}{\alpha^2\beta+\alpha\beta^2}&=\frac{(\alpha+\beta)^3-\alpha\beta(\alpha+\beta)}{\alpha\beta(\alpha+\beta)},\\&=\frac{-p_1^3+p_1p_2}{-p_2p_1},\\&=\frac{p_1^2-p_2}{p_2}.\end{aligned}$$

§ 3.] The general proposition established for symmetric functions of two variables can be extended without difficulty to symmetric functions of any number of variables.

---

* See p. 77, footnote.

We shall first prove, in its most general form, Newton's Theorem that *the sums of the integral powers of the roots of any integral equation,*

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0 \qquad (1),$$

can be expressed as integral functions of $p_1, p_2, \ldots, p_n$, *whose coefficients are all integral numbers.*

Let the n roots of (1) be $a_1, a_2, \ldots, a_n$, and let the equation whose roots are the same as those of (1), with the exception of a_1, be

$$x^{n-1} + {}_1p_1x^{n-2} + {}_1p_2x^{n-3} + \ldots + {}_1p_{n-1} = 0 \qquad (2);$$

also let the equation whose roots are the same as those of (1), with the exception of a_2, be

$$x^{n-1} + {}_2p_1x^{n-2} + {}_2p_2x^{n-3} + \ldots + {}_2p_{n-1} = 0 \qquad (3),$$

and so on.

Then

$$\begin{aligned} &x^{n-1} + {}_1p_1x^{n-2} + {}_1p_2x^{n-3} + \ldots + {}_1p_{n-1} \\ &\quad \equiv (x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n)/(x - a_1), \\ &\quad \equiv x^{n-1} + (a_1 + p_1)x^{n-2} + (a_1^2 + p_1a_1 + p_2)x^{n-3} \ldots \\ &\qquad\qquad + (a_1^r + p_1a_1^{r-1} + \ldots + p_r)x^{n-r-1} + \ldots \end{aligned}$$

by chap. v., § 13.

Hence, equating coefficients, we have

$$\left.\begin{array}{llll} {}_1p_1 = a_1 & + p_1, & & \\ {}_1p_2 = a_1^2 & + p_1a_1 & + p_2, & \\ \ldots & \ldots & \ldots & \\ {}_1p_r = a_1^r & + p_1a_1^{r-1} & + \ldots + p_r, & \\ \ldots & \ldots & \ldots & \ldots \\ {}_1p_{n-1} = a_1^{n-1} & + p_1a_1^{n-2} & + \ldots + p_{n-1} & \end{array}\right\} \qquad (3').$$

Similar values can be obtained for ${}_2p_1, {}_2p_2, {}_2p_3, \ldots, {}_2p_{n-1}$ in terms of a_2 and $p_1, p_2, \ldots, p_n$; and so on.

Taking the $(r-1)$th equation in the system (3′), and multiplying by a_1, we have

$${}_1p_{r-1}a_1 = a_1^r + p_1a_1^{r-1} + \ldots + p_{r-1}a_1.$$

Similarly

$${}_2p_{r-1}a_2 = a_2^r + p_1a_2^{r-1} + \ldots + p_{r-1}a_2;$$

and so on.

Adding the n equations thus obtained, we have

$$_1p_{r-1}a_1 + {_2p_{r-1}}a_2 + \ldots + {_np_{r-1}}a_n = s_r + p_1s_{r-1} + \ldots + p_{r-1}s_1 \quad (4).$$

Now $_1p_{r-1}$ is the sum of all the products $r-1$ at a time of the $n-1$ quantities $-a_2, -a_3, \ldots, -a_n$. Hence $_1p_{r-1}a_1$ is the sum, with the negative sign, of all those products r at a time of the n quantities $-a_1, -a_2, \ldots, -a_n$ which contain a_1. Similarly the next term contains all those products r at a time in which a_2 occurs; and so on. Hence on the left all the products r at a time of the n quantities $-a_1, -a_2, \ldots, -a_n$ occur, each as often as there are letters in any such product, that is to say, r times. Hence the equation (4) becomes

$$-rp_r = s_r + p_1s_{r-1} + \ldots + p_{r-1}s_1,$$

or

$$s_r + p_1s_{r-1} + \ldots + p_{r-1}s_1 + rp_r = 0.$$

This will hold for any value of r from 1 to $n-1$, both inclusive. We have therefore the system

$$\left.\begin{array}{lr} s_1 + p_1 & = 0 \\ s_2 + p_1s_1 + 2p_2 & = 0 \\ s_3 + p_1s_2 + p_2s_1 + 3p_3 & = 0 \\ \ldots\ldots\ldots\ldots & \\ s_{n-1} + p_1s_{n-2} + \ldots\ (n-1)p_{n-1} & = 0 \end{array}\right\} \quad (5).$$

Again, since a_1 is a root of (1), we have

$$a_1^n + p_1a_1^{n-1} + \ldots + p_n = 0.$$

Similarly

$$a_2^n + p_1a_2^{n-1} + \ldots + p_n = 0;$$

and so on.

If we first add these n equations as they stand, then multiply them by $a_1, a_2, \ldots, a_n$ and add, then multiply them by $a_1^2, a_2^2, \ldots, a_n^2$ respectively, and add, and so on, we obtain

$$\left.\begin{array}{l} s_n + p_1s_{n-1} + \ldots + np_n = 0 \\ s_{n+1} + p_1s_n + \ldots + s_1p_n = 0 \\ s_{n+2} + p_1s_{n+1} + \ldots + s_2p_n = 0 \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad (6),$$

and so on.

The equations (5) and (6) constitute *Newton's Formulæ* for

calculating s_1, s_2, s_3, ... &c., in terms of $p_1, p_2, \ldots, p_n$. It is obvious that $s_1, s_2, s_3, \ldots$ are determined as rational integral functions of $p_1, p_2, \ldots, p_n$, in which all the coefficients are integral numbers.

A little consideration of the formulæ will show that *in the expression for s_r the sum of the suffixes of the p's in each term will be r.*

Hence to find all the terms that can possibly occur in s_r we have simply to write down all the products of powers of $p_1, p_2, \ldots p_n$ in which the sum of the suffixes is r.

Example.

To find the sum of the cubes of the roots of the equation

$$x^3 - 2x^2 + 3x + 1 = 0.$$

We have

$$s_1 - 2 = 0, \quad s_2 - 2s_1 + 2 \times 3 = 0, \quad s_3 - 2s_2 + 3s_1 + 3 \times 1 = 0.$$

Hence

$$s_1 = 2, \quad s_2 = -2, \quad s_3 = -13.$$

§ 4.] We can now show that *every integral symmetric function of the roots can be expressed as an integral function of $p_1, p_2, \ldots p_n$.* The terms of every symmetric function can be grouped into types, each term of a type being derivable from every other of that type by merely interchanging the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ (see chap. iv., § 22). All the terms belonging to the same type have the same coefficient. It is sufficient, therefore, to prove the above proposition for symmetric functions containing only one type of terms. Such symmetric functions may be classed as single, double, triple, &c., according as one, two, three, &c., of the variables $\alpha_1, \alpha_2, \ldots \alpha_n$ appear in each term. Thus $\Sigma\alpha_1^p$, $\Sigma\alpha_1^p\alpha_2^q$, $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r$, &c., are single, double, triple, &c., symmetric functions.

For the single functions, which are simply sums of powers, the theorem has already been established. We can make the double function depend on this case as follows:—

Consider the distribution of the product

$$(\alpha_1^p + \alpha_2^p + \ldots + \alpha_n^p)(\alpha_1^q + \alpha_2^q + \ldots + \alpha_n^q).$$

Terms of two different types, and of two only, can occur, namely,

terms derivable from $\alpha_1^p\alpha_1^q$, that is, α_1^{p+q}, and terms derivable from $\alpha_1^p\alpha_2^q$. We have in fact

$$s_p s_q = \Sigma\alpha_1^{p+q} + \Sigma\alpha_1^p\alpha_2^q.$$

Hence
$$\Sigma\alpha_1^p\alpha_2^q = s_p s_q - s_{p+q}.$$

Now s_p, s_q, s_{p+q} can all be expressed as integral functions of $p_1, p_2, \ldots, p_n$. Hence the same is true of $\Sigma\alpha_1^p\alpha_2^q$.

Here we have supposed $p \neq q$. If $p = q$, then the term $\alpha_1^p\alpha_2^p$ will occur twice, and we have

$$s_p^2 = \Sigma\alpha_1^{2p} + 2\Sigma\alpha_1^p\alpha_2^p\,;$$

but this does not affect our reasoning.

The case of triple functions can be made to depend on that of double and single functions in a similar way. In the distribution of

$$(\alpha_1^p + \alpha_2^p + \ldots + \alpha_n^p)(\alpha_1^q + \alpha_2^q + \ldots + \alpha_n^q)(\alpha_1^r + \alpha_2^r + \ldots + \alpha_n^r)$$

every term is of the form $\alpha_u^p\alpha_v^q\alpha_w^r$, where u, v, w are, 1st, all different; 2nd, such that two are equal; 3d, all equal. Any particular term can occur only once if p, q, r be all unequal. Hence we have

$$s_p s_q s_r = \Sigma\alpha_1^p\alpha_2^q\alpha_3^r + \Sigma\alpha_1^{p+q}\alpha_2^r + \Sigma\alpha_1^{q+r}\alpha_2^p + \Sigma\alpha_1^{r+p}\alpha_2^q + \Sigma\alpha_1^{p+q+r}.$$

In the last equation every term, except $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r$, can be expressed as an integral function of $p_1, p_2, \ldots, p_n$. Hence $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r$ can be so expressed.

If two or more of the numbers p, q, r be equal, then each term of $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r$ will occur a particular number of times; and the same is true of certain of the other terms in the equation last written. But this does not affect the conclusion in any way.

We can now make the case of quadruple symmetric functions depend on the cases already established; and so on. Hence the proposition is generally true.

It is obvious, from the nature of each step in the above process, and from what has been already proved for $s_1, s_2, s_3, \ldots$, that *in the expression for any homogeneous symmetric function of degree r the sum of the suffixes of the p's will be r for each term*; so that we can at once write down all the terms that can possibly

occur in that expression, and then determine the coefficients by any means that may happen to be convenient.

It is important to remark that the *degree in* $p_1, p_2, \ldots, p_n$ *of the expression* for $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r \ldots$ *in terms of* $p_1, p_2, \ldots, p_n$ *must be equal to the highest of the indices* $p, q, r \ldots$ For, let the term of highest degree be $p_1^{\lambda_1}p_2^{\lambda_2}\ldots p_n^{\lambda_n}$, then, since $p_1 = {}_1p_1 - \alpha_1$, $p_2 = {}_1p_2 - {}_1p_1\alpha_1$, where ${}_1p_1$, ${}_1p_2$, &c., do not contain α_1,* we see that $p_1^{\lambda_1}p_2^{\lambda_2}\ldots p_n^{\lambda_n}$, when expressed in terms of $\alpha_1, \alpha_2, \ldots, \alpha_n$, will introduce the power $\alpha_1^{\lambda_1+\lambda_2+\cdots+\lambda_n}$ with the coefficient $(-1)^n{}_1p_1^{\lambda_2}{}_1p_2^{\lambda_3}\ldots{}_1p_{n-1}^{\lambda_n}$. Now, since there are no terms of higher degree than $p_1^{\lambda_1}p_2^{\lambda_2}\ldots p_n^{\lambda_n}$, if the power $\alpha_1^{\lambda_1+\lambda_2+\cdots+\lambda_n}$ occur again, it must occur as the highest power, resulting from a different term of the same degree; that is to say, it will occur with a different coefficient and cannot destroy the former term. Hence the index of the highest power of any letter in the symmetric function must be equal to the degree of the highest term in its expression in terms of $p_1, p_2, \ldots, p_n$.†

Although, in establishing the leading theorem of this paragraph, we have used the language of the theory of equations, the result is really a fundamental principle in the calculus of algebraical identities; and it is for this reason that we have introduced it here. We may state the result as follows:—

Let us call Σx_1, Σx_1x_2, $\Sigma x_1x_2x_3$, $\ldots$, $\Sigma x_1x_2x_3\ldots x_n$ the n *elementary symmetric functions* of the system of n variables $x_1, x_2, \ldots, x_n$. *Then we can express any symmetric integral function of* $x_1, x_2, \ldots, x_n$ *as an integral function of the* n *elementary symmetric functions; and therefore any rational symmetric function of these variables as a rational function of the* n *elementary symmetric functions.*

On account of its great importance we give a proof of this

---

* They are, in fact, the functions of $\alpha_2, \alpha_3, \ldots, \alpha_n$ defined in § 3. See Exercises xxxvi., 51.

† Salmon, *Higher Algebra*, § 58.

proposition not depending on Newton's Theorem (which is itself merely a particular case).*

Let ${}_nq_1, {}_nq_2, \ldots, {}_nq_n$ denote the n elementary symmetric functions of the n variables $x_1, x_2, \ldots, x_n$, that is to say, $\Sigma_n x_1, \Sigma_n x_1 x_2, \ldots, x_1 x_2 \ldots x_n$; and let ${}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1}$ denote the $n-1$ elementary symmetric functions of $x_1, x_2, \ldots, x_{n-1}$, that is, $\underset{n-1}{\Sigma} x_1, \underset{n-1}{\Sigma} x_1 x_2, \ldots, x_1 x_2 \ldots x_{n-1}$. It is obvious that, when $x_n = 0$, ${}_nq_1, {}_nq_2, \ldots, {}_nq_{n-1}$ become ${}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1}$ respectively.

Let us now assume that all symmetric integral functions not involving more than $n-1$ variables can be expressed as integral functions of ${}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1}$. Let $f(x_1, x_2, \ldots, x_{n-1}, x_n)$ be any symmetric integral function of the n variables $x_1, x_2, \ldots, x_n$. Then $f(x_1, x_2, \ldots, x_{n-1}, 0)$ is a symmetric integral function of $x_1, x_2, \ldots, x_{n-1}$, and can therefore, by hypothesis, be expressed integrally in terms of ${}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1}$. Let this expression be $\phi({}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1})$, so that ϕ is a known function.

Now assume

$$f(x_1, x_2, \ldots, x_{n-1}, x_n) = \phi({}_nq_1, {}_nq_2, \ldots, {}_nq_{n-1}) + \psi(x_1, x_2, \ldots, x_{n-1}, x_n) \quad (7).$$

Then, since $\phi({}_nq_1, {}_nq_2, \ldots, {}_nq_{n-1})$ is a symmetric integral function of $x_1, x_2, \ldots, x_n$, $\psi(x_1, x_2, \ldots, x_{n-1}, x_n)$ is obviously a symmetric integral function of these variables.

If we put $x_n = 0$ on both sides of the identity (7), then

$$f(x_1, x_2, \ldots, x_{n-1}, 0) = \phi({}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1}) + \psi(x_1, x_2, \ldots, x_{n-1}, 0) \quad (8).$$

But $f(x_1, x_2, \ldots, x_{n-1}, 0) = \phi({}_{n-1}q_1, {}_{n-1}q_2, \ldots, {}_{n-1}q_{n-1})$. Hence, by (8), $\psi(x_1, x_2, \ldots, x_{n-1}, 0) = 0$. Therefore the integral function $\psi(x_1, x_2, \ldots, x_{n-1}, x_n)$ is exactly divisible by some

* This proof is taken from a paper by Mr. R. E. Allardice, *Proc. Edinb. Math. Soc.* for 1889.

power of x_n, say x_n^{α}; hence, on account of its symmetry, also by $x_1^{\alpha}, x_2^{\alpha}, \ldots, x_{n-1}^{\alpha}$. We may therefore put

$$\psi(x_1, x_2, \ldots, x_{n-1}, x_n) = {}_nq_n^{\alpha} f_1(x_1, x_2, \ldots, x_{n-1}, x_n),$$

where f_1 is a symmetric integral function of $x_1, x_2, \ldots, x_n$ of lower degree than f.

We can now deal with f_1 in the same way as we dealt with f; and so on. We shall thus resolve $f(x_1, x_2, \ldots, x_{n-1}, x_n)$ into a closed expression of the form

$$\phi + {}_nq_n^{\alpha_1}\phi_1 + {}_nq_n^{\alpha_1+\alpha_2}\phi_2 + \ldots + {}_nq_n^{\alpha_1+\alpha_2+\ldots+\alpha_m}\phi_m \quad (9),$$

where $\phi_1, \phi_2, \ldots, \phi_m$ are, like ϕ, all known functions of ${}_nq_1$, ${}_nq_2, \ldots, {}_nq_{n-1}$, or else constants.

If, therefore, the integral expression in question be possible for $n-1$ variables, it is possible for n variables.

Now every integral function of a single variable, x_1, is a symmetric function of that variable, and can be expressed integrally in terms of ${}_1q_1$, which is simply x_1. Hence it follows by induction that every symmetric integral function of n variables can be expressed as an integral function of the n elementary symmetric functions.

Cor. *It follows at once, by induction, from the form of* (9) *that the coefficients of the expression for any symmetric integral function* $f(x_1, x_2, \ldots, x_n)$ *in terms of* ${}_nq_1, {}_nq_2, \ldots, {}_nq_n$ *are integral functions of the coefficients of* f. *In particular, if the coefficients of* f *be integral numbers, the coefficients of its expression in terms of* ${}_nq_1, {}_nq_2, \ldots, {}_nq_n$ *will also be integral numbers.*

We now give a few examples of the calculation of symmetric functions in terms of the elementary functions, and of the use of this transformation in establishing identities and in elimination.

Example 1.

If α, β, γ be the roots of the equation

$$x^3 - p_1x^2 + p_2x - p_3 = 0,$$

express $\beta^3\gamma + \beta\gamma^3 + \gamma^3\alpha + \gamma\alpha^3 + \alpha^3\beta + \alpha\beta^3$ in terms of p_1, p_2, p_3.

Here we have $p_1 = \Sigma\alpha$, $p_2 = \Sigma\alpha\beta$, $p_3 = \alpha\beta\gamma$. Remembering that no term of higher degree than the 3rd can occur in the value of $\Sigma\alpha^3\beta$, we see that

$$\Sigma\alpha^3\beta = \mathrm{A}p_1^2p_2 + \mathrm{B}p_1p_3 + \mathrm{C}p_2^2 \qquad (1),$$

where A, B, C are numbers which we have to determine.

Suppose $\gamma=0$; then $p_1=\alpha+\beta$, $p_2=\alpha\beta$, $p_3=0$; and (1) becomes

$$\alpha^3\beta+\alpha\beta^3 \equiv \mathrm{A}(\alpha+\beta)^2\alpha\beta + \mathrm{C}\alpha^2\beta^2;$$

that is to say, $$\alpha^2+\beta^2 \equiv \mathrm{A}(\alpha+\beta)^2 + \mathrm{C}\alpha\beta.$$

Hence $$\mathrm{A}=1, \quad \mathrm{C}=-2.$$

We now have $$\Sigma\alpha^3\beta = p_1^2p_2 + \mathrm{B}p_1p_3 - 2p_2^2.$$

Let $\alpha=\beta=\gamma=1$, so that $p_1=3$, $p_2=3$, $p_3=1$. We then have

$$6=27+3\mathrm{B}-18.$$

Hence $$\mathrm{B}=-1.$$

Therefore, finally, $$\Sigma\alpha^3\beta = p_1^2p_2 - p_1p_3 - 2p_2^2.$$

In other words, we have the identity

$$\Sigma\alpha^3\beta \equiv (\Sigma\alpha)^2\Sigma\alpha\beta - \alpha\beta\gamma\Sigma\alpha - 2(\Sigma\alpha\beta)^2.$$

Example 2.

To show that

$$(yz-xu)(zx-yu)(xy-zu) \equiv (yzu+zux+uxy+xyz)^2 - xyzu(x+y+z+u)^2 \qquad (2).$$

The left-hand side of (2) is a symmetric function of x, y, z, u. Let us calculate its value in terms of $p_1=\Sigma x$, $p_2=\Sigma xy$, $p_3=\Sigma xyz$, $p_4=xyzu$.

Since the degree of $\Pi(yz-xu)$ in x, y, z, u is 6, and the degree in x alone is 3, we have

$$\Pi(yz-xu) = \mathrm{A}p_1^2p_4 + \mathrm{B}p_1p_2p_3 + \mathrm{C}p_2^3 + \mathrm{D}p_2p_4 + \mathrm{E}p_3^2 \qquad (3).$$

If we put $u=0$, then $p_1=\Sigma_3 x$, $p_2=\Sigma_3 xy$, $p_3=xyz$, $p_4=0$, where the suffix 3 under the Σ means that only three variables, x, y, z, are to be considered. If p_1, p_2, p_3 have for the moment these meanings, then (3) becomes the identity

$$p_3^2 \equiv \mathrm{B}p_1p_2p_3 + \mathrm{C}p_2^3 + \mathrm{E}p_3^2.$$

Hence $$\mathrm{B}=0, \quad \mathrm{C}=0, \quad \mathrm{E}=1.$$

Hence $$\Pi(yz-xu) \equiv \mathrm{A}p_1^2p_4 + \mathrm{D}p_2p_4 + p_3^2 \qquad (4).$$

Now let $x=y=1$, and $z=u=-1$, so that $p_1=0$, $p_2=-2$, $p_3=0$, $p_4=1$. Then (4) becomes

$$0=-2\mathrm{D}.$$

Hence $$\mathrm{D}=0.$$

We now have $$\Pi(yz-xu) \equiv \mathrm{A}p_1^2p_4 + p_3^2.$$

In this put $x=y=z=u=1$, and we have

$$0=16\mathrm{A}+16$$

Hence $$\mathrm{A}=-1.$$

Hence, finally, $$\Pi(yz-xu) \equiv p_3^2 - p_1^2p_4,$$

which establishes the identity (2).

Example 3.

If $x+y+z=0$, show that

$$\frac{x^{11}+y^{11}+z^{11}}{11} = \frac{x^3+y^3+z^3}{3}\cdot\frac{x^8+y^8+z^8}{2} - \frac{(x^3+y^3+z^3)^3}{9}\cdot\frac{x^2+y^2+z^2}{2} \qquad (5).$$

(*Wolstenholme.*)

If $p_1=\Sigma x$, $p_2=\Sigma xy$, $p_3=xyz$, $s_2=\Sigma x^2$, $s_3=\Sigma x^3$, &c., then we are required to prove that

$$\frac{s_{11}}{11}=\frac{s_3 s_8}{6}-\frac{s_3{}^3 s_2}{18} \qquad (5').$$

We know that s_{11} is a rational function of p_1, p_2, p_3. In the present case $p_1=0$, and we need only write down those terms which do not contain p_1. We thus have

$$s_{11}=\mathrm{A}p_2{}^4p^3+\mathrm{B}p_2p_3{}^3 \qquad (6),$$

provided $x+y+z=0$.

A may be most simply determined by putting $z=-(x+y)$, writing out both sides of (6) as functions of x and y, dividing by xy, and comparing the coefficients of x^9. We thus find $\mathrm{A}=11$.

We have therefore

$$s_{11}=11p_2{}^4p_3+\mathrm{B}p_2p_3{}^3.$$

In this last equation we may give x, y, z any values consistent with $x+y+z=0$, say $x=2$, $y=-1$, $z=-1$. We thus get $\mathrm{B}=-11$. Hence

$$s_{11}=11p_2{}^4p_3-11p_2p_3{}^3 \qquad (7).$$

In like manner we have

$$s_8=\mathrm{A}p_2{}^4+\mathrm{B}p_2p_3{}^2.$$

Putting in this equation first $x=1$, $y=-1$, $z=0$, and then $x=2$, $y=-1$, $z=-1$, we find $\mathrm{A}=2$, $\mathrm{B}=-8$.

Hence $$s_8=2p_2{}^4-8p_2p_3{}^2 \qquad (8).$$

We also find $$s_3=3p_3 \qquad (9),$$
$$s_2=-2p_2 \qquad (10).$$

From (8), (9), and (10) we deduce

$$\begin{aligned}\frac{s_3s_8}{6}-\frac{s_3{}^3s_2}{18}&=p_3(p_2{}^4-4p_2p_3{}^2)+3p_3{}^3p_2,\\&=p_2{}^4p_3-p_2p_3{}^3,\\&=\frac{s_{11}}{11},\end{aligned}$$

which is the required equation.

Since we have four equations, (7), (8), (9), (10), and only two quantities, p_2, p_3, to eliminate, we can of course obtain an infinity of different relations, such as (5); all these will, however, be equivalent to two independent equations, say to (5), and

$$72s_8=9s_2{}^4+4s_2s_3{}^2 \qquad (11).$$

Example 4.

Eliminate x, y, z from the equations $x+y+z=0$, $x^3+y^3+z^3=a$, $x^5+y^5+z^5=b$, $x^7+y^7+z^7=c$.

Using the same notation as in last example, we can show that

$$s_3=3p_3, \quad s_5=-5p_2p_3, \quad s_7=7p_2{}^2p_3.$$

Our elimination problem is therefore reduced to the following:—

To eliminate p_2 and p_3 from the equations

$$3p_3=a, \quad -5p_2p_3=b, \quad 7p_2{}^2p_3=c.$$

This can be done at once. The result is

$$21b^2-25ac=0.$$

EXERCISES XXXVI.

α and β being the roots of the equation $x^2+px+q=0$, express the following in terms of p and q:—

(1.) $\alpha^5+\beta^5$. (2.) $(\alpha^6+\beta^6)/(\alpha-\beta)^2$. (3.) $\alpha^{-5}+\beta^{-5}$. (4.) $\alpha^{-5}-\beta^{-5}$.
(5.) $(\alpha^3+\beta^3)^{-1}+(\alpha^3-\beta^3)^{-1}$. (6.) $(1-\alpha)^2\beta^2+(1-\beta)^2\alpha^2$.

(7.) If the sum of the roots of a quadratic be A, and the sum of their cubes B^3, find the equation.

(8.) If s_n denote the sum of the nth powers of the roots of a quadratic, then the equation is

$$(s_n s_{n-2}-s_{n-1}{}^2)x^2-(s_{n+1}s_{n-2}-s_n s_{n-1})x+(s_{n+1}s_{n-1}-s_n{}^2)=0.$$

(9.) If α and β be the roots of $x^2+px+q=0$, find the equation whose roots are $(\alpha-h)^2$, $(\beta-h)^2$.

(10.) Prove that the roots of

$$x^2-(2p-q)x+p^2-pq+q^2=0$$

are $p+\omega q$, $p+\omega^2 q$, ω and ω^2 being the imaginary cube roots of 1.

(11.) If α, β be the roots of x^2+x+1, prove that $\alpha^n+\beta^n=2$, or $=-1$, according as n is or is not a multiple of 3.

(12.) Find the condition that the roots of $ax^2+bx+c=0$ may be deducible from those of $a'x^2+b'x+c'=0$ by adding the same quantity to each root.

(13.) If the differences between the roots of $x^2+px+q=0$ and $x^2+qx+p=0$ be the same, show that either $p=q$ or $p+q+4=0$. What peculiarity is there when $p=q$?

Calculate the following functions of α, β, γ in terms of $p_1=\Sigma\alpha$, $p_2=\Sigma\alpha\beta$, $p_3=\alpha\beta\gamma$:—

(14.) $\alpha^2/\beta\gamma+\beta^2/\gamma\alpha+\gamma^2/\alpha\beta$. (15.) $\alpha^{-5}+\beta^{-5}+\gamma^{-5}$.
(16.) $(\beta^2+\gamma^2)(\gamma^2+\alpha^2)(\alpha^2+\beta^2)$. (17.) $\Sigma(\alpha^2+\beta\gamma)/(\alpha^2-\beta\gamma)$.
(18.) $\Sigma(\beta-\gamma)^2$. (19.) $\Sigma(\alpha-\beta)^2(\beta-\gamma)^2$. (20.) $\Sigma(\beta+\gamma)^3$.

Calculate the following functions of α, β, γ, δ in terms of the elementary symmetric functions:—

(21.) $\Sigma\alpha^4$. (22.) $\Sigma\alpha^{-3}$. (23.) $\Sigma\alpha^2\beta^2$. (24.) $\Sigma\alpha^2\beta\gamma$. (25.) $\Sigma(\alpha+\beta)^4$.

(26.) If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic $x^4+p_1x^3+p_2x^2+p_3x+p_4=0$, find the equation whose roots are $\beta\gamma+\alpha\delta$, $\gamma\alpha+\beta\delta$, $\alpha\beta+\gamma\delta$.

(27.) If the roots of $x^2-p_1x+p_2=0$, $x^2-q_1x+q_2=0$, $x^2-r_1x+r_2=0$, be β, γ; γ, α; α, β respectively, then α, β, γ are the roots of

$$x^3-\tfrac{1}{2}(p_1+q_1+r_1)x^2+(p_2+q_2+r_2)x-\tfrac{1}{2}(p_1q_1r_1-p_1p_2-q_1q_2-r_1r_2)=0.$$

(28.) If α, β, γ be the roots of $x^3+px+q=0$, show that the equation whose roots are $\alpha+\beta\gamma$, $\beta+\gamma\alpha$, $\gamma+\alpha\beta$, is $x^3-px^2+(p+3q)x+q-(p+q)^2=0$.

(29.) If α, β, γ be the roots of

$$p/(a+x)+q/(b+x)+r/(c+x)=1,$$

show that $p=(a+\alpha)(a+\beta)(a+\gamma)/(a-b)(a-c)$.

If α, β be the roots of $x^2+p_1x+p_2=0$, and α', β' the roots of $x^2+p_1'x+p_2'=0$, express the following in terms of p_1, p_2, p_1', p_2':—

(30.) $(\alpha'-\alpha)(\beta'-\beta)+(\alpha'-\beta)(\beta'-\alpha)$.

(31.) $(\alpha' - \alpha)^2 + (\beta' - \beta)^2 + (\alpha' - \beta)^2 + (\beta' - \alpha)^2$.

(32.) $(\alpha + \alpha')(\beta + \beta')(\alpha + \beta')(\beta + \alpha')$.

(33.) $4(\alpha - \alpha')(\alpha - \beta')(\beta - \alpha')(\beta - \beta')$.

[The result in this case is

$4(p_2 - p'_2)^2 + 4(p_1 - p_1')(p_1p_2' - p_1'p_2) = (2p_2 + 2p_2' - p_1p_1')^2 - (p_1^2 - 4p_2)(p_1'^2 - 4p_2')$.]

(34.) A, A′ and B, B′ are four points on a straight line whose distances, from a fixed point O on that line (right or left according as the algebraic values are positive or negative), are the roots of the equations

$$ax^2 + bx + c = 0, \quad a'x^2 + b'x + c' = 0.$$

If $$AA' . BB' + AB' . BA' = 0,$$

show that $$2ca' + 2c'a - bb' = 0\ ;$$

and if $$AA' . BA' + AB' . BB' = 0,$$

that $$2ca'^2 - 2c'aa' + ab'^2 - a'bb' = 0.$$

(35.) α, β are the roots of $x^2 - 2ax + b^2 = 0$, and α', β' the roots of $x^2 - 2cx + d^2 = 0$. If $\alpha\alpha' + \beta\beta' = 4n^2$, show that

$$(a^2 - b^2)(c^2 - d^2) = (ac - 2n^2)^2.$$

(36.) α, β, α', β', being as in last exercise, form the equation whose roots are $\alpha\alpha' + \beta\beta'$, $\alpha\beta' + \alpha'\beta$.

(37.) If the roots of $ax^2 + bx + c = 0$ be the square roots of the roots of $a'x^2 + b'x + c' = 0$, show that $a'b^2 + a^2b' = 2aa'c$.

(38.) Show that when two roots of a cubic are equal, its roots can always be obtained by means of a quadratic equation.

Exemplify by solving the equation $12x^3 - 56x^2 + 87x - 45 = 0$.

(39.) If one of the roots of the cubic $x^3 + p_1x^2 + p_2x + p_3 = 0$ be equal to the sum of the other two, solve the cubic. Show that in this case the coefficients must satisfy the relation

$$p_1^3 - 4p_1p_2 + 8p_3 = 0.$$

(40.) If the square of one of the roots of the cubic $x^3 + p_1x^2 + p_2x + p_3 = 0$ be equal to the product of the other two, show that one of the roots is $-p_2/p_1$; and that the other two are given by the quadratic

$$p_1p_2x^2 + p_2(p_1^2 - p_2)x + p_1^2p_3 = 0.$$

As an example of this case, solve the cubic

$$x^3 - 9x^2 - 63x + 343 = 0.$$

(41.) If two roots of the biquadratic $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$ be equal, show that the repeated root is a common root of the two equations

$$4x^3 + 3p_1x^2 + 2p_2x + p_3 = 0, \quad x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0.$$

(42.) If the three variables x, y, z be connected by the relation $\Sigma x = xyz$, show that $\Sigma 2x/(1 - x^2) = \Pi 2x/(1 - x^2)$.

(43.) If $\Sigma x = 0$, show that $2\Sigma x^7 = 7xyz\Sigma x^4$.

(44.) If $\Sigma x = 0$, show that $\Sigma x^8 = 2(\Sigma yz)^4 - 8x^2y^2z^2\Sigma yz$.

(45.) If $\Sigma a = 0$ (three variables), then $(\Sigma(b^3 - c^3)/a^3)(\Sigma a^3/(b^3 - c^3)) = 36 - 4(\Sigma a^3)(\Sigma a^{-3})$.

(46.) If $\Sigma x^3 = 0$, $\Sigma x^4 = 0$ (three variables), show that $\Sigma x^5 + xyz\{(y - z)(z - x)(x - y)\}^{\frac{2}{3}} = 0$.

(47.) If $x+y+z+u=0$, show that $(\Sigma x^3)^2=9(\Sigma xyz)^2=9\Pi(yz-xu)$.

(48.) Under the hypothesis of last exercise, show that

$$ux(u+x)^2+yz(u-x)^2+uy(u+y)^2+zx(u-y)^2+uz(u+z)^2+xy(u-z)^2+4xyzu=0.$$

Eliminate x, y, z between the equations

(49.) $\Sigma\left(\frac{x}{a}\right)=l$, $\Sigma\left(\frac{x}{a}\right)^2=m^2$, $\Sigma\left(\frac{x}{a}\right)^3=n^3$, $\Sigma\left(\frac{x}{a}\right)^4=p^4$.

(50.) $\Sigma x^2=a^2$, $\Sigma xy=b^2$, $\Sigma x^2y^2=c^4$, $\Sigma x^3=d^3$.

(51.) Show that $p_1={}_sp_1-a_s$, $p_2={}_sp_2-{}_sp_1a_s$, . . ., $p_r={}_sp_r-{}_sp_{r-1}a_s$, . . ., $p_n=-{}_sp_{n-1}a_s$, where p_1, p_2, . . ., ${}_sp_1, {}_sp_2$, . . . have the same meanings as in § 3.

SPECIAL PROPERTIES OF QUADRATIC FUNCTIONS.

§ 5.] *Discrimination of Roots.*—We have already seen (chap. xvii., § 4) how, without solving a quadratic equation, to distinguish between cases where the roots are real, equal, or imaginary. There are a variety of other cases that occur in practice for which it is convenient to have criteria. These may be treated by means of the relations between the roots and the coefficients of the equation given in § 1 of the present chapter. If α, β be the roots of

$$ax^2+bx+c=0 \qquad (1),$$

then
$$\alpha+\beta=-b/a, \quad \alpha\beta=c/a.$$

If both α and β be positive, then both $\alpha+\beta$ and $\alpha\beta$ are positive. Conversely, if $\alpha\beta$ be positive, α and β must have like signs; and if $\alpha+\beta$ be also positive, each of the two signs must be positive; but if $\alpha+\beta$ be negative, each of the two signs must be negative. Hence *the necessary and sufficient condition that both roots of* (1) *be positive is that b/a be negative and c/a positive; and the necessary and sufficient condition that both roots be negative is that b/a be positive and c/a positive.* This presupposes, of course, that the condition for the reality of the roots be fulfilled, namely, $b^2-4ac>0$.

Reality being presupposed, *the necessary and sufficient condition that the roots have opposite signs is obviously that c/a be negative.*

The necessary and sufficient condition that the two roots be numerically equal, but of opposite sign, is $\alpha+\beta=0$, that is, $b/a=0$.

If one root vanish, then $\alpha\beta=0$; and, conversely, if $\alpha\beta=0$, then at least one of the two, α, β, must vanish. *Hence the neces-*

sary and sufficient condition for one zero root is $c/a = 0$, that is, $c = 0$, a being supposed finite.

If both roots vanish, then $\alpha\beta = 0$ and $\alpha + \beta = 0$; and, conversely, if $\alpha\beta = 0$ and $\alpha + \beta = 0$, then both $\alpha = 0$ and $\beta = 0$; for the first equation requires that either $\alpha = 0$ or $\beta = 0$, say $\alpha = 0$; then the second gives $0 + \beta = 0$, that is, $\beta = 0$ also. *Hence the necessary and sufficient condition for two zero roots is $c/a = 0$, $b/a = 0$, that is,* a being supposed finite, $c = 0$, $b = 0$.

The two last conclusions have already been arrived at in chap. xvii., § 2. Perhaps they will be more fully understood by considering the case as a limit. Let us suppose that the root α remains finite, and that the root β becomes very small. Then $\alpha\beta$ becomes very small, and approaches zero as its limit, while $\alpha + \beta$ approaches α as its limit. In other words, c/a becomes very small, and $-b/a$ remains finite, becoming in the limit equal to the finite root of the quadratic.

If both α and β become infinitely small, then both $\alpha + \beta$ and $\alpha\beta$, that is to say, both $-b/a$ and c/a, become infinitely small.

Infinite Roots.—If the quadratic (1) have no zero root, it is equivalent to

$$a + b\left(\frac{1}{x}\right) + c\left(\frac{1}{x}\right)^2 = 0,$$

that is, if $\xi = 1/x$, to

$$c\xi^2 + b\xi + a = 0 \qquad (2).$$

The roots of (2) are $1/\alpha$ and $1/\beta$; and we have $1/\alpha + 1/\beta = -b/c$, $1/\alpha\beta = a/c$. Let us suppose that one of the two, α, β, say β, becomes infinitely great, while the other, α, remains finite; then $1/\beta$ becomes infinitely small, and $1/\alpha\beta$, that is, a/c, becomes infinitely small, while $1/\alpha + 1/\beta$, that is, $-b/c$, approaches the finite value $1/\alpha$. Hence the necessary and sufficient condition that one root of (1) be infinite is $a = 0$, c being supposed finite.

In like manner, the condition that two roots of (1) become infinite, that is, that two roots of (2) become zero, is $a = 0$, $b = 0$.

If therefore in any case WHERE A QUADRATIC EQUATION IS IN QUESTION *we obtain an equation of the form $bx + c = 0$, or an equation of the paradoxical form $c = 0$, we conclude that one root of the*

quadratic has become infinite in the one case, and that the two roots have become infinite in the other.

For convenience of reference we collect the criteria for discriminating the roots in the following table:—

Roots real . .	$b^2-4ac>0$.	Both roots negative	$c/a+$, $b/a+$.
Roots imaginary .	$b^2-4ac<0$.	Roots of opposite signs . . .	$c/a-$.
Roots equal . .	$b^2-4ac=0$.	One root $=0$.	$c=0$.
Roots equal with opposite signs .	$b=0$.	Two roots $=0$.	$b=0$, $c=0$.
Both roots positive	$c/a+$. $b/a-$.	One root $=\infty$.	$a=0$.
		Two roots $=\infty$.	$b=0$, $a=0$.

§ 6.] The reader should notice that some of the results embodied in the table of last paragraph can be easily generalised. Thus, for example, it can be readily shown that *if in the equation*

$$p_0x^n+p_1x^{n-1}+\ .\ .\ .\ +p_n=0 \qquad (1)$$

the last r coefficients all vanish, then the equation will have r zero roots; and if the first r coefficients all vanish it will have r infinite roots.

Again, if $p_1=0$, the algebraic sum of the roots will be zero; and so on.

It is not difficult to find the condition that two roots of any equation be equal. We have only to express, by the methods already explained, the symmetric function $\Pi(\alpha_1-\alpha_2)^2$ of the roots in terms of $p_0, p_1, \ldots, p_n$, and equate this to zero. For it is obvious that if the product of the squares of all the differences of the roots vanish, two roots at least must be equal, and conversely.

For example, in the case of the cubic

$$x^3+p_1x^2+p_2x+p_3=0 \qquad (2),$$

whose roots are α, β, γ, we find

$$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-4p_1^3p_3+p_1^2p_2^2+18p_1p_2p_3-4p_2^3-27p_3^2.$$

The condition for equal roots is therefore

$$-4p_1^3p_3+p_1^2p_2^2+18p_1p_2p_3-4p_2^3-27p_3^2=0.$$

Further, if all the roots of the cubic be real, $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$ will be positive, and if two of them be imaginary, say $\beta=\lambda+\mu i$, $\gamma=\lambda-\mu i$, then $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-4\mu^2\{(\lambda-\alpha)^2+\mu^2\}^2$, that is, $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$ is negative. Hence the roots of (2) are real and unequal, such that two at least are equal, or such that two are imaginary, according as

$$-4p_1^3p_3+p_1^2p_2^2+18p_1p_2p_3-4p_2^3-27p_3^2$$

is positive, zero, or negative.

The further pursuit of this matter belongs to the higher theory of equations.

§ 7.] *If the two quadratic equations*

$$ax^2+bx+c=0, \quad a'x^2+b'x+c'=0$$

be equivalent, then $b/a=b'/a'$ *and* $c/a=c'/a'$. For, if the roots of each be a and β, then

$$b/a=-(a+\beta)=b'/a', \quad c/a=a\beta=c'/a';$$

and this condition is obviously sufficient.

The above proposition leads to the following: *A quadratic function of x is completely determined when its roots are given, and also the value of the function corresponding to any value of x which is not a root.* This we may prove independently as follows. Let the roots of the function y be a and β; then $y\equiv A(x-a)(x-\beta)$. Now, if V be the value of y when $x=\lambda$, say, then we must have

$$V=A(\lambda-a)(\lambda-\beta).$$

This equation determines the value of A, and we have, finally,

$$y\equiv V\frac{(x-a)(x-\beta)}{(\lambda-a)(\lambda-\beta)} \qquad (1).$$

The result thus arrived at is only a particular case of the following: *An integral function of the nth degree is uniquely determined when its $n+1$ values, $V_1, V_2, \ldots, V_{n+1}$, corresponding respectively to the $n+1$ values $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ of its variable x, are given.* To prove this we may consider the case of a quadratic function.

Let the required function be ax^2+bx+c; then, by the conditions of the problem, we have

$$a\lambda_1^2+b\lambda_1+c=V_1, \quad a\lambda_2^2+b\lambda_2+c=V_2, \quad a\lambda_3^2+b\lambda_3+c=V_3.$$

These constitute a linear system to determine the unknown coefficients a, b, c. This system cannot have more than one definite solution. Moreover, there is in general one definite solution, for we can construct synthetically a function to satisfy the required conditions, namely,

$$y \equiv V_1 \frac{(x-\lambda_2)(x-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + V_2 \frac{(x-\lambda_1)(x-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + V_3 \frac{(x-\lambda_1)(x-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \qquad (2).$$

The reasoning and the synthesis are obviously general. We obtain, as the solution of the corresponding problem for an integral function of x of the nth degree,

$$y \equiv \Sigma V_1 \frac{(x-\lambda_2)(x-\lambda_3)\ldots(x-\lambda_{n+1})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_{n+1})} \qquad (3).$$

This result is called Lagrange's Interpolation Formula.

Example 1.

Find the quadratic equation with real coefficients one of whose roots is $5+6i$.

Since the coefficients are real, the other root must be $5-6i$. Hence the required equation is

$$(x-\overline{5+6i})(x-\overline{5-6i})=0,$$

that is,
$$(x-5)^2+6^2=0,$$
that is,
$$x^2-10x+61=0.$$

Example 2.

Find the quadratic equation with rational coefficients one of whose roots is $3+\sqrt{7}$.

Since the coefficients are rational,* it follows that the other root must be $3-\sqrt{7}$. Hence the equation is

$$(x-\overline{3+\sqrt{7}})(x-\overline{3-\sqrt{7}})=0,$$

that is,
$$x^2-6x+2=0.$$

Example 3.

Find the equation of lowest degree with rational coefficients one of whose roots is $\sqrt{2}+\sqrt{3}$.

By the principles of chap. x.* it follows that each of the quantities $\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$, $-\sqrt{2}-\sqrt{3}$ must be a root of the required equation. Hence the equation is

$$(x-\sqrt{2}-\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x+\sqrt{2}+\sqrt{3})=0,$$

that is,
$$x^4-10x^2+1=0.$$

Example 4.

Construct a quadratic function of x whose values shall be 4, 4, 6, when the values of x are 1, 2, 3 respectively.

---

* This we have not explicitly proved; but we can establish, by reasoning similar to that employed in chap. xii., § 5, Cor. 4, that, if $a+b\sqrt{p}$ be a root of $f(x)=0$, and if a and b and also all the coefficients of $f(x)$ be rational so far as $\sqrt{p}$ is concerned, then $a-b\sqrt{p}$ is also a root of $f(x)=0$.

The required function is

$$4\frac{(x-2)(x-3)}{(1-2)(1-3)}+4\frac{(x-1)(x-3)}{(2-1)(2-3)}+6\frac{(x-1)(x-2)}{(3-1)(3-2)},$$

that is,

$$x^2-3x+6.$$

§ 8.] The condition that the two equations

$$ax^2+bx+c=0, \quad a'x^2+b'x+c'=0$$

have one root in common is the same as the condition that the two integral functions

$$y=ax^2+bx+c, \quad y'=a'x^2+b'x+c'$$

shall have a linear factor in common. Now any common factor of y and y' is a common factor of

$$c'y-cy', \quad \text{and} \quad ay'-a'y;$$

that is, if we denote $ac'-a'c$ by (ac'), &c., a common factor of

$$(ac')x^2+(bc')x, \quad \text{and} \quad (ab')x+(ac');$$

that is, since x is not a common factor of y and y' unless $c=0$ and $c'=0$, any common factor of y and y' is a common factor of

$$(ac')x+(bc'), \quad \text{and} \quad (ab')x+(ac').$$

Now, if these two linear functions have a common factor of the 1st degree in x, the one must be the other multiplied by a constant factor.

Hence *the required condition is*

$$\frac{(ac')}{(ab')}=\frac{(bc')}{(ac')},$$

or

$$(ac'-a'c)^2=(bc'-b'c)(ab'-a'b).$$

The common root of the two equations is, of course,

$$x=-\frac{bc'-b'c}{ac'-a'c}=-\frac{ac'-a'c}{ab'-a'b}.$$

By the process here employed we could find the r conditions that two integral equations should have r roots in common.

It is important to notice that the process used in the demonstration is simply that for finding the G.C.M. of two integral functions—a process in which no irrational operations occur. Hence

Cor. 1. *If two integral equations have r roots in common, these roots are the roots of an integral equation of the rth degree, whose coefficients are rational functions of the coefficients of the given equations.*

In particular, if the coefficients of the two equations be real rational numbers, the r common roots must be the roots of an equation of the rth degree with rational coefficients.

For example, two quadratics whose coefficients are all rational cannot have a single root in common unless it be a rational root.

Cor 2. We may also infer that *if two integral equations whose coefficients are rational have an odd number of roots in common, then one at least of these must be real.*

Exercises XXXVII.

Discriminate the roots of the following quadratic equations without solving them :—

(1.) $4x^2-8x+3=0$. (2.) $9x^2-12x-1=0$. (3.) $4x^2-4x+6=0$.

(4.) $9x^2-36x+36=0$. (5.) $4x^2-4x-3=0$. (6.) $4x^2+8x+3=0$.

(7.) $(x-3)(x+4)+(x-2)(x+3)=0$.

(8.) Show that the roots of $(b^2-4ac)x^2+4(a+c)x-4=0$ are always real; and find the conditions—1° that both be positive, 2° that they have opposite signs, 3° that they be both negative, 4° that they be equal, 5° that they be equal but of opposite sign.

(9.) Show that the roots of $x^2+2(p+q)x+2(p^2+q^2)=0$ are imaginary.

(10.) Show that the roots of

$$\{c^2-2bc+b^2\}x^2-2\{c^2-(a+b)c+ab\}x+\{2a^2-2(b+c)a+b^2+c^2\}=0$$

are imaginary.

(11.) Show that the roots of

$$(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0$$

are real; and that they cannot be equal unless $a=b=c$.

(12.) The roots of $a/(x-a)+b/(x-b)+c/(x-c)=0$ are real; and cannot be equal unless either two of the three, a, b, c, are zero, or else $a=b=c$.

(13.) Find the condition that the cubic $x^3+qx+r=0$ have equal roots.

(14.) Show that the cubic $12x^3-52x^2+75x-36=0$ has equal roots; and solve it.

(15.) If two of the roots of a cubic be equal, and its coefficients be all rational, show that all its roots must be rational.

(16.) Find the condition that two roots of the biquadratic $ax^4+dx+e=0$ be equal.

(17.) If $a/(x+a)+b/(x+b)=c/(x+c)+d/(x+d)$ have a pair of equal roots, then either one of the quantities a or b is equal to c or d, or else $1/a+1/b=1/c+1/d$. Prove also that the roots are then $-a$, $-a$, 0, $-b$, $-b$, 0, or 0, 0, $-2ab/(a+b)$.

Write down and simplify the equations whose roots are as follows:—

(18.) 1, 0. (19.) $\frac{1}{2}$, $-\frac{2}{3}$. (20.) $3+\sqrt{2}$, $3-\sqrt{2}$.

(21.) $(a+\sqrt{a^2-1})/(a-\sqrt{a^2-1})$, $(a-\sqrt{a^2-1})/(a+\sqrt{a^2-1})$.

Find the equations of lowest degree with real rational coefficients which have respectively the following for one root:—

(22.) $\alpha+\beta i$. (23.) $1+\sqrt{2}-\sqrt{3}$. (24.) $\sqrt{2}+i\sqrt{3}$.

(25.) $\sqrt[3]{2}+\sqrt[3]{4}$. (Result, $x^3-6x-6=0$.)

(26.) $\sqrt[3]{2}+\sqrt[3]{3}$. (Result, $x^9-15x^6-87x^3-125=0$.)

(27.) $\sqrt{(qr)}+\sqrt{(rp)}+\sqrt{(pq)}$.

(28.) Find the equation of the 6th degree two of whose roots are $1+\sqrt{2}$ and $1+\sqrt[4]{-1}$.

(29.) Find an equation with rational coefficients one of whose roots is $ap^{2/3}+bp^{1/3}+c$.

Hence show how to find the greatest integer in $ap^{2/3}+bp^{1/3}+c$ without extracting the cube roots.

(30.) Form the equation whose roots are $p+\alpha_1 q$, $p+\alpha_2 q$, . . ., $p+\alpha_{2n}q$, where α_1, α_2, . . ., α_{2n} are the imaginary $(2n+1)$th roots of 1, showing that the coefficients are all rational, and finding the general term of the equation.

(31.) Construct a quadratic function whose roots shall be equal with opposite signs, and whose values shall be 23 and 67 when $x=5$ and when $x=6$ respectively.

(32.) Construct a cubic function y corresponding to the following table of values:—

$$\begin{array}{lllll} x= & 2\cdot5, & 3, & 3\cdot5, & 4\,; \\ y= & 6, & 8, & 15, & 18. \end{array}$$

(33.) If $x^2+ax+bc=0$, $x^2+bx+ca=0$ have a common root, then their other roots satisfy $x^2+cx+ab=0$.

(34.) If $2(p+q+r)=\alpha^2+\beta^2+\gamma^2$, and the roots of $x^2+\alpha x-p=0$ be β and γ, and the roots of $x^2+\beta x-q=0$ be γ and α, then the equation whose roots are α and β is $x^2+\gamma x-r=0$.

VARIATION OF QUADRATIC FUNCTIONS FOR REAL VALUES OF THE VARIABLE.

§ 9.] The quadratic function

$$y=ax^2+bx+c$$

may be put in one or other of the three forms

$$y=a\{(x-l)^2-m\} \qquad \text{I.,}$$
$$y=a\{(x-l)^2\} \qquad \text{II.,}$$
$$y=a\{(x-l)^2+m\} \qquad \text{III.,}$$

according as its roots α and β are real (say $\alpha=l+\sqrt{m}$,

$\beta = l - \sqrt{m}$), equal (say $\alpha = l, \beta = l$), or imaginary (say $\alpha = l + i\sqrt{m}$, $\beta = l - i\sqrt{m}$). l and m are both essentially real quantities, and m is positive.

Each of these three cases may be farther divided into two, according as a is positive or negative.

In all three cases when x is very great $(x-l)^2$ is very great and positive. Hence, in all three cases, y is infinite when x is infinite, and it has the same sign as a.

In all three cases the function within the crooked bracket diminishes in algebraical value when x diminishes, so long as $x > l$, and has an algebraically least value when $x = l$; for $(x-l)^2$, the only variable part, being essentially positive, cannot be less than zero. When x is diminished beyond the value $x = l$, $(x-l)^2$ continually increases in numerical value.

We conclude, therefore, that in all three cases *the quadratic function y has an algebraical minimum or maximum value when $x = l$, according as a is positive or negative; and that the function has no other turning value.*

In Case I., where the roots are real and unequal, y will have the same sign as a or not, according as the value of x does not or does lie between the roots.

For $y = a(x-\alpha)(x-\beta)$; and $(x-\alpha)(x-\beta)$ will be positive if x be algebraically greater than both α and β, for then $x-\alpha$ and $x-\beta$ are both positive; and the same will be true if x be algebraically less than both α and β, for then $x-\alpha$ and $x-\beta$ are both negative. If x lie between α and β, then one of the two, $x-\alpha$, $x-\beta$, is positive and the other negative.

In Cases II. and III., where the roots are either equal or imaginary, the function y will have the same sign as a for all values of x.

For in these cases the function within the crooked brackets has clearly a positive value for all real values of x.

§ 10.] The above conclusions may be reached by a different but equally instructive method as follows:—

Let us trace the graph of the function

$$y = ax^2 + bx + c \qquad (1);$$

and, for the present, suppose a to be positive.

To find the general character of the graph, let us inquire where it cuts a parallel to the axis of x, drawn at any given distance y from that axis. In other words, let us seek for the abscissæ of all points on the graph whose ordinates are equal to y.

We have

$$y = ax^2 + bx + c,$$

that is,

$$ax^2 + bx + (c - y) = 0 \qquad (2).$$

We have, therefore, a quadratic equation to determine the abscissæ of points on the parallel. Hence the parallel cuts the graph in two real distinct points, in two coincident real points, or in no real point, according as the roots of (2) are real and unequal, real and equal, or imaginary.

Since a is positive, it follows that when $x = -\infty$, $y = +\infty$; and when $x = +\infty$, $y = +\infty$. Moreover, the quadratic function y is continuous, and can only become infinite when x becomes infinite. Hence there must be one minimum turning point on the graph. There cannot be more than one, for, if there were, it would be possible to draw a parallel to the x-axis to meet the graph in more than two points.

The graph therefore consists of a single festoon, beginning and ending at an infinite distance above the axis of x.

The main characteristic point to be determined is the minimum point. To obtain this we have only to diminish y until the parallel UV (Fig. 1) just ceases to meet the graph. At this stage it is obvious that the two points U and V run together; that is to say, the two abscissæ corresponding to y become equal. Hence, to find y, we have simply to express the condition that the roots of (2) be equal. This condition is

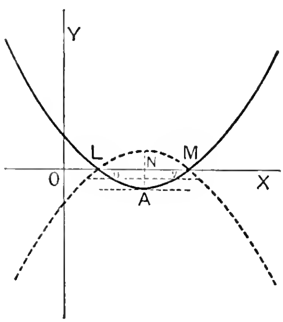


FIG. 1.

$$b^2 - 4a(c - y) = 0.$$

Hence

$$y = -\frac{b^2 - 4ac}{4a} \qquad (3).$$

The corresponding value of x is easily obtained, if we notice that the sum of the roots of (2) is in all cases $-b/a$, and that when the two are equal each must be equal to $-b/2a$. Hence the abscissa of the minimum point is given by

$$x = -\frac{b}{2a} \qquad (4).$$

There are obviously three possible cases—

I. The value of y given by (3) may be negative. Since a is supposed positive, this will happen when $b^2 - 4ac$ is positive.

In this case the minimum point A will lie below the axis of x, and the graph will be like the fully drawn curve in Fig. 1.

Here the graph must cut the x-axis, hence the function y must have two real and unequal roots, namely, $x = \mathrm{OL}$, $x = \mathrm{OM}$; and it is obvious that y is positive or negative, that is, has the same sign as a or the opposite, according as x does not or does lie between OL and OM.

II. The value of y given by (3) will be zero, provided $b^2 - 4ac = 0$.

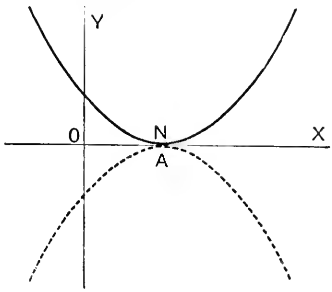


Fig. 2.

In this case the minimum point A falls on the axis of x, and the graph will be like the fully drawn curve in Fig. 2.

Here the two roots of the function are equal, namely, each is equal to OA.

It is obvious that here y is always positive, that is, has the same sign as a.

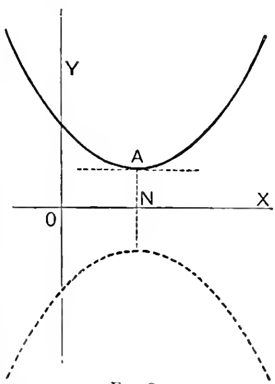


Fig. 3.

III. The value of y given by (3) will be positive, provided $b^2 - 4ac$ be negative.

In this case the graph will be like the fully drawn curve in Fig. 3.

Here the graph does not cut the axis of x, so that the function has no real roots. Also y is always positive, that is, has the same sign as a.

If we suppose a to be negative, the discussion proceeds exactly as before, except that for positive we must say negative, and for minimum maximum. The typical graphs in the three cases will be obtained by taking the mirror-images in the axis of x of those already given. These graphs are indicated by dotted lines in Figs. 1, 2, 3.

For simplicity we have supposed the abscissæ of the points L, M, N, A to be positive in all cases. It will of course happen in certain cases that one or more of these are negative. The corresponding figures are obtained in all cases simply by shifting the axis of y through a proper distance to the right.

Example 1.

To find for what values of x the function $y=2x^2-12x+13$ is negative, and to find its turning value.

We have
$$\begin{aligned} y &= 2(x^2-6x+9)-5, \\ &= 2\{(x-3)^2-\tfrac{5}{2}\}, \\ &= 2\{x-(3-\sqrt{\tfrac{5}{2}})\}\{x-(3+\sqrt{\tfrac{5}{2}})\}. \end{aligned}$$

Hence y will be negative if x lie between $3-\sqrt{(5/2)}$ and $3+\sqrt{(5/2)}$, and will be positive for all other values of x.

Again, it is obvious, from the second form of the function, that y is algebraically least when $(x-3)^2=0$. Hence $y=-5$ is a minimum value of y corresponding to $x=3$.

Example 2.

To find the turning values of $(x^2-8x+15)/x$.

We have
$$y=x+\frac{15}{x}-8.$$

First, suppose x to be positive, then we may write
$$y=\left(\sqrt{x}-\sqrt{\frac{15}{x}}\right)^2-8+2\sqrt{15},$$
from which it appears that y has a minimum value, $-8+2\sqrt{15}$, when $\sqrt{x}-\sqrt{(15/x)}=0$, that is, when $x=\sqrt{15}$.

Next, let x be negative, $=-\xi$ say, then we may write
$$\begin{aligned} y &= -\xi-\frac{15}{\xi}-8, \\ &= -8-2\sqrt{15}-\left(\sqrt{\xi}-\sqrt{\frac{15}{\xi}}\right)^2, \end{aligned}$$

from which we see that $-8-2\sqrt{15}$ is a maximum value of y corresponding to $\xi=\sqrt{15}$, that is, to $x=-\sqrt{15}$.

Example 3.

If x and y be both positive, then—

If $x+y$ be given, the greatest and least values of xy correspond to the least and greatest values of $(x-y)^2$; so that the maximum value of xy is obtained by putting $x=y$, if that be possible under the circumstances of the problem.

If xy be given, the greatest and least values of $x+y$ correspond to the greatest and least values of $(x-y)^2$; so that the minimum of $x+y$ is obtained by putting $x=y$, if that be possible under the circumstances of the problem.

These statements follow at once from the identity

$$(x+y)^2-(x-y)^2\equiv 4xy.$$

For, if $x+y=c$, then

$$4xy=c^2-(x-y)^2.$$

And, if $xy=d^2$, then

$$(x+y)^2=4d^2+(x-y)^2.$$

Hence the conclusions follow immediately, provided x and y, and therefore xy and $x+y$, be both positive.

These results might also be arrived at by eliminating the value of y by means of the given relation. Thus, if $x+y=c$, then $xy=x(c-x)=cx-x^2$ $=c^2/4-(c/2-x)^2$. Hence xy is made as large as possible by making x as nearly $=c/2$ as possible, and so on.

Many important problems in geometry regarding maxima and minima may be treated by the simple method illustrated in Example 3.

Example 4.

To draw through a point A within a circle a chord such that the sum of the squares of its segments shall be a maximum or a minimum.

Let r be the radius of the circle, d the distance of A from the centre, x and y the lengths of the segments of the chords.

Then, by a well-known geometrical proposition,

$$xy=r^2-d^2 \qquad (1).$$

Under this condition we have to make

$$u=x^2+y^2 \qquad (2)$$

a maximum or minimum.

Now, if we denote x^2 and y^2 by ξ and η, then ξ and η are two positive quantities; and, by (1), we have

$$\xi\eta=(r^2-d^2)^2 \qquad (3).$$

Hence, by Example 3, $\xi+\eta$ is a minimum when $=\eta$, and is a maximum when $(\xi-\eta)^2$ is made as great as possible. If we diminish η, it follows, by (3), that ξ increases. Hence $(\xi-\eta)^2$ will be made as great as possible by making ξ as great as possible.

Hence the sum of the squares on the segments of the chord is a minimum when it is bisected, and a maximum when it passes through the centre of the circle.

Example 5.

A and B are two points on the diameter of a circle, PQ a chord through B. To find the positions of PQ for which the area APQ is a maximum or a minimum.

Let O be the centre of the circle. The area OPQ bears to the area APQ the constant ratio OB : AB. Hence we have merely to find the turning values of the area OPQ.

Let $OB=a$, and let x denote the perpendicular from O on PQ. Then, if u denote the area OPQ, $u=x\sqrt{(r^2-x^2)}$.

We have therefore to find the turning values of u. Since u is positive, this is the same thing as finding the turning values of u^2. Now

$$u^2=x^2(r^2-x^2)=\frac{r^4}{4}-\left(x^2-\frac{r^2}{2}\right)^2.$$

There are two cases to consider. First, suppose $a>r/\sqrt{2}$. Then, since the least and greatest values of x allowable under the circumstances are 0 and a, we have, confining ourselves to half a revolution of the chord about A, three turning values. If we put $x=0$ we give to $(x^2-r^2/2)^2$ the greatest value which we can give it by diminishing x below $r/\sqrt{2}$. Hence $x=0$ gives a minimum value of OPQ.

If we put $x=r/\sqrt{2}$, we give $(x^2-r^2/2)^2$ its least possible numerical value. Hence, for $x=r/\sqrt{2}$, OPQ is a maximum.

If we put $x=a$, we give $(x^2-r^2/2)^2$ the greatest value which we can give it by increasing x beyond $r/\sqrt{2}$. Hence to $x=a$ corresponds a minimum value of OPQ.

Next, suppose $a<r/\sqrt{2}$. In this case we cannot make $x=$ or $>r/\sqrt{2}$. Hence, corresponding to $x=0$, we have, as before, OPQ a minimum. But now $(x^2-r^2/\sqrt{2})^2$ diminishes continually as x increases up to a. Hence, for $x=a$, OPQ is a maximum.

Remark.—This example has been chosen to illustrate a peculiarity that very often arises in practical questions regarding maxima and minima, namely, that all the theoretically possible values of the variable may not be admissible under the circumstances of the problem.

Example 6.

Given the perimeter of a right-angled triangle, to show that the sum of the sides containing the right angle is greatest when the triangle is isosceles.

Let x and y denote the two sides, p the given perimeter. Then the hypotenuse is $p-x-y$; and we have, by the condition of the problem,

$$\{p-(x+y)\}^2=x^2+y^2.$$

Hence
$$xy-p(x+y)=-\frac{p^2}{2}.$$

This again may be written

$$(p-x)(p-y)=\frac{p^2}{2} \qquad (1).$$

Under the condition (1) we have to make

$$u = x + y \tag{2}$$

a maximum.

If we put $\xi = p - x$, $\eta = p - y$, we have

$$\xi\eta = \frac{p^2}{2} \tag{3};$$

and we have to make $u = 2p - (\xi + \eta)$

a maximum; this is, to make $\xi + \eta$ a minimum. Now, under the condition (3), $\xi + \eta$ is a minimum when $\xi = \eta$. Hence $x + y$ is a maximum when $x = y$.

§ 11.] The method employed in § 10 for finding the turning points of a quadratic function is merely an example of the

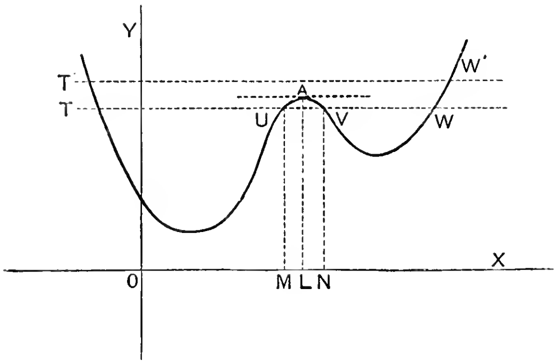


Fig. 4.

general method indicated in chap. xv., § 13. Consider any function whatever, say

$$y = f(x) \tag{1}.$$

Let A be a maximum turning point on its graph, whose abscissa and ordinate are x and y. If we draw a parallel to OX a little below A, it will intersect the graph in a certain number of points, TUVW say. Two of these will be in the neighbourhood of A, left and right of AL. If we move the parallel upwards until it pass through A, the two points U and V will run together at A, and their two abscissæ will become equal. If we move the parallel a little farther upwards, we lose two of the real intersections altogether.

Hence to find y we have simply to express the condition that the roots of the equation

$$f(x) - y = 0 \tag{2}$$

be equal, and then examine whether, if we increase y by a small amount, we lose two real roots or not. If we do, then y is a maximum value.

If it appears that two real roots are lost, not by increasing but by diminishing y, then y is a minimum value.

Example 1.

To find the turning values of

$$y=x^3-9x^2+24x+3.$$

The values of x corresponding to a given ordinate y are given by

$$x^3-9x^2+24x+(3-y)=0.$$

If D denote the product of the squares of the differences of the roots of this cubic, then all its roots will be real, two roots will be equal or two imaginary, according as D is positive, zero, or negative.

Using the value of D calculated in § 6, and putting $p_1=-9$, $p_2=24$, $p_3=3-y$, we find

$$D=-27(y-19)(y-23).$$

Hence $y=19$, $y=23$ are turning values of y. If we make y a little less than 19, D is negative, that is, two real roots of the cubic are lost. Hence 19 is a minimum value of y. If we make y a little greater than 23, D is again negative; hence 23 is a maximum value of y.

It is easy to obtain the corresponding values of x, if we remember that two of the roots of the cubic become equal when there is a turning value. In fact, if the two equal roots be α, α, and the third root γ, we have, by § 1,

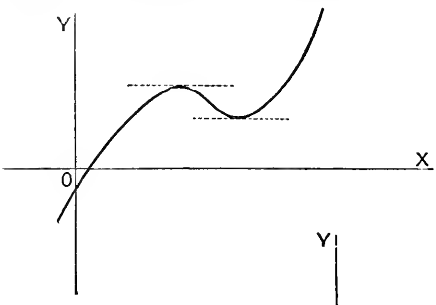


FIG. 5.

$$2\alpha+\gamma=9, \quad \alpha^2+2\alpha\gamma=24.$$

Hence

$$\alpha^2-6\alpha+8=0,$$

which gives

$$\alpha=2, \text{ or } \alpha=4.$$

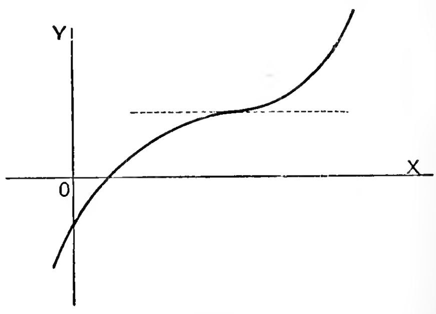


FIG. 6.

It will be found that $x=4$ corresponds to the minimum value $y=19$; and that $x=2$ corresponds to the maximum $y=23$.

Remark.—The above method is obviously applicable to any cubic integral function whatsoever, and we see that such a function has in general two turning values, which are the roots of a certain quadratic equation easily obtainable by means of the function D.

If the roots of this quadratic be real and unequal, there are two distinct turning points, one a maximum, the other a minimum.

If the roots be equal, we have a point which may be regarded as an amalgamation of a maximum point with a minimum, which is sometimes called a maximum-minimum point.

If the roots be imaginary, the function has no real turning point.

If the coefficient of x^3 be positive, the graphs in the first two cases have the general characters shown in Figs. 5 and 6 respectively.

Example 2.

To discuss the turning values of

$$y=\frac{x^2-8x+15}{x} \qquad (1).$$

The equation for the values of x corresponding to any given value of y is

$$x^2-(y+8)x+15=0.$$

Let D be the function b^2-4ac of § 5, whose sign discriminates the roots of a quadratic. In the present instance we have

$$D=(y+8)^2-60=\{y-(-8-\sqrt{60})\}\{y-(-8+\sqrt{60})\} \qquad (2).$$

Hence the turning values of y are

$$y=-8-\sqrt{(60)}, \text{ and } y=-8+\sqrt{(60)}.$$

If y has any value between these, D is negative, and the roots of (1) are imaginary. Hence the algebraically less of the two, namely, $-8-\sqrt{(60)}$, is a maximum; and the algebraically greater, namely, $-8+\sqrt{(60)}$, a minimum.

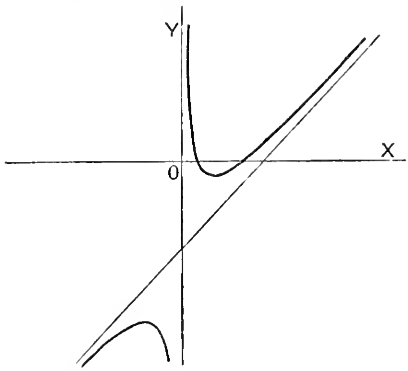


Fig. 7.

The values of x corresponding to these are at once obtained from the equation $x=(y+8)/2$. They are $x=-\sqrt{(15)}$ and $x=+\sqrt{(15)}$ respectively.

The reader should examine carefully the graph of this function, which has a discontinuity when $x=0$ (see chap. XV., § 5). We have the following series of corresponding values:—

$x=-\infty$,	-1,	-0,	$+0$,	$+3$,	>3 <5,	5,	>5,	$+\infty$,
$y=-\infty$,	-24,	$-\infty$,	$+\infty$,	0,	$-$	0,	$+$	$+\infty$.

Hence the graph is represented by Fig. 7.

Example 3.

To discuss generally the turning values of the function

$$y=\frac{ax^2+bx+c}{a'x^2+b'x+c'} \qquad (1).$$

The equation which gives the values of x corresponding to any given value of y is

$$(a-a'y)x^2+(b-b'y)x+(c-c'y)=0.$$

Let
$$\begin{aligned} D&=(b-b'y)^2-4(a-a'y)(c-c'y),\\ &=(b'^2-4a'c')y^2+2(2a'c+2ac'-bb')y+(b^2-4ac),\\ &=Ay^2+By+C, \text{ say.}\end{aligned}$$

Then we have
$$x=\frac{-(b-b'y)\pm\sqrt{D}}{2(a-a'y)} \qquad (2).$$

The turning values of y are therefore given by the equation

$$Ay^2+By+C=0 \qquad (3).$$

I. If $B^2-4AC>0$, this equation will have real unequal roots, and there will be two real turning values of y.

If A be positive, then, for real values of x, y cannot lie between the roots of the equation (3). Hence the less root will be a maximum and the greater a minimum value of y.

If A be negative, then, for real values of x, y must lie between the roots of (3). Hence the less root will be a minimum and the greater a maximum value of y.

II. If $B^2-4AC<0$, the equation has no real root, and D has always the same sign as A. In this case the sign of A must of necessity be positive; for, if it were not, there would be no real value of x corresponding to any value of y whatever.

Hence there is a real value of x corresponding to any given value of y whatever; and y has no turning values.

III. If $B^2-4AC=0$, we may apply the same general reasoning as in Case II. The present case has, however, a special peculiarity, as we shall see immediately.

The criteria for distinguishing these three cases may be expressed in terms of the roots α, β and α', β' of the two functions ax^2+bx+c and $a'x^2+b'x+c'$, and in this form they are very useful.

We have

$$\begin{aligned} B^2-4AC&=4(2ac'+2a'c-bb')^2-4(b^2-4ac)(b'^2-4a'c'),\\ &=4a^2a'^2\left\{\left(2\frac{c'}{a'}+2\frac{c}{a}-\frac{b}{a}\frac{b'}{a}\right)^2-\left(\frac{b^2}{a^2}-4\frac{c}{a}\right)\left(\frac{b'^2}{a'^2}-4\frac{c'}{a'}\right)\right\},\\ &=4a^2a'^2\{[2\alpha'\beta'+2\alpha\beta-(\alpha+\beta)(\alpha'+\beta')]^2-(\alpha-\beta)^2(\alpha'-\beta')^2\},\\ &=4a^2a'^2\{2\alpha'\beta'+2\alpha\beta-(\alpha+\beta)(\alpha'+\beta')-(\alpha-\beta)(\alpha'-\beta')\}\\ &\qquad\times\{2\alpha'\beta'+2\alpha\beta-(\alpha+\beta)(\alpha'+\beta')+(\alpha-\beta)(\alpha'-\beta')\},\\ &=16a^2a'^2(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta').\end{aligned}$$

Hence it appears that the sign of B^2-4AC depends merely on the sign of

$$E\equiv(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta') \qquad (4)$$

Since a, b, c, a', b', c' are all real, the roots of ax^2+bx+c and of $a'x^2+b'x+c'$, if imaginary, must be conjugate imaginaries. Hence, by reasoning as in § 6, we see that, if the roots of ax^2+bx+c, or of $a'x^2+b'x+c'$, or of both, be imaginary, E is positive.

The same is true if the roots of either or of both of these functions be equal.

Consider, next, the case where α, β, α', β' are all real and all unequal.

Since the sign of E is not altered if we interchange both α with α' and β with β', or both α with β and α' with β', we may, without losing generality, suppose that α is the algebraically least of the four, α, β, α', β', and that α' is algebraically less than β'. If we now arrange the four roots in ascending order of magnitude, there are just three possible cases, namely, $\alpha, \beta, \alpha', \beta'$; $\alpha, \alpha', \beta', \beta$; $\alpha, \alpha', \beta, \beta'$. In the first case, $\alpha-\alpha'$, $\alpha-\beta'$, $\beta-\alpha'$, $\beta-\beta'$ have all negative signs; in the second, two have negative signs, and two positive; in the third, three have negative signs, and one the positive sign. It is, therefore, in the third case alone that E has the negative sign. The peculiarity of this case is that each pair of roots is separated as to magnitude by one of the other pair. We shall describe this by saying that the roots *interlace*.

Lastly, suppose $E=0$. In this case one at least of the four factors, $\alpha-\alpha'$, $\beta-\beta'$, $\beta-\alpha'$, $\beta-\beta'$, must vanish; that is to say, the two functions ax^2+bx+c and $a'x^2+b'x+c'$ must have at least one root, and therefore at least one linear factor in common.*

Hence, in this case, (1) reduces to

$$y=\frac{a(x-\alpha)}{a'(x-\alpha')} \qquad (5),$$

say. Hence we have

$$y=a\cdot\frac{x-\alpha'+\alpha'-\alpha}{a'(x-\alpha')}=\frac{a}{a'}+\frac{a(\alpha'-\alpha)}{a'(x-\alpha')} \qquad (6).$$

From (6) it appears that y has a discontinuity when $x=\alpha'$, passing from the value $+\infty$ to $-\infty$, or the reverse, as x passes through that value; but that, for all other values of x, y either increases or decreases continuously as x increases. Hence y has no real turning values in this case, unless we choose to consider the value $y=a/a'$, which corresponds to $x=\pm\infty$, as a maximum-minimum value.

The graph in this case, supposing a/a', α, and $\alpha'-\alpha$ to be both positive, is like Fig. 8, where $OA=\alpha$, $OA'=\alpha'$, $OB=a/a'$.

To sum up—

Case I. occurs when the roots of either or of both of the functions ax^2+bx+c, $a'x^2+b'x+c'$ are imaginary or equal, and when all the roots are real but not interlaced.

Case II. occurs when the roots of both quadratic functions are real and interlaced.

---

* In the case where they have two linear factors in common, y reduces to a constant, a case too simple to require any discussion.

Case III. occurs when the two quadratic functions have one or both roots in common. In this case y reduces to the quotient of two linear functions, or to a constant, and has no maximum or minimum value properly so called.

In the above discussion we have assumed that neither a nor a' vanish; in other words, that neither of the two quadratic functions has an infinite root. The cases where infinite roots occur are, however, really covered by the above

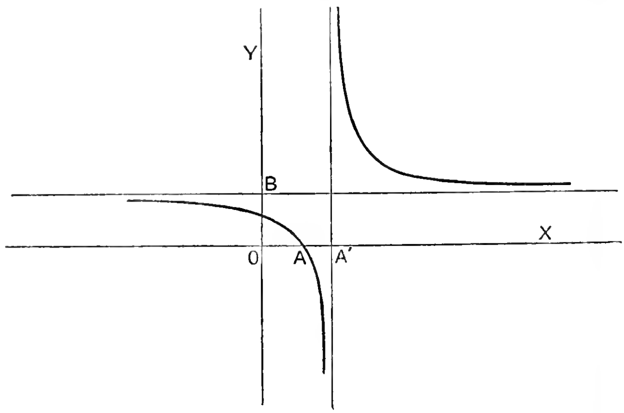


FIG. 8.

statements, as may be seen either by considering them as limits, or by working out the expression for B^2-4AC in terms of the finite roots in the particular instances in question.

In stating the above conclusions so generally as this, the student must remember that one of the turning values may either become infinite or correspond to an infinite value of x; otherwise he may find himself at a loss in certain cases to account for the apparent disappearance of a turning value.

A great variety of particular cases are included under the general case of this example. If we put $a'=0$, $c'=0$, for instance, we have the special case of Example 2.

As our object here is merely to illustrate methods, it will be sufficient to give the results in two more particular cases.

Example 4.

To trace the variation of the function

$$y=\frac{x^2-7x+6}{x^2-8x+15}.$$

The quadratic for x in terms of y is

$$(1-y)x^2-(7-8y)x+(6-15y)=0.$$

Hence

$$D=(7-8y)^2-4(1-y)(6-15y)=4\{y-(\tfrac{7}{2}-\sqrt{6})\}\{y-(\tfrac{7}{2}+\sqrt{6})\}.$$

Hence $7/2-\sqrt{6}$ and $7/2+\sqrt{6}$ are maximum and minimum values of y respectively. The corresponding values of x are given by

$$x = \tfrac{1}{2}\,\frac{7-8y}{1-y}\,;$$

and are $9+2\sqrt{6}$ and $9-2\sqrt{6}$ respectively. We observe farther that y is discontinuous when $x=3$ and when $x=5$; that when $x=+\infty$ or $=-\infty$, $y=1$; and that the other value of x for which $y=1$ is $x=9$.

We have thus the following table of corresponding values :—

$x=$	$-\infty$,	0,	$+1$,	$+3-0$,	$+3+0$,	$+4{\cdot}1$,
$y=$	$+1$,	$+{\cdot}4$,	0,	$-\infty$,	$+\infty$,	$+5{\cdot}9$, min.
$x=$	$+5-0$,	$+5+0$,	$+6$,	$+9$,	$+13{\cdot}9$,	$+\infty$,
$y=$	$+\infty$,	$-\infty$,	0,	$+1$,	$+\ 1{\cdot}05$, max.	$+1$.

The graph has the general form indicated in Fig. 9, which is not drawn to scale, but distorted in order to bring out more clearly the maximum point B.

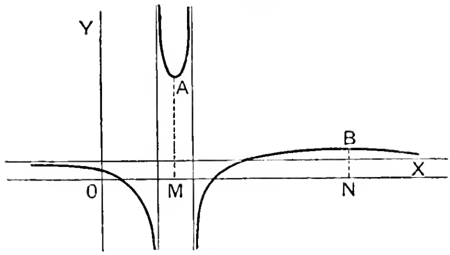


FIG. 9.

Example 5.

To trace the variation of the function

$$y=\frac{x^2-5x+4}{x^2-8x+15}.$$

The quadratic for x is

$$(1-y)x^2-(5-8y)x+(4-15y)=0.$$

Here we find

$$D=4\{(y-\tfrac{1}{2})^2+2\}.$$

Hence there are no real turning values.

The graph will be found to be as in Fig. 10.

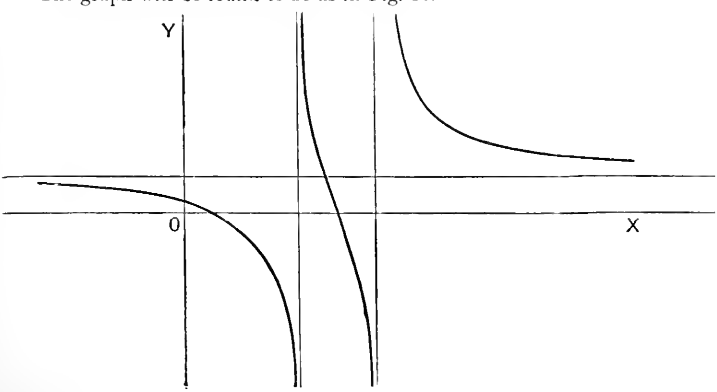


FIG. 10.

Example 6.

To find the turning values of $z=x^2+y^2$, given that $ax^2+bxy+cy^2=1$.

We have, since $ax^2+bxy+cy^2=1$,

$$z=\frac{x^2+y^2}{ax^2+bxy+cy^2}=\frac{\xi^2+1}{a\xi^2+b\xi+c},$$

where $\xi=x/y$.

We have now to find the turning values of z considered as a function of ξ. The quadratic for ξ is

$$(az-1)\xi^2+bz\xi+(cz-1)=0.$$

Hence the turning values of z are given by

$$b^2z^2=4(az-1)(cz-1),$$

that is, by

$$(b^2-4ac)z^2+4(a+c)z-4=0.$$

The result thus arrived at constitutes an analytical solution of the well-known problem to find the greatest and least central radii (that is, the semi-axes) of the ellipse whose equation is $ax^2+bxy+cy^2=1$.

Remark.—The artifice used in this example will obviously enable us to find the turning values of $u=f(x, y)$, when $\phi(x, y)=c$, provided $f(x, y)$ and $\phi(x, y)$ be homogeneous functions of x and y whose degree does not exceed the 2nd. Indeed it has a general application to all cases where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions; the only difficulty is in discriminating the roots of the resulting equation.

§ 12.] *Examination of the Increment.*—There is yet another method which is very useful in discussing the variation of integral functions. Suppose we give x any small increment, h, then the corresponding increment of the function $f(x)$ is $f(x+h)-f(x)$. If this is positive, the function increases when x increases; if it is negative, the function decreases when x increases. The condition that $x=a$ corresponds to a maximum value of $f(x)$ is therefore that, as x passes through the value a, $f(x+h)-f(x)$ shall cease to be positive and begin to be negative, and for a minimum shall cease to be negative and begin to be positive.

The practical application of the method will be best understood by studying the following example:—

Example.

To find the turning values of

$$y=x^3-9x^2+24x+3.$$

Let I denote the increment of y corresponding to a very small increment, h, of x; then

$$\begin{aligned}\mathrm{I}&=(x+h)^3-9(x+h)^2+24(x+h)+3-x^3+9x^2-24x-3,\\&=(3x^2-18x+24)h+(3x-9)h^2+h^3.\end{aligned}$$

Now, since for our present purpose it does not matter how small h may be, we may make it so small that $(3x-9)h^2+h^3$ is as small a fraction of $(3x^2-18x+24)h$ as we please. Hence, so far as determining the sign of I is concerned, we may write

$$I=(3x^2-18x+24)h.$$

Here h is supposed positive, hence the sign of I depends merely on the sign of $3x^2-18x+24$. Hence I will change sign when, and only when, x passes through a root of the equation

$$3x^2-18x+24=0.$$

Hence the turning values of y correspond to $x=2$ and $x=4$.

Moreover, we have

$$I=3(x-2)(x-4)h.$$

Therefore, when x is a little less than 2, I is positive; and when x is a little greater than 2, I is negative. Hence the value of y corresponding to $x=2$ is a maximum.

In like manner we may show that the value of y corresponding to $x=4$ is a minimum.

Exercises XXXVIII.

(1.) Find the limits within which x must lie in order that $8(x^2-a^2)-65xa$ may be negative.

Trace the graphs of

(2.) $y=x^2-5x+6$.

(3.) $y=-3x^2+12x-6$.

(4.) $y=-4x^2+20x-25$.

Find the turning values of the following; and discriminate between maxima and minima:—

(5.) $ae^{kx}+be^{-kx}$.

(6.) $a/x+a/(a-x)$.

(7.) $\sqrt{(1+x)}+\sqrt{(1-x)}$.

(8.) $x-1+\sqrt{(x+1)}$.

Trace the graphs of the following, and mark, in particular, the points where the graph cuts the axes, and the points where y has a turning value:—

(9.) $y=(x^2+8x+16)/(x^2-7x+12)$.

(10.) $y=(x^2-7x+12)/(x^2+8x+16)$.

(11.) $y=(x^2+8x+16)/(x^2-6x+9)$.

(12.) $y=(x^2-10x+27)/(x^2-8x+15)$.

(13.) $y=(x^2-8x+15)/(x^2-10x+27)$.

(14.) $y=(x^2-10x+27)/(x^2-14x+52)$.

(15.) $y=(x^2-9x+14)/(x^2+2x-15)$.

(16.) $y=(x^2+x-6)/(x^2-1)$.

(17.) $y=(x^2+5x+6)/(2x+3)$.

(18.) $y=1/(x^2+3x+5)$.

(19.) $y=(2x^2+x-6)/(2x^2+5x-12)$.

(20.) Show that the algebraically greatest and least values of $(x^2+2x-2)/(x^2+3x+5)$ are $\sqrt{(12/11)}$ and $-\sqrt{(12/11)}$; and find the corresponding values of x.

(21.) Show that $(ax-b)(dx-c)/(bx-a)(cx-d)$ may have all real values, provided $(a^2-b^2)(c^2-d^2)>0$.

(22.) Show that $(ax^2+bx+c)/(cx^2+bx+a)$ is capable of all values if

$b^2>(a+c)^2$; that there are two values between which it cannot lie if $(a+c)^2>b^2>4ac$; and that there are two values between which it must lie if $b^2<4ac$ (Wolstenholme).

(23.) If $ra>pb$, then the turning value of $(ax+b)/(px+r)^2$ is $a^2/4p(ra-pb)$.

Find the turning values of the following; and discriminate maxima and minima:—

(24.) $(x-1)(x-3)/x^2$. (25.) $(x-3)/(x^2+x-3)$.

(26.) $l(ax+b)^2+l'(a'x+b')^2+l''(a''x+b'')^2$.

(27.) $ax+by$, given $x^2+y^2=c^2$. (28.) $a^2x^2+b^2y^2$, given $x+y=a$.

(29.) xy, given $a^2/x^2+b^2/y^2=1$. (30.) $x^3y+x^2y^2+xy^3$, given $xy=a^2$.

(31.) $ax^2+2hxy+by^2$, given $Ax^2+2Hxy+By^2=1$.

(32.) $xy/\sqrt{(x^2+y^2)}$. (33.) $(2x-1)(3x-4)(x-3)$.

(34.) $1/\sqrt{x}+1/\sqrt{y}$, given $x+y=c$.

(35.) To inscribe in a given square the square of minimum area.

(36.) To circumscribe about a given square the square of maximum area.

(37.) To inscribe in a triangle the rectangle of maximum area.

(38.) P and Q are two points on two given parallel straight lines. PQ subtends a right angle at a fixed point O. To find P and Q so that the area POQ may be a minimum.

(39.) ABC is a right-angled triangle, P a movable point on its hypotenuse. To find P so that the sum of the squares of the perpendiculars from P on the two sides of the triangle may be a minimum.

(40.) To circumscribe about a circle the isosceles trapezium of minimum area.

(41.) Two particles start from given points on two intersecting straight lines, and move with uniform velocities u and v along the two straight lines. Show how to find the instant at which the distance between the particles is least.

(42.) OX, OY are two given straight lines; A, B fixed points on OX; P a movable point on OY. To find P so that AP^2+BP^2 shall be a minimum.

(43.) To find the rectangle of greatest area inscribed in a given circle.

(44.) To draw a tangent to a given circle which shall form with two given perpendicular tangents the triangle of minimum area.

(45.) Given the aperture and thickness of a biconvex lens, to find the radii of its two surfaces when its volume is a maximum or a minimum.

(46.) A box is made out of a square sheet of cardboard by cutting four equal squares out of the corners of the sheet, and then turning up the flaps. Show how to construct in this way the box of maximum capacity.

(47.) Find the cylinder of greatest volume inscribed in a given sphere.

(48.) Find the cylinders of greatest surface and of greatest volume inscribed in a given right circular cone.

(49.) Find the cylinder of minimum surface, the volume being given.

(50.) Find the cylinder of maximum volume, the surface being given.

CHAPTER XIX.

Solution of Arithmetical and Geometrical Problems by means of Equations.

§ 1.] The solution of isolated arithmetical and geometrical problems by means of conditional equations is one of the most important parts of a mathematical training. This species of exercise can be taken, and ought to be taken, before the student commences the study of algebra in the most general sense. It is chiefly in the applications of algebra to the systematic investigation of the properties of space that the full power of formal algebra is seen. All that we need do here is to illustrate one or two points which the reader will readily understand after what has been explained in the foregoing chapters.

§ 2.] The two special points that require consideration in solving problems by means of conditional equations are the *choice of variables, and the discussion* or *interpretation of the solution.*

With regard to the choice of variables it should be remarked that, while the selection of one set of variables in preference to another will never alter the order of the system of equations on whose solution any given problem depends, yet, as we have already had occasion to see in foregoing chapters, a judicious selection may very greatly diminish the complexity of the system, and thus materially aid in suggesting special artifices for its solution.

With regard to the interpretation of the solution, it is important to notice that it is by no means necessarily true that all the solutions, or even that any of the solutions, of the system of equations to which any problem leads are solutions of the

problem. Every algebraical solution furnishes numbers which satisfy certain abstract requirements; but these numbers may in themselves be such that they do not constitute a solution of the concrete problem. They may, for example, be imaginary, whereas real numbers are required by the conditions of the concrete case; they may be negative, whereas positive numbers are demanded; or (as constantly happens in arithmetical problems involving discrete quantity) they may be fractional, whereas integral solutions alone are admissible.

In every concrete case an examination is necessary to settle the admissibility or inadmissibility of the algebraical solutions. All that we can be sure of, *a priori*, is that, if the concrete problem have any solution, it will be found among the algebraical solutions; and that, if none of these are admissible, there is no solution of the concrete problem at all.

These points will be illustrated by the following examples. For the sake of such as may not already have had a sufficiency of this kind of mental gymnastic, we append to the present chapter a collection of exercises for the most part of no great difficulty.

Example 1.

There are three bottles, A, B, C, containing mixtures of three substances, P, Q, R, in the following proportions:—

$$\begin{aligned} &A, \quad aP+a'Q+a''R\,; \\ &B, \quad bP+b'Q+b''R\,; \\ &C, \quad cP+c'Q+c''R. \end{aligned}$$

It is required to find what proportions of a mixture must be taken from A, B, C, in order that its constitution may be $dP+d'Q+d''R$ (Newton, *Arithmetica Universalis*).

Let x, y, z be the proportions in question; then the constitution of the mixture is

$$(ax+by+cz)P+(a'x+b'y+c'z)Q+(a''x+b''y+c''z)R.$$

Hence we must have

$$ax+by+cz=d, \quad a'x+b'y+c'z=d', \quad a''x+b''y+c''z=d''.$$

The system of equations to which we are thus led is that discussed in chap. xvi., § 11, with the sole difference that the signs of d, d', d'' are reversed.

If, therefore, $ab'c''-ab''c'+bc'a''-bc''a'+ca'b''-ca''b'\neq0$, we shall obtain a unique finite solution. Unless, however, the values of x, y, z all come out positive, there will be no proper solution of the concrete problem. It is in

fact obvious, *a priori*, that there are restrictions; for it is clearly impossible, for instance, to obtain, by mixing from A, B, C, any mixture which shall contain one of the substances in a proportion greater than the greatest in which it occurs in A, B, or C.

Example 2.

A farmer bought a certain number of oxen (of equal value) for £350. He lost 5, and then sold the remainder at an advance of £6 a head on the original price. He gained £365 by the transaction; how many oxen did he buy?

Let x be the number bought; then the original price in pounds is $350/x$. The selling price is therefore $350/x+6$. Since the number sold was $x-5$, we must therefore have

$$(x-5)\left(\frac{350}{x}+6\right)-350=365.$$

This equation is equivalent to

$$6x^2-395x-1750=0,$$

which has the two roots $x=70$ and $x=-25/6$. The latter number is inadmissible, both because it is negative and because it is fractional; hence the only solution is $x=70$.

Example 3.

A′OA is a limited straight line such that $OA=OA'=a$. P is a point in OA, or in OA produced, such that $OP=p$. To find a point Q in A′A such that $PQ^2=AQ.QA'$. Discuss the different positions of Q as p varies from 0 to its greatest admissible value.

Let $OQ=x$, x denoting a positive or negative quantity, according as Q is right or left of O. Then $PQ=\pm(x-p)$, $A'Q=a+x$, $AQ=a-x$; and we have in all cases

$$(x-p)^2=(a+x)(a-x)=a^2-x^2 \qquad (1).$$

Hence

$$x^2-px+\tfrac{1}{2}(p^2-a^2)=0 \qquad (2).$$

The roots of (2) are $\frac{1}{2}p\pm\sqrt{(\frac{1}{2}a^2-\frac{1}{4}p^2)}$.

These roots will be real if $p^2<2a^2$; that is to say, confining ourselves to positive values of p, if $p<\sqrt{2}a$.

From (1) we see that in all cases where x is real it must be numerically less than a. Hence Q always lies between A′ and A.

When $p=0$, the roots of (2) are $\pm a\sqrt{}/2$; that is to say, the two positions of Q are equidistant from O.

So long as p is $<a$, $\frac{1}{2}(p^2-a^2)$ will be negative, and the roots of (2) will be of opposite sign; that is to say, the two positions of Q will lie on opposite sides of O. Since the sum of the two roots is $p(=OP)$, if Q_1Q_2 be the two positions of Q, the relative positions of the points will be as in Fig. 1, where $OQ_2=PQ_1$.

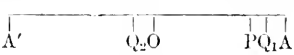


Fig. 1.

When $p=a$, Q_2 moves up to O, and Q_1 up to A.

If $p>a$, then both roots are positive, and the points will be as in Fig. 2, where $OQ_2=Q_1P$.

Fig. 2.

If $OB=\sqrt{2}a$, then B is the limiting position of P for which a solution of the problem is possible. When P moves up to B, Q_1 and Q_2 coincide at C (OC being $\frac{1}{2}$OB).

Example 4.

To find four real positive numbers in continued proportion such that their sum is a and the sum of their squares b^2.

Let us take for variables the first of the four numbers, say x, and the common value, say y, of the ratio of each number to the preceding. Then the four numbers are x, xy, xy^2, xy^3. Hence, by our data,

$$x+xy+xy^2+xy^3=a,$$
$$x^2+x^2y^2+x^2y^4+x^2y^6=b^2;$$

that is to say,
$$x(1+y)(1+y^2)=a \qquad (1),$$
$$x^2(1+y^2)(1+y^4)=b^2 \qquad (2).$$

From (1) we derive
$$x^2(1+y)^2(1+y^2)^2=a^2 \qquad (3),$$

and from (2) and (3), rejecting the factor y^2+1, which is clearly irrelevant, we derive

$$a^2(1+y^4)=b^2(1+y^2)(1+y)^2 \qquad (4).$$

The equation (4) is a reciprocal biquadratic in y, which can be solved by the methods of chap. xvii., § 8.

For every value of y (1) gives a corresponding value of x.

The student will have no difficulty in showing that there will be two proper solutions of the problem, provided a be $>b$. Since, however, the two values of y are reciprocals, and since $x(1+y)(1+y^2)\equiv xy^3(1+1/y)(1+1/y^2)$, these two solutions consist merely of the same set of four numbers read forwards and backwards. There is, therefore, never more than one distinct solution.

Newton, in his *Arithmetica Universalis,* solves this problem by taking as variables the sum of the two mean numbers, and the common value of the product of the two means and of the two extremes. He expresses the four numbers in terms of these and of a and b, then equates the product of the second and fourth to the square of the third, and the product of the first and third to the square of the second. It will be a good exercise to work out the problem in this way.

Example 5.

In a circle of given radius a to inscribe an isosceles triangle the sum of the squares of whose sides is $2b^2$.

Let x be the length of one of the two equal sides of the triangle, $2y$ the length of the base.

If ABC be the triangle, and if AD, the diameter through A, meet BC in E, then, since ABD is a right angle, we have $AB^2=AD\,.\,AE$. Hence

$$x^2=2a\sqrt{(x^2-y^2)} \qquad (1).$$

Again, by the conditions of the problem, we have

$$2x^2+4y^2=2b^2,$$

that is,

$$x^2+2y^2=b^2 \qquad (2).$$

From (1) and (2) we derive

$$x^4-6a^2x^2+2a^2b^2=0 \qquad (3).$$

The roots of (3) are

$$\pm a\sqrt{\left\{3\pm 3\sqrt{\left(1-\frac{2b^2}{9a^2}\right)}\right\}};$$

and the corresponding values of y are given by (2).

The necessary and sufficient condition that the values of x and of y be real is that $b<3a/\sqrt{2}$. When this condition is satisfied, there are two real positive values of x, and if $b>2a$ there are two corresponding real positive values of y.

It follows from the above that, for the inscribed isosceles triangle the sum of the squares of whose sides is a maximum, $b=3a/\sqrt{2}$. Corresponding to this we have $x=\sqrt{3}a$, $2y=\sqrt{3}a$; that is to say, the inscribed triangle, the sum of the squares of whose sides is a maximum, is equilateral, as is well known.

Example 6.

Find the isosceles triangle of given perimeter $2p$ inscribed in a circle of radius a; show that, if $2p$ be less than $3\sqrt{3}$, and greater than $2a$, there are two solutions of the problem; and that the inscribed triangle of maximum perimeter is equilateral.

Taking the variables as in last example, we find

$$x^2=2a\sqrt{(x^2-y^2)} \qquad (1),$$

$$x+y=p \qquad (2).$$

Hence

$$x^4-8a^2px+4a^2p^2=0 \qquad (3).$$

We cannot reduce the biquadratic (3) to quadratics, as in last example; but we can easily show that, provided p be less than a certain value, it has two real positive roots.

Let us consider the function

$$y=x^4-8a^2px+4a^2p^2 \qquad (4);$$

and let I be the increment of y corresponding to a very small positive increment (h) of x. Then we find, as in chap. xviii., § 12, that

$$\mathrm{I}=4(x^3-2a^2p)h \qquad (5).$$

Hence, so long as $x^3<2a^2p$, I is negative; and when $x^3>2a^2p$, I is positive. Hence, observing that $y=+\infty$ when $x=\pm\infty$, we see that the minimum value of y corresponds to $x=\sqrt[3]{(2a^2p)}$, and that the graph of (4) consists of a single festoon. Hence (3) will have two real roots, provided the minimum point be below the x-axis; that is, provided y be negative when $x=\sqrt[3]{(2a^2p)}$; that is, provided

$$4a^2p^{\frac{4}{3}}\left(-\frac{3}{2^{\frac{2}{3}}}a^{\frac{2}{3}}+p^{\frac{2}{3}}\right)$$

be negative; that is, provided $2p < 3\sqrt{3}a$. It is obvious that both the roots are positive; for when $x=0$ we have $y=4a^2p^2$, which is positive; hence the graph does not descend below the axis of x until it reaches the right-hand side of the axis of y.

From the above reasoning it follows that the greatest admissible perimeter is $3\sqrt{3}a$. When $2p$ has this value, the minimum point of the graph lies on the axis of x, and $x=\sqrt[3]{(2a^2p)}=\sqrt[3]{(3\sqrt{3}a^3)}=\sqrt{3}a$ corresponds to two equal roots of (3). The corresponding value of $2y$ is given by $2y=2p-2x=3\sqrt{3}a-2\sqrt{3}a=\sqrt{3}a$; in other words, the inscribed isosceles triangle of maximum perimeter is equilateral.

Another interesting way of showing that (3) has two equal roots is to discuss the graphs (referred to one and the same pair of axes) of the functions

$$y=x^4, \text{ and } y=8a^2px-4a^2p^2.$$

These can be easily constructed; and it is obvious that the abscissæ of their intersections are the real roots of (3).

EXERCISES XXXIX.

(1.) How long will an up and a down train take to pass each other, each being 44 yards long, and each travelling 30 miles an hour?

(2.) Diophantus passed in infancy the sixth part of his life, in adolescence a twelfth, then he married and in this state he passed a seventh of his life and five years more. Then he had a son whom he survived four years and who only reached the half of his father's age. How old was Diophantus when he died?

(3.) A man met several beggars and wished to give 25 pence to each; but, on counting his money, he found that he had 10 pence too little for that; and then made up his mind to give each 20 pence. After doing this he had 25 pence over. What had he at first, and how many beggars were there?

(4.) Two bills on the same person are sent to a banker, the first for £580 payable in 7 months, the second for £730 payable in 4 months. The banker gives £1300 for the two. What was the rate of discount, simple interest being allowed in lieu of discount?

(5.) A basin containing 1200 cubic metres of water is fed by three fountains, and can be emptied by a discharging pipe in 4 hours. The basin is emptied and the three fountains set on; how long does it take to fill with the discharging pipe open?—given that the three fountains each running alone would fill the basin in 3, 6, and 7 hours respectively.

(6.) If I subtract from the double of my present age the treble of my age 6 years ago, the result is my present age. What is my age?

(7.) A vessel is filled with a mixture of spirit and water, 70°/$_\circ$ of which is spirit. After 9 gallons is taken out and the vessel filled up with water, there remains $58\frac{1}{3}$°/$_\circ$ of spirit: find the contents of the vessel.

(8.) Find the time between 8 and 9 o'clock when the hour and minute hands of a clock are perpendicular.

(9.) A and B move on two paths intersecting at O. B is 500 yards short

of O when A is at O; in two minutes they are equidistant from O, and in eight minutes more they are again equidistant from O. Find the speeds of A and B.

(10.) I have a sum to buy a certain number of nuts. If I buy at the rate of 40 a penny, I shall spend 5d. too much, if at the rate of 50 a penny, 10d. too little. How much have I to spend?

(11.) If two numbers be increased by 1 and diminished by 1 respectively, their product is diminished by 4. If they be diminished by 1 and increased by 2 respectively, their product is increased by 16. Find the numbers.

(12.) A is faster than B by p miles an hour. He overtakes B, who has a start of h miles, after a run of q miles. Required the speeds of A and B.

(13.) To divide a given number a into two parts whose squares shall be in the ratio $m:1$.

(14.) Four apples are worth as much as five plums; three pears as much as seven apples; eight apricots as much as fifteen pears; and five apples sell for twopence. I wish to buy an equal number of each of the four fruits, and to spend an exact number of pence; find the least sum I can spend.

(15.) A man now living said he was x years of age in the year x^2. What is his age and when was he born?

Remark on the nature of this and the preceding problem.

(16.) OABCD are five points in order on a straight line. If $OA=a$, $OB=b$, $OC=c$, $OD=d$, find the distance of P from O in order that $PA:PD=PB:PC$. (Assume P to lie between B and C.)

(17.) A man can walk from P to Q and back in a certain time at the rate of $3\frac{1}{2}$ miles an hour. If he walks 3 miles an hour to and 4 miles an hour back, he takes 5 minutes longer; find the distance PQ.

(18.) A starts to walk from P to Q half an hour after B; overtakes B midway between P and Q; and arrives at Q at 2 P.M. After resting $7\frac{1}{2}$ minutes, he starts back and meets B in 10 minutes more. When did each start from P?

(19.) At two stations, A and B, on a line of railway the prices of coals are £p per ton and £q per ton respectively. If the distance between A and B be d, and the rate for the carriage of coal be £r per ton per mile, find the distance from A of a station on the line at which it is indifferent to a consumer whether he buys coals from A or from B.

(20.) A merchant takes every year £1000 out of his income for personal expenses. Nevertheless his capital increases every year by a third of what remains; and at the end of three years it is doubled. How much had he at first?

(21.) A takes m times as long to do a piece of work as B and C together; B n times as long as C and A together; C x times as long as A and B together. Find x; and show that $1/(x+1)+1/(m+1)+1(n+1)=1$.

(22.) The total increase in the number of patients in a certain hospital in a certain year over the number in the preceding year was $2\frac{1}{2}\%$. In the number of out-patients there was an increase of 4%; but in the number of in-patients a decrease of 11%. Find the ratio of the number of out to the number of in-patients.

(23.) The sum of the ages of A and B is now 60; 10 years ago their ages were as 5 to 3. Find their ages now.

(24.) Divide 111 into three parts, so that one-third of the first part is greater by 4 than one-fourth of the second, and less by 5 than one-fifth of the third.

(25.) In a hundred yards' race A can beat B by $\frac{1}{5}''$; but he is handicapped by 3 yards, and loses by $1\frac{1}{13}$ yards. Find the times of A and B.

(26.) A and B run a mile, and A beats B by 100 yards. A then runs with C, and beats him by 200 yards. Finally, B runs with C; by how much does he beat him?

(27.) A person rows a miles down a river and back in t hours. He can row b miles with the stream in the same time as c miles against. Find the times of going and returning, and the velocity of the stream.

(28.) A mixture of black and green tea sold at a certain price brings a profit of 4°/$_\circ$ on the cost price. The teas sold separately at the same price would bring 5°/$_\circ$ and 3°/$_\circ$ profit respectively. In what proportion were the two mixed?

(29.) If a rectangle were made a feet longer and b feet narrower, or a' feet longer and b' feet narrower, its area would in each case be unaltered. Find its area.

(30.) Two vessels, A and B, each contain 1 oz. of a mixture of spirit and water. If $1/m$th oz. of spirit be added to A and $1/n$th oz. of spirit to B, or if $1/n$th oz. of water be added to A and $1/m$th oz. of water to B, the percentages of spirit in A and B in each case become equal. What percentage of spirit is there in each?

(31.) A wine-merchant mixes wine at 10s. per gallon with spirit at 20s. per gallon and with water, and makes 25°/$_\circ$ profit by selling the mixture at 11s. 8d. per gallon. If he had added twice as much spirit and twice as much water, he would have made the same profit by selling at 11s. 3d. per gallon. How much spirit and how much water does he add to each 100 gallons of wine?

(32.) Find the points on the dial of a watch where the two hands cross.

(33.) Three gamesters agree that the loser shall always double the capital of the two others. They play three games, and each loses one. At the end they have each £a. What had they at first?

(34.) A cistern can be filled in 6 hours by one pipe, and in 8 hours by another. It was filled in 5 hours by the two running partly together and partly separately. The time they ran together was two-thirds of the time they ran separately. How long did each run?

(35.) A horse is sold for £24, and the number expressing the profit per cent expresses also the cost price. Find the cost price.

(36.) I spent £18 in cigars. If I had got one box more for the money, each box would have been 5s. cheaper. How many boxes did I buy?

(37.) A person about to invest in 3°/$_\circ$ consols observed that, if the price had been £5 less, he would have received $\frac{1}{4}$°/$_\circ$ more interest on his money. Find the price of consols.

(38.) Out of a cask containing 360 quarts of pure alcohol a quantity is drawn and replaced by water. Of the mixture a second quantity, 84 quarts

more than the first, is drawn and replaced by water. The cask now contains as much alcohol as water. What quantity was drawn out at first?

(39.) Find four consecutive integers such that the product of two of them may be a number which has the other two for digits.

(40.) The consumption of an important commodity is found to increase as the square of the decrease of its price below a certain standard price (p). If the customs' duty be levied at a given percentage on the value (a) of the commodity before the duty is paid, show that, provided the rate be below a certain limit, there are two other rates which will yield the same total revenue, and determine the rates which will yield the greatest and least revenues.

(41.) A number has two digits, the sum of the squares of which is 130. If the order of the digits be reversed the number is increased by 18. Find the number.

(42.) Three numbers are in arithmetical progression. The square of the first, together with the product of the second and third, is 16; and the square of the second, together with the product of the first and third, is 14.

(43.) To find three numbers in arithmetical progression such that their sum is $2a$, and the sum of their squares $4b^2$.

(44.) The sides of a triangle are the roots of $x^3 - ax^2 + bx - c = 0$. Show that its area is $\frac{1}{4}\sqrt{\{a(4ab - a^3 - 8c)\}}$.

(45.) The hypotenuse of a right-angled triangle is h, and the radius of the inscribed circle r. Find the sides of the triangle. Find the greatest admissible value of r for a given value of h.

The following are from Newton's *Arithmetica Universalis*, *q.v.*, pp. 119 *et seq.* :—

(46.) Given the sides of a triangle, to find the segments of any side made by the foot of the perpendicular from the opposite vertex.

(47.) Given the perimeter and area of a right-angled triangle, to find the hypotenuse.

(48.) Given the perimeter and altitude of a right-angled triangle, to find its sides.

(49.) The same, given the hypotenuse and the sum of the altitude and the two sides.

(50.) Find the sides of a triangle which is such that the three sides, a, b, c, and the perpendicular on a form an arithmetical progression.

(51.) The same, the progression being geometric.

(52.) To find a point in a given straight line such that the difference of its distances from two given points shall be a given length.

CHAPTER XX.

Arithmetic and Geometric Progressions and the Series allied to them.

§ 1.] *By a series is meant the sum of a number of terms formed according to some common law.*

For example, if $f(n)$ be any function of n whatsoever, the function

$$f(1)+f(2)+f(3)+\ .\ .\ .+f(r)+\ .\ .\ .+f(n) \qquad (1)$$

is called a series.

$f(1)$ is called the *first term*; $f(2)$ *the second term*, &c.; and $f(r)$ is called the *rth, or general, term.*

For the present we consider only series which have a finite number n of terms.

As examples of this new kind of function, let $f(n)=n$, then we have the series

$$1+2+3+\ .\ .\ .+n \qquad (2);$$

let $f(n)=1/(a+bn)$, and we have the series

$$\frac{1}{a+b}+\frac{1}{a+b2}+\frac{1}{a+b3}+\ .\ .\ .+\frac{1}{a+bn} \qquad (3);$$

let $f(n)=\sqrt{n}/(2-\sqrt{n})$, and we have the series

$$\frac{\sqrt{1}}{2-\sqrt{1}}+\frac{\sqrt{2}}{2-\sqrt{2}}+\frac{\sqrt{3}}{2-\sqrt{3}}+\ .\ .\ .+\frac{\sqrt{n}}{2-\sqrt{n}} \qquad (4);$$

and so on.

It is obvious that when the nth term of a series is given we can write down all the terms by simply substituting for n 1, 2, 3, . . . successively.

Thus, if the nth term be n^2+2n, the series is

$$(1^2+2.1)+(2^2+2.2)+(3^2+2.3)+\ .\ .\ .+(n^2+2n),$$

or

$$3+8+15+\ .\ .\ .+(n^2+2n).$$

It is not true, however, that when the first few terms are

given we can in general find the nth term, if nothing is told us regarding the form of that term. This is sufficiently obvious from the second form in which the last series was written; for in the earlier terms all trace of the law of formation is lost.

If we have some general description of the nth term, it may in certain cases be possible to find it from the values of a certain number of particular terms. If, for example, we were told that the nth term is an integral function of n of the 2nd degree, then, by chap. xviii., § 7, we could determine that function if the values of three terms of the series of known order were given.

§ 2.] If we regard the series

$$f(1) + f(2) + \ . \ . \ . + f(n)$$

as a function of n, and call it $\phi(n)$, it has a striking peculiarity, shared by no function of n that we have as yet fully discussed, namely, that *the number of terms in the function $\phi(n)$ depends on the value of its variable. For example,*

$$\phi(1) = 1, \quad \phi(2) = 1 + 2, \quad \phi(3) = 1 + 2 + 3,$$

and so on.

It happens in certain cases that an expression can be found for $\phi(n)$ which has not this peculiarity; for example, we shall show presently that

$$1^2 + 2^2 + 3^2 + \ . \ . \ . + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

On the left of this equation the number of terms is n; on the right we have an ordinary integral function of n, the number of terms in which is independent of n, and which is therefore called a *closed* function of n.

When, as in the example quoted, we can find for the sum of a series an expression involving only known functions and constructed by a fixed number of steps, then the series is said to admit of summation; and the closed expression in question is spoken of as the sum, par excellence, of the series.

The property of having a sum in the sense just explained is an exceptional one; and the sum, where it exists, must always be found by some artifice depending on the nature of the series. What the student should endeavour to do is to group together,

and be sure that he can recognise, all the series that can be summed by any given artifice. This is not so difficult as might be supposed; for the number of different artifices is by no means very large.

In this chapter we shall discuss two very important cases, leaving the consideration of several general principles and of several interesting particular cases to the second part of this work.

SERIES WHOSE nTH TERM IS AN INTEGRAL FUNCTION OF n.

§ 3.] An *Arithmetic Series*, or an *Arithmetic Progression*, as it is often called, is a series in which each term exceeds the preceding by a fixed quantity, called the common difference. Let a be the first term, and b the common difference; then the terms are a, $a+b$, $a+2b$, $a+3b$, &c., the nth term being obviously $a+(n-1)b$.

Here a and b may be any algebraical quantities whatsoever, the word "exceed" in the definition being taken in the algebraical sense.

Since the nth term may be written $(a-b)+bn$, where $a-b$ and b are constants, we see that the nth term of an arithmetical series is an integral function of n of the 1st degree. Such a series is therefore the simplest of the general class to be considered in this section.

The usual method of summing an A.P. is as follows. Let Σ denote the sum of n terms, then

$$\Sigma = \quad a + \quad \{a+b\} + \quad \{a+2b\} + \ldots + \quad \{a+(n-1)b\}.$$

If we write the terms in the reverse order, we have

$$\Sigma = \{a+(n-1)b\} + \{a+(n-2)b\} + \{a+(n-3)b\} + \ldots + \quad a.$$

If we now add, taking the pairs of terms in the same vertical line together, we find

$$2\Sigma = \{2a+(n-1)b\} + \{2a+(n-1)b\} + \{2a+(n-1)b\} + \ldots + \{2a+(n-1)b\}.$$

Hence, since there are n terms,

$$\Sigma = \frac{n}{2}\{2a+(n-1)b\} \qquad (1).$$

This gives Σ in terms of n, a, b.

If we denote the last term of the series by l, we have $l=a+(n-1)b$. Hence

$$\Sigma = n\frac{a+l}{2} \qquad (2).$$

That is to say, *the sum of n terms of an A.P. is n times the average of the first and last terms*, a proposition which is convenient in practice.

Example 1.

To sum the arithmetical series $5+3+1-1-3-\ldots$ to 100 terms. Here $a=5$ and $b=-2$. Hence

$$\begin{aligned}\Sigma &= \tfrac{100}{2}\{2\times 5+(100-1)(-2)\},\\ &=50(10-198),\\ &=-9400.\end{aligned}$$

Example 2.

To find the sum of the first n odd integers.

The nth odd integer is $2n-1$. Hence

$$\begin{aligned}\Sigma &= 1+3+5+\ldots+(2n-1),\\ &= n\frac{1+(2n-1)}{2},\\ &= n^2.\end{aligned}$$

It appears, therefore, that the sum of any number of consecutive odd integers, beginning with unity, is the square of their number. This proposition was known to the Greek geometers.

Example 3.

Sum the series $1-2+3-4+5\ldots$ to n terms. First suppose n to be even, $=2m$ say. Then the series is

$$\begin{aligned}\Sigma &= 1-2+3-4+\ldots+(2m-1)-2m,\\ &= 1+3+\ldots+(2m-1)\\ &\quad -2-4-\ldots-2m.\end{aligned}$$

In each line there are m terms. The first line has for its sum m^2, by Example 2. The second gives $-m(2+2m)/2$, that is, $-m(m+1)$. Hence

$$\Sigma = m^2 - m(m+1) = -m = -\frac{n}{2}.$$

Next suppose n to be odd, $=2m-1$, say.

Then we have

$$\Sigma = 1-2+3-4+\ldots+(2m-1).$$

To find the sum in this case, all we have to do is to add $2m$ to our former result. We thus find

$$\begin{aligned}\Sigma &= 2m-m=m,\\ &= \frac{n+1}{2}.\end{aligned}$$

This result might be obtained even more simply by associating the terms of the given series in pairs.

§ 4.] The artifice of § 3 will not apply to the case where the nth term of a series is an integral function of n of higher degree than the 1st. We proceed, therefore, to develop a more general method.

Let the nth term of the series be

$$p_0n^r + p_1n^{r-1} + p_2n^{r-2} + \ldots + p_r \qquad (1),$$

where $p_0, p_1, p_2, \ldots, p_r$ are independent of n.

And let us denote the sums of the first, second, third, . . ., rth powers of the first n integral numbers by ${}_ns_1, {}_ns_2, {}_ns_3, \ldots, {}_ns_r$; so that

$$\begin{aligned} {}_ns_1 &= 1 + 2 + 3 + \ldots + n, \\ {}_ns_2 &= 1^2 + 2^2 + 3^2 + \ldots + n^2, \\ {}_ns_3 &= 1^3 + 2^3 + 3^3 + \ldots + n^3, \end{aligned}$$

and so on.

If Σ denote the sum of n terms of the series whose nth term is (1), we have,

$$\begin{aligned} \Sigma = {} & p_0 1^r + p_1 1^{r-1} + p_2 1^{r-2} + \ldots + p_r, \\ & + p_0 2^r + p_1 2^{r-1} + p_2 2^{r-2} + \ldots + p_r, \\ & + p_0 3^r + p_1 3^{r-1} + p_2 3^{r-2} + \ldots + p_r, \\ & \ldots \ldots \ldots \ldots \ldots \\ & + p_0 n^r + p_1 n^{r-1} + p_2 n^{r-2} + \ldots + p_r. \end{aligned}$$

Hence, adding in vertical columns, we have

$$\Sigma = p_0\,{}_ns_r + p_1\,{}_ns_{r-1} + p_2\,{}_ns_{r-2} + \ldots + np_r \qquad (2).$$

From this formula we see that we could sum the series whose general term is (1) if only we knew the sums of the first, second, third, . . ., rth powers of the first n integers.

These sums can be calculated successively by a uniform process, as we shall now show.

§ 5.] *To calculate* ${}_ns_1$.

If in the identity $(x+1)^2 - x^2 = 2x + 1$ we put successively $x = n, x = n-1, \ldots, x = 2, x = 1$, we have the following equations—

$$\begin{aligned} (n+1)^2 - n^2 &= 2n \qquad\quad + 1, \\ n^2 - (n-1)^2 &= 2(n-1) + 1, \\ &\ldots \ldots \ldots \\ 3^2 - 2^2 &= 2\cdot 2 \qquad\; + 1, \\ 2^2 - 1^2 &= 2\cdot 1 \qquad\; + 1. \end{aligned}$$

If we add all these equations, the terms on the left mutually destroy each other, with the exception of two, which give $(n+1)^2 - 1$; and those on the right, added in vertical columns, give $2_n s_1 + n$. Hence

$$(n+1)^2 - 1 = 2_n s_1 + n \qquad (1).$$

From this we have

$$\begin{aligned} 2_n s_1 &= (n+1)^2 - (n+1), \\ &= (n+1)n, \\ {}_n s_1 &= \frac{n(n+1)}{2} \qquad (2); \end{aligned}$$

a result which we might have obtained by the method of § 3, for $1 + 2 + \ldots + n$ is an A.P.

Cor. *The sum of the first powers of the first n integers is an integral function of n of the 2nd degree.*

§ 6.] *To calculate ${}_n s_2$.*

In the identity $(x+1)^3 - x^3 = 3x^2 + 3x + 1$ put successively $x = n$, $x = n-1$, . . ., $x = 2$, $x = 1$, and we have

$$\begin{aligned} (n+1)^3 - n^3 &= 3n^2 + 3n + 1, \\ n^3 - (n-1)^3 &= 3(n-1)^2 + 3(n-1) + 1, \\ &\cdots\cdots \\ 3^3 - 2^3 &= 3.2^2 + 3.2 + 1, \\ 2^3 - 1^3 &= 3.1^2 + 3.1 + 1. \end{aligned}$$

Hence, adding all these equations, we have

$$(n+1)^3 - 1 = 3_n s_2 + 3_n s_1 + n \qquad (1).$$

Using in (1) the value already found for ${}_n s_1$, we have

$$\begin{aligned} 3_n s_2 &= (n+1)^3 - \tfrac{3}{2}n(n+1) - (n+1), \\ &= \frac{n+1}{2}\{2(n+1)^2 - 3n - 2\}, \\ &= \frac{n+1}{2}(2n^2 + n). \end{aligned}$$

Hence

$${}_n s_2 = \frac{n(n+1)(2n+1)}{6} \qquad (2)$$

Cor. *The sum of the squares of the first n integers is an integral function of n of the 3rd degree.*

§ 7.] *To calculate* ${}_ns_3$.

In the identity $(x+1)^4-x^4=4x^3+6x^2+4x+1$ put successively $x=n$, $x=n-1$, . . ., $x=2$, $x=1$; add the n equations so obtained, and we find, as before,

$$(n+1)^4-1=4{}_ns_3+6{}_ns_2+4{}_ns_1+n,$$

or

$$(n+1)^4-(n+1)=4{}_ns_3+6{}_ns_2+4{}_ns_1 \qquad (1).$$

Using the values of ${}_ns_2$ and ${}_ns_1$ already found, we have

$$\begin{aligned}4{}_ns_3&=n(n+1)(n^2+3n+3)-n(n+1)(2n+1)-2n(n+1),\\&=n(n+1)(n^2+3n+3-2n-1-2),\\&=n^2(n+1)^2.\end{aligned}$$

Hence

$${}_ns_3=\left\{\frac{n(n+1)}{2}\right\}^2 \qquad (2).$$

Cor. 1. ${}_ns_3$ *is an integral function of n of the 4th degree.*

Cor. 2. *The sum of the cubes of the first n integers is the square of the sum of their first powers.*

§ 8.] Exactly as in § 7 we can show that

$$(n+1)^5-(n+1)=5{}_ns_4+10{}_ns_3+10{}_ns_2+5{}_ns_1 \qquad (1);$$

and from this equation, knowing ${}_ns_1$, ${}_ns_2$, ${}_ns_3$, we can calculate ${}_ns_4$. The result is

$${}_ns_4=\frac{n(n+1)(6n^3+9n^2+n-1)}{30} \qquad (2).$$

§ 9.] This process may be continued indefinitely, and the functions ${}_ns_1$, ${}_ns_2$, . . ., ${}_ns_{r-1}$ . . . calculated one after the other.

Suppose, in fact, that ${}_ns_1$, ${}_ns_2$, . . ., ${}_ns_{r-1}$ had all been calculated. Then, just as in §§ 5-8, we deduce the equation

$$(n+1)^{r+1}-(n+1)={}_{r+1}C_1\,{}_ns_r+{}_{r+1}C_2\,{}_ns_{r-1}+\ldots+{}_{r+1}C_r\,{}_ns_1 \quad (1),$$

where ${}_{r+1}C_1$, ${}_{r+1}C_2$, &c., are the binomial coefficients of the $r+1$th order.

The equation (1) enables us to calculate ${}_ns_r$.

Cor. 1. ${}_ns_r$ *is an integral function of n of the $r+1$th degree, so that we may write*

$${}_ns_r=q_0n^{r+1}+q_1n^r+q_2n^{r-1}+\ldots+q_{r+1};$$

and it is obvious from (1) *that*

$$q_0 = \frac{1}{{}_{r+1}C_1} = \frac{1}{r+1}.$$

Cor. 2. ${}_ns_r$ *is divisible by* $n(n+1)$, *so that we may write*

$${}_ns_r = n(n+1)\left\{\frac{n^{r-1}}{r+1} + p_1n^{r-2} + p_2n^{r-3} + \ldots + p_{r-1}\right\};$$

for this is true when $r=1$, $r=2$, $r=3$, $r=4$; hence it must be true for $r=5$, for we have

$$(n+1)^6 - (n+1) = {}_6C_1\,{}_ns_5 + {}_6C_2\,{}_ns_4 + {}_6C_3\,{}_ns_3 + {}_6C_2\,{}_ns_2 + {}_6C_1\,{}_ns_1,$$

and $(n+1)^6 - (n+1)$ is divisible by $n(n+1)$; and so on.

§ 10.] We can now sum any series whose nth term is reducible to an integral function of n. By § 4 and § 9, Cor. 1, we see that *the sum of n terms of any series whose nth term is an integral function of n of the rth degree is an integral function of n of the r + 1th degree.* We may, therefore, if we choose, in summing any such series, assume the sum to be $An^{r+1} + Bn^r + \ldots + K$; and determine the coefficients A, B, . . ., K by giving particular values to n. If $S_1, S_2, \ldots, S_{r+2}$ be the sums of 1, 2, . . ., $r+2$ terms of the series, then it is obvious, by Lagrange's Theorem, chap. xviii., § 7, that the sum is

$$\sum_1^{r+2} S_s \frac{(n-1)(n-2)\ldots(n-s+1)(n-s-1)\ldots(n-r-2)}{(s-1)(s-2)\ldots\ldots 1 \quad (-1) \quad \ldots(s-r-2)}.$$

The following are a few examples :—

Example 1.

To sum the series $\Sigma = a + (a+b) + (a+2b) + \ldots + \{a+(n-1)b\}$.

The nth term is $(a-b)+nb$.

The $n-1$th term is $(a-b)+(n-1)b$.

. . . .

The 2nd term is $(a-b)+2b$.

The 1st term is $(a-b)+1b$.

Hence

$$\Sigma = (a-b)n + b\,{}_ns_1,$$

$$= (a-b)n + b\frac{n(n+1)}{2},$$

$$= \frac{n}{2}\{2a+(n-1)b\},$$

as was found in § 3.

Example 2. $\Sigma = 1^2 + 3^2 + 5^2 +$ to n terms.

The nth term is $(2n-1)^2 = 4n^2 - 4n + 1$.

Hence
$$\begin{array}{rlll} \Sigma = & 4n^2 & -4n & +1 \\ & +4(n-1)^2 & -4(n-1) & +1 \\ & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ & +4.2^2 & -4.2 & +1 \\ & +4.1^2 & -4.1 & +1. \end{array}$$

Hence, adding in vertical columns, we have
$$\begin{aligned} \Sigma &= 4{}_ns_2 - 4{}_ns_1 + n, \\ &= 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n, \\ &= \frac{(2n-1)n(2n+1)}{3}. \end{aligned}$$

Example 3.
$$\Sigma = 2.3.4 + 3.4.5 + 4.5.6 + \ .\ .\ .\ \text{to } n \text{ terms.}$$

The nth term is $(n+1)(n+2)(n+3) = n^3 + 6n^2 + 11n + 6$.

Hence
$$\begin{aligned} \Sigma &= {}_ns_3 + 6{}_ns_2 + 11{}_ns_1 + 6n, \\ &= \tfrac{1}{4}(n^4 + 10n^3 + 35n^2 + 50n). \end{aligned}$$

Example 4.

A wedge-shaped pile of shot stands on a rectangular base. There are m and n shot respectively in the two sides of the lowest rectangular layer, $m-1$ and $n-1$ in the two sides of the next rectangular layer, and so on, the uppermost layer being a single line of shot. Find the whole number of shot in the pile, m being greater than n.

The number in the lowest layer is mn; in the next $(m-1)(n-1)$; in the next $(m-2)(n-2)$, and so on; the number in the last layer is $(m-\overline{n-1})(n-\overline{n-1})$, that is, $(m-n+1)$.

Hence we have to sum the series
$$\Sigma = mn + (m-1)(n-1) + (m-2)(n-2) + .\ .\ . + (m-\overline{n-1})(n-\overline{n-1}),$$
in which there are n terms.

The rth term of the series is $(m-\overline{r-1})(n-\overline{r-1})$, that is, $(\overline{m+1}-r)(\overline{n+1}-r)$, that is, $(m+1)(n+1)-(m+n+2)r+r^2$.

Hence we may write the series as follows:—
$$\begin{array}{rlll} \Sigma = & (m+1)(n+1) & -(m+n+2)n & +n^2 \\ & +(m+1)(n+1) & -(m+n+2)(n-1) & +(n-1)^2 \\ & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ & +(m+1)(n+1) & -(m+n+2)2 & +2^2 \\ & +(m+1)(n+1) & -(m+n+2)1 & +1^2, \\ = & n(m+1)(n+1) & -(m+n+2){}_ns_1 & +{}_ns_2, \\ = & \multicolumn{3}{l}{(m+1)n(n+1) - \tfrac{1}{2}(m+n+2)n(n+1) + \tfrac{1}{6}n(n+1)(2n+1),} \\ = & \multicolumn{3}{l}{\tfrac{1}{6}n(n+1)(3m-n+1).} \end{array}$$

Remark.—In working examples by this method the student must be careful to see that the series is complete; in other words, that there are exactly n terms, *all formed according to the same law.* If any terms are wanting, or if there are redundant terms, allowance must be made by adding or subtracting terms, as the case may be.

SERIES WHOSE nTH TERM IS THE PRODUCT OF AN INTEGRAL FUNCTION OF n AND A SIMPLE EXPONENTIAL FUNCTION OF n.

§ 11.] The typical form of the nth term in the class of series now to be considered is

$$(p_0 n^s + p_1 n^{s-1} + \ldots + p_s) r^n,$$

where $p_0, p_1, \ldots, p_s, r$ are all independent of n, and s is any positive integer.

The simplest case is that in which the integral function reduces to a constant. The nth term is then of the form $p_s r^n$, or say $p_s r \cdot r^{n-1}$, that is, ar^{n-1}, where $a = p_s r$ is a constant.

The ratio of the nth to the $(n-1)$th term in this special case is $ar^n/ar^{n-1} = r$, that is to say, is constant.

A series in which the ratio of each term to the preceding is constant is called a GEOMETRIC SERIES *or* GEOMETRIC PROGRESSION; *and the constant ratio in question is called the* COMMON RATIO.

If the first term be a and the common ratio r, the second term is ar; the third $(ar)r$, that is, ar^2; the fourth $(ar^2)r$, that is, ar^3; and so on. The nth term is ar^{n-1}. A geometric series is therefore neither more nor less general than that particular case of the general class of series now under discussion which introduced it to our notice.

§ 12.] *To sum a geometrical series.*

Let
$$\Sigma = a + ar + ar^2 + \ldots + ar^{n-1} \qquad (1).$$

Multiply both sides of (1) by $1 - r$ and we have

$$\begin{aligned}(1-r)\Sigma &= a + ar + ar^2 + \ldots + ar^{n-1}\\ &\qquad - ar - ar^2 - \ldots - ar^{n-1} - ar^n,\\ &= a - ar^n \qquad (2).\end{aligned}$$

Hence
$$\Sigma = a\frac{1 - r^n}{1 - r} \qquad (3).$$

Since the number of operations on the right-hand side of (3) is independent of n,* we have thus obtained the sum of the series (1).

Cor. If l be the last term of the series, then $l = ar^{n-1}$ and $ar^n = rl$. Hence (3) may be written

* Here we regard the raising of r to the nth power as a single operation.

$$\Sigma = \frac{a - rl}{1 - r} \tag{4}.$$

Example 1.

$$\Sigma = \tfrac{3}{2} + \tfrac{3}{4} + \tfrac{3}{8} + . \; . \; . \text{ to 10 terms.}$$

In this case $a = \frac{3}{2}$, $r = \frac{1}{2}$. Hence

$$\Sigma = \tfrac{3}{2}\,\frac{1 - (\frac{1}{2})^{10}}{1 - \frac{1}{2}} = 3\left(1 - \frac{1}{2^{10}}\right).$$

Example 2.

$$\Sigma = 1 - 2 + 4 - 8 + 16 \; . \; . \; . \text{ to } n \text{ terms.}$$

Here $a = 1$, $r = -2$. Hence

$$\Sigma = 1 \,.\, \frac{1 - (-2)^n}{1 - (-2)} = \frac{1 - (-1)^n 2^n}{3},$$
$$= \tfrac{1}{3}(1 - 2^n), \text{ if } n \text{ be even,}$$
$$= \tfrac{1}{3}(1 + 2^n), \text{ if } n \text{ be odd.}$$

Example 3.

$$\Sigma = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + . \; . \; . \text{ to } n \text{ terms,}$$
$$= \frac{x^2 - y^2}{x - y} + \frac{x^3 - y^3}{x - y} + \frac{x^4 - y^4}{x - y} + . \; . \; . + \frac{x^{n+1} - y^{n+1}}{x - y},$$
$$= \frac{1}{x - y}(x^2 + x^3 + . \; . \; . + x^{n+1}) - \frac{1}{x - y}(y^2 + y^3 + . \; . \; . + y^{n+1}),$$
$$= \frac{x^2}{x - y}(1 + x + . \; . \; . + x^{n-1}) - \frac{y^2}{x - y}(1 + y + . \; . \; . + y^{n-1}).$$

Now $\quad 1 + x + . \; . \; . + x^{n-1} = (1 - x^n)/(1 - x)$,

and $\quad 1 + y + . \; . \; . + y^{n-1} = (1 - y^n)/(1 - y)$,

hence $\quad \Sigma = \dfrac{x^2(1 - x^n)}{(x - y)(1 - x)} - \dfrac{y^2(1 - y^n)}{(x - y)(1 - y)}.$

§ 13.] We next proceed to consider *the case where the integral function which multiplies r^n is of the 1st degree.*

The general term in this case is

$$(a + bn)r^n \tag{1},$$

where a and b are constants.

It will be observed that a term of this form would result if we multiplied together the nth term of any arithmetic series by the nth term of any geometric series. For this reason a series whose nth term has the form (1) is often called an *arithmetico-geometric series.*

The series may be summed by an extension of the artifice employed to sum a G.P.

Let

$$\Sigma = (a + b.1)r^1 + (a + b.2)r^2 + (a + b.3)r^3 + . \; . \; . + (a + b.n)r^n.$$

Multiply by $1-r$, and we have

$$
\begin{aligned}
&(1-r)\Sigma \\
&= (a+b.1)r^1 + (a+b.2)r^2 + (a+b.3)r^3 + \dots + (a+bn)r^n \\
&\qquad - (a+b.1)r^2 - (a+b.2)r^3 - \dots - (a+b\overline{n-1})r^n \\
&\qquad\qquad - (a+bn)r^{n+1}, \\
&= (a+b.1)r + \left| br^2 + br^3 + \dots + br^n \right| - (a+bn)r^{n+1} \qquad (1).
\end{aligned}
$$

Looking merely at the terms within the two vertical lines, we see that these constitute a geometric series. Hence, if we multiply by $1-r$ a second time, there will be no series left on the right-hand side; and we shall in effect have found the required expression for Σ. We have, in fact,

$$
\begin{aligned}
&(1-r)^2\Sigma \\
&\quad = (1-r)(a+b)r + br^2 + br^3 + \dots + br^n \\
&\qquad\qquad - br^3 - \dots - br^n - br^{n+1} \\
&\qquad\qquad\qquad - (1-r)(a+bn)r^{n+1}, \\
&\quad = (1-r)(a+b)r + br^2 - br^{n+1} - (1-r)(a+bn)r^{n+1}, \\
&\quad = (a+b)r - (a+b)r^2 + br^2 - \{a+(n+1)b\}r^{n+1} + (a+bn)r^{n+2} \quad (2).
\end{aligned}
$$

Hence

$$
\Sigma = \frac{(a+b)r - (a+b)r^2 + br^2 - \{a+(n+1)b\}r^{n+1} + (a+bn)r^{n+2}}{(1-r)^2} \qquad (3).
$$

§ 14.] If the reader has not already perceived that the artifice of multiplying repeatedly by $1-r$ will sum any series of the general form indicated in § 11, probably the following argument will convince him that such is the case.

Let $f_s(n)$ denote an integral function of n of the sth degree; then the degree of $f_s(n)-f_s(n-1)$ is the $(s-1)$th, since the two terms in n^s destroy each other. Hence we may denote $f_s(n)-f_s(n-1)$ by $f_{s-1}(n)$. Similarly, $f_{s-1}(n)-f_{s-1}(n-1)$ will be an integral function of n of the $(s-2)$th degree, and may be denoted by $f_{s-2}(n)$, and so on.

Consider now the series

$$
\Sigma = f_s(1)r^1 + f_s(2)r^2 + \dots + f_s(n)r^n \qquad (1).
$$

Multiply by $1-r$, and we have

$$\begin{aligned}(1-r)\Sigma &= f_s(1)r^1 + f_s(2)r^2 + f_s(3)r^3 \ldots\ldots + f_s(n)r^n\\ &\quad - f_s(1)r^2 - f_s(2)r^3 \ldots - f_s(n-1)r^n - f_s(n)r^{n+1},\\ &= f_s(1)r^1 + |\, f_{s-1}(2)r^2 + f_{s-1}(3)r^3 + \ldots + f_{s-1}(n)r^n \,| - f_s(n)r^{n+1} \quad (2).\end{aligned}$$

The series between the vertical lines in (2) is now simpler than that in (1); since the integral function which multiplies r^n is now of the $(s-1)$th degree only.

If we multiply once more by $1-r$ we shall find on the right certain terms at the beginning and end, together with a series whose nth term is now $f_{s-2}(n)r^n$.

Each time we multiply by $1-r$ we reduce the degree of the multiplier of r^n by unity. Hence by multiplying by $(1-r)^{s+1}$ we shall extirpate the *series* on the right-hand side altogether, and there will remain only a fixed number of terms.

It follows that any series whose nth term consists of an integral function of n of the sth degree multiplied by r^n can be summed by simply multiplying by $(1-r)^{s+1}$.

This simple proposition contains the whole theory of the summation of the class of series now under discussion.

Example 1. $\Sigma = 1^2r + 2^2r^2 + 3^2r^3 + \ldots + n^2r^n$.

Here the degree of the multiplier of r^n is 2. Hence, in order to effect the summation, we must multiply by $(1-r)^3$. We thus find

$$\begin{aligned}(1-r)^3\Sigma &= 1^2r + 2^2r^2 + 3^2r^3 + 4^2r^4 + \ldots + n^2r^n\\ &\quad - 3.1^2r^2 - 3.2^2r^3 - 3.3^2r^4 - \ldots - 3(n-1)^2r^n - 3n^2r^{n+1}\\ &\quad + 3.1^2r^3 + 3.2^2r^4 - \ldots + 3(n-2)^2r^n + 3(n-1)^2r^{n+1} + 3n^2r^{n+2}\\ &\quad - 1^2r^4 - \ldots - (n-3)^2r^n - (n-2)^2r^{n+1} - (n-1)^2r^{n+2}\\ &\quad - n^2r^{n+3},\\ &= r + r^2 - (n+1)^2r^{n+1} + (2n^2+2n-1)r^{n+2} - n^2r^{n+3}.\end{aligned}$$

Hence

$$\Sigma = \frac{r + r^2 - (n+1)^2r^{n+1} + (2n^2+2n-1)r^{n+2} - n^2r^{n+3}}{(1-r)^3}.$$

Example 2. $\Sigma = 1 - 2r + 3r^2 - 4r^3 + \ldots - 2nr^{2n-1}$.

Multiply by $(1+r)^2$, and we have

$$\begin{aligned}(1+r)^2\Sigma &= 1 - 2r + 3r^2 - 4r^3 + \ldots - 2nr^{2n-1}\\ &\quad + 2r - 2.2r^2 + 2.3r^3 - \ldots + 2(2n-1)r^{2n-1} - 2.2nr^{2n}\\ &\quad + r^2 - 2r^3 + \ldots - (2n-2)r^{2n-1} + (2n-1)r^{2n} - 2nr^{2n+1},\\ &= 1 - (2n+1)r^{2n} - 2nr^{2n+1}.\end{aligned}$$

Hence

$$\Sigma = \frac{1-(2n+1)r^{2n} - 2nr^{2n+1}}{(1+r)^2}.$$

If we put $r=1$, we deduce

$$1-2+3-4 \ . \ . \ . -2n=-n,$$

which agrees with § 3, Example 3, above.

CONVERGENCY AND DIVERGENCY OF THE ABOVE SERIES.

§ 15.] We have seen that the sum of n terms of a series whose nth term is an integral function of n is an integral function of n; and we have seen that every integral function becomes infinite for an infinite value of its variable. Hence the sum of n terms of any series whose nth term is an integral function of n may be made to exceed (numerically) any quantity, however great, by sufficiently increasing n.

This is expressed by saying that every such series is *divergent.*

§ 16.] Consider the geometric series

$$\Sigma = a + ar + ar^2 + \ . \ . \ . + ar^{n-1}.$$

If $r=1$, the series becomes

$$\Sigma = a + a + a + . \ . \ . + a = na.$$

Hence, by sufficiently increasing n, we may cause Σ to surpass any value, however great.

If r be numerically greater than 1, the same is true, for we have

$$\Sigma = \frac{a(r^n-1)}{r-1},$$

$$= \frac{ar^n}{r-1} - \frac{a}{r-1}.$$

Now, since $r > 1$, we can, by sufficiently increasing n, make r^n, and therefore $ar^n/(r-1)$, as great as we please. Hence, by sufficiently increasing n, we can cause Σ to surpass any value, however great (see Ex. ix. 46).

In these two cases the geometric series is said to be divergent.

If r be numerically less than 1, we can, by sufficiently increasing n, make r^n as small as we please, and therefore $ar^n/(1-r)$ as small as we please. Hence, by sufficiently increasing n, we can cause Σ to differ from $a/(1-r)$ as little as we please. This is often expressed by saying that *when r is numerically less than* 1, *the sum to infinity of the series $a + ar + ar^2 + . \ . \ .$ is $a/(1-r)$.*

In this case the series is said to be convergent, and to converge to the value $a/(1-r)$.

There is yet another case worthy of notice.

If $r = -1$, the series becomes

$$\Sigma = a - a + a - a + \ . \ . \ .$$

Hence the sum of an odd number of its terms is always a, and the sum of an even number of them always 0. The sum, therefore, does not become infinite when an infinite number of terms are taken; but neither does it converge to one definite value. A series having this property is sometimes said to *oscillate*.

Example 1.

Find the limit of the sum of an infinite number of terms of the series

$$\Sigma = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ . \ . \ .$$

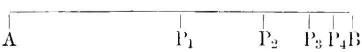
For n terms we have

$$\Sigma = \tfrac{1}{2}\frac{1-1/2^n}{1-\frac{1}{2}} = 1 - \frac{1}{2^n}.$$

Hence, when n is made infinitely great,

$$\Sigma = 1.$$

This case may be illustrated geometrically as follows:—



Let AB be a line of unit length. Bisect AB in P_1; bisect P_1B in P_2, P_2B in P_3; and so on indefinitely.

It is obvious that by a sufficient number of these operations we can come nearer to B than any assigned distance, however small. In other words, if we take a sufficient number of terms of the series

$$AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \ . \ . \ .,$$

we shall have a result differing from AB, that is, from unity, as little as we please.

This is simply a geometrical way of saying that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ . \ . \ . \ ad\ \infty = 1.$$

Example 2.

To evaluate the recurring decimal $\cdot\dot{3}\dot{4}$.

Let

$$\Sigma = \cdot\dot{3}\dot{4} = \frac{34}{100} + \frac{34}{100^2} + \frac{34}{100^3} + \ . \ . \ . \ ad\ \infty.$$

Then Σ is obviously a geometric series, whose common ratio, 1/100, is less than 1. Hence

$$\Sigma = \frac{34}{100}\frac{1}{1-\frac{1}{100}} = \frac{34}{99}.$$

PROPERTIES OF QUANTITIES WHICH ARE IN ARITHMETIC, GEOMETRIC, OR HARMONIC PROGRESSION.

§ 17.] If a be the first term, b the common difference, n the number of terms, and Σ the sum of an arithmetic progression, we have

$$\Sigma = \frac{n}{2}\{2a + (n-1)b\} \qquad (1).$$

This equation enables us to determine any one of the four quantities, Σ, a, b, n, when the other three are given. The equation is an integral equation of the 1st degree in all cases, except when n is the unknown quantity, in which case the equation is a quadratic. This last case presents some points of interest, which we may illustrate by a couple of examples.

Example 1.

Given $\Sigma=36$, $a=15$, $b=-3$, to find n. We have by the formula (1) above

$$36=\frac{n}{2}\{30-(n-1)3\}.$$

Hence
$$n^2-11n+24=0.$$

The roots of this equation are $n=3$ and $n=8$. It may seem strange that there should be two different numbers of terms for which the sum is the same. The mystery is explained by the fact that the common difference is negative. The series is, in fact,

$$15+12+9 \mid +6+3+0-3-6 \mid -9-\ldots:$$

and, inasmuch as the sum of the part between the vertical lines is zero, the sum of 8 terms of the series is the same as the sum of 3 terms.

Example 2.

$$\Sigma=14, \quad a=3, \quad b=2.$$

The equation for n in this case is

$$n^2+2n=14.$$

Hence
$$n=-1\pm\sqrt{(15)}=+2\cdot87\ldots, \text{ or } -4\cdot87\ldots$$

The second of the roots, being negative, has no immediate reference to our problem. The first root is admissible so far as its sign is concerned, but it is open to objection because it is fractional, for, from the nature of the case, n must be integral. It may be conjectured, therefore, that we have set ourselves an impossible problem. Analytically considered, the function n^2+2n varies continuously, and there is in the abstract no difficulty in giving to it any value whatsoever. The sum of an arithmetic series, on the other hand, varies *per saltum*; and it so happens that 14 is not one of the values that Σ can assume when $a=3$ and $b=2$. There are, however, two values which Σ

can assume between which 14 lies; and we should expect that the integers next lower and next higher than 2·87 would correspond to these values of Σ. So, in fact, it is; for, when $n=2$ $\Sigma=8$, and when $n=3$ $\Sigma=15$.

§ 18.] An arithmetic progression is determined when its first term and common difference are given; that is to say, when these are given we can write down as many terms of the progression as we please. An arithmetic progression is therefore what mathematicians call a *twofold manifoldness*; that is, it is determined by any two independent data.

Bearing this in mind, we can write the most general arithmetic progressions of 3, 4, 5, &c. terms as follows:—

$$\alpha-\beta, \quad \alpha, \quad \alpha+\beta,$$
$$\alpha-3\beta, \quad \alpha-\beta, \quad \alpha+\beta, \quad \alpha+3\beta,$$
$$\alpha-2\beta, \quad \alpha-\beta, \quad \alpha, \quad \alpha+\beta, \quad \alpha+2\beta,$$
$$\&c.,$$

where α and β are any quantities whatsoever. It will be observed that in the cases where we have an odd number of terms the common difference is β, in the cases where we have an even number 2β. These formulæ are sometimes useful in establishing equations of condition between quantities in A.P.

Example 1.

Given that the pth term of an A.P. is P, and that the qth term is Q, to find the A.P. Let a be the first term and b the common difference; then the pth and qth terms are $a+(p-1)b$ and $a+(q-1)b$ respectively. Hence

$$a+(p-1)b=\text{P}, \quad a+(q-1)b=\text{Q}.$$

These are two equations of the 1st degree to determine a and b.

We find

$$b=(\text{P}-\text{Q})/(p-q), \quad a=\{(p-1)\text{Q}-(q-1)\text{P}\}/(p-q).$$

Example 2.

If a, b, c be in A.P., show that

$$a^2(b+c)+b^2(c+a)+c^2(a+b)=\frac{2}{9}(a+b+c)^3.$$

We may put $\qquad a=\alpha-\beta, \quad b=\alpha, \quad c=\alpha+\beta.$

The equation to be established is now

$$(\alpha-\beta)^2(2\alpha+\beta)+\alpha^2.2\alpha+(\alpha+\beta)^2(2\alpha-\beta)=\frac{2}{9}(3\alpha)^3,$$
$$=6\alpha^3$$

Since α and β are independent of one another, this equation must be an identity. The left-hand side reduces to

$$2\alpha\{(\alpha-\beta)^2+(\alpha+\beta)^2\}+\beta\{(\alpha-\beta)^2-(\alpha+\beta)^2\}+2\alpha^3,$$
$$\equiv 2\alpha\{2\alpha^2+2\beta^2\}+\beta\{-4\alpha\beta\}+2\alpha^3,$$
$$\equiv 6\alpha^3.$$

Hence the required result is established.

§ 19.] If three quantities, a, b, c, be in A.P., we have $b-a=c-b$ by definition. Hence

$$b=(c+a)/2.$$

In this case b is spoken of as the *arithmetic mean* between a and c. The arithmetic mean between two quantities is therefore merely what is popularly called their average.

If a and c be any two quantities whatsoever, and $A_1, A_2, \ldots, A_n$ n others, such that $a, A_1, A_2, \ldots, A_n, c$ form an A.P., then $A_1, A_2, \ldots, A_n$ are said to be n arithmetic means inserted between a and c.

There is no difficulty in finding $A_1, A_2, \ldots, A_n$ when a and c are given. For, if b be the common difference of the A.P., $a, A_1, A_2, \ldots, A_n, c$, then

$$A_1=a+b, \quad A_2=a+2b, \quad \ldots, \quad A_n=a+nb,$$

and
$$c=a+(n+1)b.$$

From the last of these we deduce $b=(c-a)/(n+1)$. Hence we have

$$A_1=a+\frac{c-a}{n+1}, \quad A_2=a+2\,\frac{c-a}{n+1}, \quad \&c.$$

N.B.—By the arithmetic mean or average of n quantities $a_1, a_2, \ldots, a_n$ is meant $(a_1+a_2+\ldots+a_n)/n$.

In the particular case where two quantities only are in question, the arithmetic mean in this sense agrees with the definitions given above; but in other cases the meanings of the phrases have nothing in common.

Example 1.

Insert 30 arithmetic means between 5 and 90; and find the arithmetic mean of these means.

Let b be the common difference of the A.P. 5, A_1, A_2, . . ., A_{30}, 90. Then

$$b=(90-5)/(30+1)=85/31.$$

Hence the means are

$$5+\frac{85}{31}, \quad 5+2.\frac{85}{31}, \quad 5+3.\frac{85}{31}, \quad \&c.;$$

that is,
$$\frac{240}{31}, \quad \frac{325}{31}, \quad \frac{410}{31}, \quad \&c.$$

We have
$$\frac{A_1+A_2+\ .\ .\ .\ +A_n}{n}=\frac{1}{n}\left\{n\frac{A_1+A_n}{2}\right\},$$
$$=\frac{A_1+A_n}{2},$$
$$=\left(5+\frac{85}{31}+90-\frac{85}{31}\right)\Big/2,$$
$$=(5+90)/2=95/2.$$

Remark.—It is true generally that the arithmetic mean of the n arithmetic means between a and c is the arithmetic mean between a and c.

Example 2.

The arithmetic mean of the squares of n quantities in A.P. exceeds the square of their arithmetic mean by a quantity which depends only upon n and upon their common difference.

Let the n quantities be
$$a+b,\quad a+2b,\quad .\ .\ .,\quad a+nb.$$
Then, by §§ 5 and 6,
$$\frac{(a+b)^2+(a+2b)^2+\ .\ .\ .\ +(a+nb)^2}{n}$$
$$=\frac{1}{n}\left\{a^2n+abn(n+1)+b^2\frac{n(n+1)(2n+1)}{6}\right\},$$
$$=a^2+ab(n+1)+\frac{b^2}{6}(2n^2+3n+1).$$
Again,
$$\left\{\frac{(a+b)+(a+2b)+\ .\ .\ .\ +(a+nb)}{n}\right\}^2=\left(a+\frac{n+1}{2}b\right)^2$$
$$=a^2+ab(n+1)+\frac{b^2}{4}(n^2+2n+1).$$
Hence
$$\text{A.M. of squares} - \text{square of A.M.}=\frac{n^2-1}{12}b^2,$$
which proves the proposition.

§ 20.] If Σ be the sum of n terms of a geometric progression whose first term and common ratio are a and r respectively, we have
$$\Sigma=a\frac{r^n-1}{r-1} \qquad (1).$$

When any three of the four, Σ, a, r, n, are given, this equation determines the fourth. When either Σ or a is the unknown quantity, we have to solve an equation of the 1st degree. When r is the unknown quantity, we have to solve an integral equation of the nth degree, which, if n exceeds 2, will in general be effected by graphical or other approximative methods. If n be the unknown quantity, we have to solve an exponential equation of the form $r^n=s$, where r and s are known. This may be

accomplished at once by means of a table of logarithms, as we shall see in the next chapter.

§ 21.] Like an A.P., *a G.P. is a twofold manifoldness*, and may be determined by means of its first term and common ratio, or by any other two independent data.

In establishing any equation between quantities in G.P., it is usual to express all the quantities involved in terms of the first term and common ratio. Since these two are independent, the equation in question must then become an identity.

Example 1.

The pth term of a G.P. is P, and the qth term is Q; find the first term and common ratio.

Let a be the first term, r the common ratio. Then we have, by our data,

$$ar^{p-1}=\mathrm{P}, \quad ar^{q-1}=\mathrm{Q}.$$

From these, by division, we deduce

$$r^{p-q}=\mathrm{P}/\mathrm{Q}, \text{ whence } r=(\mathrm{P}/\mathrm{Q})^{1/(p-q)}.$$

Using this value of r in the first equation, we find

$$a=\mathrm{P}/(\mathrm{P}/\mathrm{Q})^{(p-1)/(p-q)}=\mathrm{P}^{(1-q)/(p-q)}\mathrm{Q}^{(1-p)/(q-p)}.$$

Hence we have

$$a=\mathrm{P}^{(1-q)/(p-q)}\mathrm{Q}^{(1-p)/(q-p)}, \quad r=\mathrm{P}^{1/(p-q)}\mathrm{Q}^{1/(q-p)}.$$

Example 2.

If a, b, c, d be four quantities in G.P., prove that

$$4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2=(a-b)^2+(c-d)^2+2(a-d)^2.$$

If the common ratio be denoted by r, we may put $b=ra$, $c=r^2a$, $d=r^3a$. The equation to be established then becomes

$$4a^2(1+r^2+r^4+r^6)-a^2(1+r+r^2+r^3)^2\equiv a^2(1-r)^2+a^2r^4(1-r)^2+2a^2(1-r^3)^2,$$

that is,

$$4(1+r^2+r^4+r^6)-(1+2r+3r^2+4r^3+3r^4+2r^5+r^6)$$
$$\equiv 1-2r+r^2+r^4-2r^5+r^6+2-4r^3+2r^6,$$

which is obviously true.

§ 22.] *When three quantities, a, b, c, are in G.P., b is called the geometric mean between a and c.*

We have, by definition, $c/b=b/a$. Hence $b^2=ac$. Hence, if we suppose a, b, c to be all positive real quantities, $b= + \sqrt{(ac)}$. That is to say, *the geometric mean between two real positive quantities is the positive value of the square root of their product.*

If a and c be two given positive quantities, and $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_n$ *n quantities, such that a,* $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_n$, *$c$ form a G.P., then* $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_n$ *are said to be n geometric means inserted between a and c.*

Let r be the common ratio of the supposed progression. Then we have $G_1 = ar$, $G_2 = ar^2$, . . ., $G_n = ar^n$, $c = ar^{n+1}$. From the last of these equations we deduce $r = (c/a)^{1/(n+1)}$, the real positive value of the root being, of course, taken. Since r is thus determined, we can find the value of all the geometric means.

The geometric mean of n positive real quantities is the positive value of the nth root of their product. This definition agrees with the former definition when there are two quantities only.

Example.

The geometric mean of the n geometric means between a and c is the geometric mean between a and c.

Let the n geometric means in question be ar, ar^2, . . ., ar^n, so that $c = ar^{n+1}$. Then

$$\begin{aligned}(ar.\,ar^2 \dots ar^n)^{1/n} &= (a^n r^{1+2+\dots+n})^{1/n},\\ &= (a^n r^{n(n+1)/2})^{1/n},\\ &= ar^{(n+1)/2},\\ &= (a^2 r^{n+1})^{1/2},\\ &= (ac)^{1/2},\end{aligned}$$

which proves the proposition.

§ 23.] *A series of quantities which are such that their reciprocals form an arithmetic progression are said to be in harmonic progression.*

From this definition we can deduce the following, which is sometimes given as the defining property :—

If a, b, c be three consecutive terms of a harmonic progression, then

$$a/c = (a-b)/(b-c) \qquad (1).$$

For, by definition, $1/a$, $1/b$, $1/c$ are in A.P., therefore

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Hence
$$\frac{a-b}{ab} = \frac{b-c}{bc}.$$

Hence
$$\frac{a-b}{b-c} = \frac{ab}{bc} = \frac{a}{c},$$

which proves the property in question.

§ 24.] *A harmonic progression*, like the arithmetic progression, from which it may be derived, is *a twofold manifoldness.* The following is therefore a perfectly general form for a harmonic

series, $1/(a+b)$, $1/(a+2b)$, $1/(a+3b)$, . . ., $1/(a+nb)$, . . ., for it contains two independent constants a and b; and the reciprocals of the terms are in A.P.

The following forms (see § 18) are perfectly general for harmonic progressions consisting of 3, 4, 5, . . . terms respectively :—

$$1/(a-\beta),\quad 1/a,\quad 1/(a+\beta);$$
$$1/(a-3\beta),\quad 1/(a-\beta),\quad 1/(a+\beta),\quad 1/(a+3\beta);$$
$$1/(a-2\beta),\quad 1/(a-\beta),\quad 1/a,\quad 1/(a+\beta),\quad 1/(a+2\beta),$$
&c.

The above formulæ may be used like those in § 18.

§ 25.] *If* a, b, c *be in H.P.*, b *is called the harmonic mean between* a *and* c. We have, by definition, $1/c-1/b=1/b-1/a$. Hence $2/b=1/a+1/c$, and $b=2ac/(a+c)$.

If a, H_1, H_2, . . ., H_n, c *form a harmonic progression*, H_1, H_2, . . ., H_n *are said to be* n *harmonic means inserted between* a *and* c.

Since $1/a$, $1/H_1$, $1/H_2$, . . ., $1/H_n$, $1/c$ in this case form an A.P., whose common difference is d, say, we have

$$d=(1/c-1/a)/(n+1)=(a-c)/(n+1)ac.$$

Hence

$$1/H_1=1/a+(a-c)/(n+1)ac,\quad 1/H_2=1/a+2(a-c)/(n+1)ac,\ \&c.;$$

and $\quad H_1=(n+1)ac/(a+nc),\quad H_2=(n+1)ac/(2a+(n-1)c)$, &c.

If a quantity H *be such that its reciprocal is the arithmetic mean of the reciprocals of* n *given quantities*, H *is said to be the harmonic mean of the* n *quantities.*

It is easy to see, from the corresponding proposition regarding arithmetic means, that the *harmonic mean of the* n *harmonic means between* a *and* c *is the harmonic mean of* a *and* c.

§ 26.] *The geometric mean between two real positive quantities* a *and* c *is the geometric mean between the arithmetic and the harmonic means between* a *and* c; *and the arithmetic, geometric, and harmonic means are in descending order of magnitude.*

Let A, G, H be the arithmetic, geometric, and harmonic means between a and c, then

$$A=(a+c)/2,\quad G=+\sqrt{(ac)},\quad H=2ac/(a+c).$$

Hence $$AH = \frac{a+c}{2} \times \frac{2ac}{a+c} = ac = G^2,$$

which proves the first part of the proposition.

Again, $$A - G = \frac{a+c}{2} - \sqrt{(ac)} = \tfrac{1}{2}(\sqrt{a} - \sqrt{c})^2.$$

$$G - H = \sqrt{(ac)} - \frac{2ac}{a+c} = \frac{\sqrt{(ac)}}{a+c}(\sqrt{a} - \sqrt{c})^2.$$

Now, since a and c are both positive, $\sqrt{a}$ and $\sqrt{c}$ are both real, therefore $(\sqrt{a} - \sqrt{c})^2$ is an essentially positive quantity; also $\sqrt{(ac)}$ and $a+c$ are both positive. Hence both $A - G$ and $G - H$ are positive.

Therefore $A > G > H$.

The proposition of this paragraph (which was known to the Greek geometers) is merely a particular case of a more general proposition, which will be proved in chap. xxiv.

§ 27.] Notwithstanding the comparative simplicity of the law of its formation, *the harmonic series does not belong to the category of series that can be summed.* Various expressions can be found to represent the sum to n terms, but all of them partake of the nature of a series in this respect, that the number of steps in their synthesis is a function of n.

It will be a good exercise in algebraic logic to prove that the sum of a harmonic series to n terms cannot be represented by any rational algebraical function of n. The demonstration will be found to require nothing beyond the elementary principles of algebraic form laid down in the earlier chapters of this work.

Exercises XL.

Sum the following arithmetical progressions :—

(1.) $5+9+13+\ldots$ to 15 terms. (2.) $3+3\frac{1}{2}+4+\ldots$ to 30 terms.

(3.) $13+12+11+\ldots$ to 24 terms. (4.) $\frac{1}{3}+\frac{1}{5}+\ldots$ to 16 terms.

(5.) $\frac{1}{n}+\frac{n-1}{n}+\ldots$ to n terms.

(6.) $(a-n)^2+(a^2+n^2)+(a+n)^2+\ldots$ to n terms.

(7.) $\frac{1+l}{1-l}+\frac{4l}{1-l^2}+\ldots$ to l terms.

(8.) The 20th term of an A.P. is 100, and the sum of 30 terms is 500; find the sum of 1000 terms of the progression.

(9.) The first term of an A.P. is 5, the number of its terms is 15, and the sum is 390; find the common difference.

(10.) How many of the natural numbers, beginning with unity, amount to 500500?

(11.) Show that an infinite number of A.P.'s can be found which have the property that the sum of the first $2m$ terms is equal to the sum of the next m terms, m being a given integer. Find that particular A.P. having the above property whose first term is unity.

(12.) An author wished to buy up the whole 1000 copies of a work which he had published. For the first copy he paid 1s. But the demand raised the price, and for each successive copy he had to pay 1d. more, until the whole had been bought up. What did it cost him?

(13.) 100 stones are placed on the ground at intervals of 5 yards apart. A runner has to start from a basket 5 yards from the first stone, pick up the stones, and bring them back to the basket one by one. How many yards has he to run altogether?

(14.) AB is a straight line 100 yards long. At A and B are erected perpendiculars, AL, BM, whose lengths are 4 yards and 46 yards respectively. At intervals of a yard along AB perpendiculars are erected to meet the line LM. Find the sum of the lengths of all these perpendiculars, including AL and BM.

(15.) Two travellers start together on the same road. One of them travels uniformly 10 miles a day. The other travels 8 miles the first day, and increases his pace by half a mile a day each succeeding day. After how many days will the latter overtake the former?

(16.) Two men set out from the two ends of a road which is l miles long. The first travels a miles the first day, $a+b$ the next, $a+2b$ the next, and so on. The second travels at such a rate that the sum of the number of miles travelled by him and the number travelled by the first is always the same for any one day, namely c. After how many days will they meet?

(17.) Insert 15 arithmetic means between 3 and 30.

(18.) Insert 10 arithmetic means between -3 and $+3$.

(19.) A certain even number of arithmetic means are inserted between 30 and 40, and it is found that the ratio of the sum of the first half of these means to the second half is 137 : 157. Find the number of means inserted.

(20.) Find the number of terms of the A.P. $1+8+15+\ldots$ the sum of which approaches most closely to 1356.

(21.) If the common difference of an A.P. be double the first term, the sum of m terms : the sum of n terms $=m^2:n^2$.

(22.) Find four numbers in A.P. such that the sum of the squares of the means shall be 106, and the sum of the squares of the extremes 170.

(23.) If four quantities be in A.P., show that the sum of the squares of the extremes is greater than the sum of the squares of the means, and that the product of the extremes is less than the product of the means.

(24.) Find the sum of n terms of the series whose rth term is $\frac{3}{4}(3r+1)$.

(25.) Find the sum of n terms of the series obtained by taking the 1st, rth, $2r$th, $3r$th, &c. terms of the A.P. whose first term and common difference are a and b respectively.

(26.) If the sum of n terms of a series be always $n(n+2)$, show that the series is an A.P.; and find its first term and common difference.

(27.) Show by general reasoning regarding the form of the sum of an A.P. that if the sum of p terms be P, and the sum of q terms Q, then the sum of n terms is $\mathrm{P}n(n-q)/p(p-q)+\mathrm{Q}n(n-p)/q(q-p)$.

(28.) Any even square, $(2n)^2$, is the sum of n terms of one arithmetic series of integers; and any odd square, $(2n+1)^2$, is the sum of n terms of another arithmetic series increased by 1.

(29.) Find n consecutive odd numbers whose sum shall be n^p.

Show that any integral cube is the difference of two integral squares.

(30.) Find the nth term and the sum of the series

$$1-3+6-10+15-21+\ldots.$$

(31.) Sum the series $3+6+\ldots+3n$.

(32.) If $s_1, s_2, \ldots, s_p$ be the sums of p arithmetical progressions, each having n terms, the first terms of which are $1, 2, \ldots, p$, and the common differences $1, 3, \ldots, \overline{2p-1}$ respectively, show that $s_1+s_2+\ldots+s_p$ is equal to the sum of the first np integral numbers.

(33.) The series of integral numbers is divided into groups as follows:— $1, | 2, 3, | 4, 5, 6, | 7, 8, 9, 10, | \ldots$, show that the sum of the nth group is $\frac{1}{2}(n^3+n)$.

If the series of odd integers be divided in the same way, find the sum of the nth group.

Sum the following series:—

(34.) $4^2+7^2+\ldots+(3n+1)^2$. (35.) $\Sigma_n(n^2-1)(n-1)$.

(36.) $\Sigma_n\{p+q(n-1)\}\{p+q(n-2)\}$.

(37.) $1^2-2^2+3^2-\ldots+(2n-1)^2-(2n)^2$.

(38.) $a^3+(a+b)^3+\ldots+(a+\overline{n-1}b)^3$.

(39.) $(1^3-1)+(2^3-2)+(3^3-3)+\ldots$ to n terms.

(40.) $1.2^2+2.3^2+3.4^2+\ldots$ to n terms.

(41.) $1+2.3^2+3.5^2+4.7^2+5.9^2+\ldots$ to n terms.

(42.) $1.3.7+3.5.9+5.7.11+\ldots$ to n terms.

(43.) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$ to n terms.

(44.) A pyramid of shot stands on an equilateral triangular base having 30 shot in each side. How many shot are there in the pyramid?

(45.) A pyramid of shot stands on a square base having m shot in each side. How many shot in the pyramid?

(46.) A symmetrical wedge-shaped pile of shot ends in a line of m shot and consists of l layers. How many shot in the pile?

(47.) If ${}_n\mathrm{S}_r=1^r+2^r+\ldots+n^r$, then ${}_n\mathrm{S}_r=p_0n^{r+1}+p_1n^r+\ldots+p_{r+1}$, where $p_0, p_1, \ldots$ can be calculated by means of the equations

$$\begin{aligned}{}_{r+1}\mathrm{C}_1\,p_0&={}1,\\ {}_{r+1}\mathrm{C}_2\,p_0+{}_r\mathrm{C}_1\,p_1&={}_r\mathrm{C}_1,\\ {}_{r+1}\mathrm{C}_3\,p_0+{}_r\mathrm{C}_2\,p_1+{}_{r-1}\mathrm{C}_1\,p_2&={}_r\mathrm{C}_2,\\ \ldots\ldots\ldots\ldots&\end{aligned}$$

${}_n\mathrm{C}_r$ denoting, as usual, the rth binomial coefficient of the nth rank.

(48.) Show that

$${}_n\mathrm{S}_r = n(n+1)(n-1)(n-2)\ldots(n-r)\left\{\frac{{}_r\mathrm{S}_r}{r(r+1)(n-r)(r-1)!}\right.$$
$$-\frac{{}_{r-1}\mathrm{S}_r}{(r-1)r(n-r+1)1!(r-2)!}+\frac{{}_{r-2}\mathrm{S}_r}{(r-2)(r-1)(n-r+2)2!(r-3)!}$$
$$\left.-\ldots\frac{(-)^r{}_1\mathrm{S}_r}{1.2(n-1)(r-1)!}\right\},$$

where $r\,!$ stands for $1.2.3\ .\ .\ .\ r$.

(49.) If P_r denote the sum of the products r at a time of $1, 2, 3, \ldots, n$, and S_r denote $1^r+2^r+\ldots+n^r$, show that $r\mathrm{P}_r=\mathrm{S}_1\mathrm{P}_{r-1}-\mathrm{S}_2\mathrm{P}_{r-2}+\mathrm{S}_3\mathrm{P}_{r-3}-\ldots$ Hence calculate P_2 and P_3.

(50.) If $f(x)$ be an integral function of x of the $(r-1)$th degree, show that $f(x)-{}_r\mathrm{C}_1f(x-1)+{}_r\mathrm{C}_2f(x-2)\ldots(-)^rf(n-r)\equiv 0$, ${}_r\mathrm{C}_1$, ${}_r\mathrm{C}_2\ldots$ being binomial coefficients.

Exercises XLI.

Sum the following geometric progressions :—

(1.) $6+18+54+\ldots$ to 12 terms. (2.) $6-18+54-\ldots$ to 12 terms.

(3.) $\cdot3333\ldots$ to n terms. (4.) $1-\frac{1}{4}+\frac{1}{4^2}-\ldots$ to n terms.

(5.) $6-4+\ldots$ to 10 terms.

(6.) $\frac{\sqrt{3}}{\sqrt{3}+1}+\frac{\sqrt{3}}{\sqrt{3}+2}+\ldots$ to 20 terms.

(7.) $1+\frac{1}{3}+\frac{1}{3^2}+\ldots$ to n terms. (8.) $1-\frac{1}{2}+\frac{1}{2^2}-\ldots$ to ∞.

(9.) $1-x+x^2-x^3+\ldots$ to ∞, $x<1$.

(10.) $\sqrt{2}+\frac{1}{\sqrt{2}}+\ldots$ to ∞. (11.) $\frac{\sqrt{3}}{\sqrt{3}+1}+\frac{\sqrt{3}}{\sqrt{3}+3}+\ldots$ to ∞.

(12.) $\frac{a+x}{a-x}-\frac{a-x}{a+x}+\ldots$ to ∞.

Sum, by means of the formula for a G.P., the following :—

(13.) $1+x-x^2-x^3+x^4+x^5-x^6-x^7+\ldots$ to ∞, $x<1$.

(14.) $(x-y)+\left(\frac{y^2}{x}-\frac{y^3}{x^2}\right)+\left(\frac{y^4}{x^3}-\frac{y^5}{x^4}\right)+\ldots$ to n terms.

(15.) $1+(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$ to n terms.

(16.) $\cdot33+\cdot333+\cdot3333+\ldots$ to n terms.

Sum the series whose nth terms are as follows :—

(17.) 2^n3^{n+1}. (18.) $(x^n+n)(x^n-n)$.

(19.) $\left(x^n-\frac{1}{x^n}\right)\left(y^n-\frac{1}{y^n}\right)$. (20.) $(p^n-q^n)(p^n+q^n)$.

(21.) $2^n(3^{n-1}+3^{n-2}+\ldots+1)$. (22.) $(-1)^na^{3n}$.

(23.) Sum to n terms the series $(r^n+1/r^n)^2+(r^{n+1}+1/r^{n+1})^2+\ldots$.

(24.) Sum to n terms $(1+1/r)^2+(1+1/r^2)^2+\ldots$.

(25.) Show that $(a+b)^n-b^n=ab^{n-1}+ab^{n-2}(a+b)+\ldots+a(a+b)^{n-1}$.

(26.) If S_t denote the sum of n terms of a G.P. beginning with the tth term, sum the series $\mathrm{S}_1+\mathrm{S}_2+\ldots+\mathrm{S}_t$.

(27.) Show that $\sqrt[3]{(\cdot\dot{0}3\dot{7})}=\cdot\dot{3}$.

Sum the following series to n terms, and, where admissible, to infinity :—

(28.) $1-2x+3x^2-4x^3+\ldots$ (29.) $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots$

(30.) $1-\frac{2^2}{2}+\frac{3^2}{2^2}-\frac{4^2}{2^3}+\ldots$ (31.) $1+\frac{1.2}{x}+\frac{2.3}{x^2}+\frac{3.4}{x^3}+\ldots$

(32.) $1^3+2^3x+3^3x^2+\ldots$

(33.) $\frac{1}{7}+\frac{2}{7^2}+\frac{3}{7^3}+\frac{1}{7^4}+\frac{2}{7^5}+\frac{3}{7^6}+\ldots$ to ∞, where the numerators recur with the period 1, 2, 3.

(34.) $\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\frac{a}{r^4}+\frac{b}{r^5}+\frac{c}{r^6}+\ldots$ to $3n$ terms, where the numerators recur with the period a, b, c.

(35.) A servant agrees to serve his master for twelve months, his wages to be one farthing for the first month, a penny for the second, fourpence for the third, and so on. What did he receive for the year's service?

(36.) A precipitate at the bottom of a beaker of volume V always retains about it a volume v of liquid. It was originally precipitated in an alkaline solution ; find what percentage of this solution remains about it after it has been washed n times by filling the beaker with distilled water and emptying it. Neglect the volume of the precipitate itself.

(37.) The middle points of the sides of a triangle of area Δ_1 form the vertices of a second triangle of area Δ_2; from Δ_2 a third triangle of area Δ_3 is derived in the same way ; and so on, *ad infinitum*. Find the sum of the areas of all the triangles thus formed.

(38.) OX, OY are two given straight lines. From a point in OX a perpendicular is drawn to OY ; from the foot of that perpendicular a perpendicular on OX ; and so on, *ad infinitum*. If the lengths of the first and second perpendicular be a and b respectively, find the sum of the lengths of all the perpendiculars ; and also the sum of the areas of all the right-angled triangles in the figure whose vertices lie on OY and whose bases lie on OX.

(39.) The population of a certain town is P at a certain epoch. Annually it loses d per cent by deaths, and gains b per cent by births, and annually a fixed number E emigrate. Find the population after the lapse of n years.

Σ, a, r, n, l, having the meanings assigned to them in § 12, solve the following problems :—

(40.) Express Σ in terms of a, n, l ; and also in terms of r, n, l.

(41.) $\Sigma=4400$, $a=11$, $n=4$, find r.

(42.) $\Sigma=180$, $r=3$, $n=5$, find a.

(43.) $\Sigma=95$, $a=20$, $n=3$, find r.

(44.) $\Sigma=155$, $a=5$, $n=3$, find r.

(45.) $\Sigma=605$, $a=5$, $r=3$, find n.

(46.) If the second term of a G.P. be 40, and the fourth term 1000, find the sum of 10 terms.

(47.) Insert one geometric mean between $\sqrt{3}/\sqrt{2}$ and $3\sqrt{3}/2\sqrt{2}$.

(48.) Insert three geometric means between 27/8 and 2/3.

(49.) Insert four geometric means between 2 and 64.

(50.) Find the geometric mean of 4, 48, and 405.

(51.) The geometric mean between two numbers is 12, and the arithmetic mean is $25\frac{1}{2}$: find the numbers.

(52.) Four numbers are in G.P., the sum of the first two is 44, and of the last two 396: find them.

(53.) Find what common quantity must be added to a, b, c to bring them into G.P.

(54.) To each of the first two of the four numbers 3, 35, 190, 990 is added x, and to each of the last two y. The numbers then form a G.P.: find x and y.

(55.) Given the sum to infinity of a convergent G.P., and also the sum to infinity of the squares of its terms, find the first term and the common ratio.

(56.) If $S=a_1+a_2+\ldots+a_n$ be a G.P., then $S'=1/a_1+1/a_2+\ldots+1/a_n$ is a G.P., and $S/S'=a_1a_n$.

(57.) If four quantities be in G.P., the sum of the squares of the extremes is greater than the sum of the squares of the means.

(58.) Sum $2n$ terms of a series in which every even term is a times the term before it, and every odd term c times the term before it, the first term being 1.

(59.) If

$$x=a+a/r+a/r^2+\ldots \text{ ad } \infty,$$
$$y=b-b/r+b/r^2-\ldots \text{ ad } \infty,$$
$$z=c+c/r^2+c/r^4+\ldots \text{ ad } \infty,$$

then

$$xy/z=ab/c.$$

(60.) Find the sum of all the products three and three of the terms of an infinite G.P., and if this be one-third the sum of the cubes of the terms, show that $r=\frac{1}{2}$.

Exercises XLII.

(1.) Insert two harmonic means between 1 and 3, and five between 6 and 8.

(2.) Find the harmonic mean of 1 and 10, and also the harmonic mean of 1, 2, 3, 4, 5.

(3.) Show that 4, 6, 12 are in H.P., and continue the progression both ways.

(4.) Find the H.P. whose 3rd term is 5 and whose 5th term is 9.

(5.) Find the H.P. whose pth term is P and whose qth term is Q.

(6.) Show that the harmonic mean between the arithmetic and geometric means of a and b is $2(a+b)/\{(a/b)^{\frac{1}{4}}+(b/a)^{\frac{1}{4}}\}^2$.

(7.) Four numbers are proportionals; show that, if the first three are in G.P., the last three are in G.P.

(8.) Three numbers are in G.P.; if each be increased by 15, they are in H.P.: find them.

(9.) Between two quantities a harmonic mean is inserted; and between each adjacent pair of the three thus obtained is inserted a geometric mean.

It is now found that the three inserted means are in A.P.: show that the ratio of the two quantities is unity.

(10.) The sides of a right-angled triangle are in A.P.: show that they are proportional to 3, 4, 5.

(11.) a, b, c are in A.P., and a, b, d in H.P.: show that $c/d = 1 - 2(a-b)^2/ab$.

(12.) If x be any term in an A.P. whose two first terms are a, b, y, the term of the same order in a H.P. commencing with the same two terms, then $(x-a)/(y-x) = b/(y-b)$.

(13.) If a^2, b^2, c^2 be in A.P., then $1/(b+c)$, $1/(c+a)$, $1/(a+b)$ are in A.P.

(14.) If P be the product of n quantities in G.P., S their sum, and S′ the sum of their reciprocals, then $P^2 = (S/S')^n$.

(15.) If a, b, c be the pth, qth, and rth terms both of an A.P. and of a G.P., then $a^{b-c}b^{c-a}c^{a-b} = 1$.

(16.) If P, Q, R be the pth, qth, rth terms of a H.P., then $\Sigma\{PQ(p-q)\} = 0$, and $\{\Sigma(q^2-r^2)/P^2\}^2 = 4\{\Sigma(q-r)/P^2\}\{\Sigma qr(q-r)/P^2\}$.

(17.) If the sum of m terms of an A.P. be equal to the sum of the next n, and also to the sum of the next p, then $(m+n)(1/n - 1/p) = (n+p)(1/m - 1/n)$.

(18.) If the squared differences of p, q, r be in A.P., then the differences taken in cyclical order are in H.P.

(19.) If $a+b+c$, $a^2+b^2+c^2$, $a^3+b^3+c^3$ be in G.P., prove that the common ratio is $\frac{1}{2}\Sigma a - 3abc/2\Sigma bc$.

(20.) If x, y, z be in A.P., ax, by, cz in G.P., and a, b, c in H.P., then H.M. of a, c : G.M. of a, c = H.M. of x, z : G.M. of x, z.

(21.) If $a^2+b^2-c^2$, $b^2+c^2-a^2$, $c^2+a^2-b^2$ be in G.P., then $a^2/c^2+c^2/b^2$, $b^2/c^2+c^2/b^2$, $a^2/b^2+b^2/c^2$ are in A.P.

(22.) If a, b, c, d be in G.P., then $abcd\left(\Sigma\frac{1}{a}\right)^2 = (\Sigma a)^2$, and

$$\frac{\sqrt{(a^4+b^4)}+\sqrt{(b^4+c^4)}+\sqrt{(c^4+d^4)}}{(a^2+b^2)^{-1}+(b^2+c^2)^{-1}+(c^2+d^2)^{-1}} = \frac{b^4}{a^4}(a^2+b^2)\sqrt{(a^4+b^4)}.$$

(23.) The sum of the n geometric means between a and k is $(a^{1/(n+1)}k - ak^{1/(n+1)})/(k^{1/(n+1)} - a^{1/(n+1)})$.

(24.) If A_1, A_2, . . ., A_n be the n arithmetic means, and H_1, H_2, . . ., H_n the n harmonic means, between a and c, sum to n terms the series whose rth term is $(A_r - a)(H_r - a)/H_r$.

(25.) If a_1, a_2, . . ., a_n be in G.P., then

$$(a_1+a_2+a_3)^2+(a_2+a_3+a_4)^2+\ .\ .\ .\ +(a_{n-2}+a_{n-1}+a_n)^2$$
$$= (a_1^2+a_1a_2+a_2^2)^2(a_1^{2n-4}-a_2^{2n-4})/(a_1^2-a_2^2)a_1^{2n-4}.$$

(26.) If a_r, b_r be the arithmetic and geometric means respectively between a_{r-1} and b_{r-1}, show that

$$a_{n-2} = \{a_n^{\frac{1}{2}} \pm (a_n^2 - b_n^2)^{\frac{1}{4}}\}^2,$$
$$b_{n-2} = \{a_n^{\frac{1}{2}} \mp (a_n^2 - b_n^2)^{\frac{1}{4}}\}^2.$$

(27.) If a_1, a_2, . . ., a_n be real, and if

$$(a_1^2+a_2^2+\ .\ .\ .\ +a_{n-1}^2)(a_2^2+a_3^2+\ .\ .\ .\ +a_n^2) = (a_1a_2+a_2a_3+\ .\ .\ .\ +a_{n-1}a_n)^2,$$

then a_1, a_2, . . ., a_n are in G.P.

CHAPTER XXI.

Logarithms.

§ 1.] It is necessary for the purposes of this chapter to define and discuss more closely than we have yet done the properties of the exponential function a^x. For the present we shall suppose that a is a positive real quantity greater than 1. Whatever positive value, commensurable or incommensurable, we give to x, we can always find two commensurable values, m/n and $(m+1)/n$ (where m and n are positive integers), between which x lies, and which differ from one another as little as we please, see chap. xiii., § 15. In defining a^x for positive values of x, we suppose x replaced by one (say m/n) of these two values, which we may suppose chosen so close together that, for the purpose in hand, it is indifferent which we use. We thus have merely to consider $a^{m/n}$; and the understanding is that, as in the chapter on fractional indices, we regard only the real positive value of the nth root; so that $a^{m/n}$ may be read indifferently as $(\sqrt[n]{a})^m$, or as $\sqrt[n]{a^m}$.

For negative values of x we define a^x by the equation $a^x = 1/a^{-x}$, in accordance with the laws of negative indices.

§ 2.] We shall now show that a^x, *defined as above, is a continuous function of x susceptible of all positive values between* 0 *and* $+\infty$.

1*st*. Let y be any positive quantity greater than 1, and let n be any positive integer. Since $a > 1$, $a^{1/n} > 1$; but, by sufficiently increasing n, we may make $a^{1/n}$ exceed 1 by as little as we please. Also, when n is given, we can, by sufficiently increasing m, make $a^{m/n}$ as great as we please.* Hence, whatever

* See chap. xi., § 14.

may be the value of y, we can so choose n that $a^{1/n} < y$; and then y will lie between two consecutive integral powers of $a^{1/n}$; say $a^{m/n} < y < a^{(m+1)/n}$. Now the difference between these two values of a^x is $a^{m/n}(a^{1/n} - 1)$; and this, by sufficiently increasing n, we may make as small as we please. Hence, given any positive quantity $y > 1$, we can find a value of x such that a^x shall be as nearly equal to y as we please.

2nd. Let y be positive and < 1; then $1/y$ is positive and greater than 1. Hence we can find a value of x, say x', such that $a^{x'} = 1/y$ as nearly as we please. Hence $a^{-x'} = y$.

We may make y pass continuously through all possible values from 0 to $+\infty$. Hence a^x is susceptible of all positive values from 0 to $+\infty$. It is obviously a continuous function, since the difference of two finite values corresponding to $x = m/n$ and $x = (m+1)/n$ is $a^{m/n}(a^{1/n} - 1)$, which can be made as small as we please by sufficiently increasing n.

Cor. We have the following set of corresponding values:—

$$\begin{array}{lcccccc} x = -\infty, & -, & -1, & 0, & +, & 1, & +\infty; \\ y = a^x = 0, & <1, & 1/a, & 1, & >1, & a, & +\infty. \end{array}$$

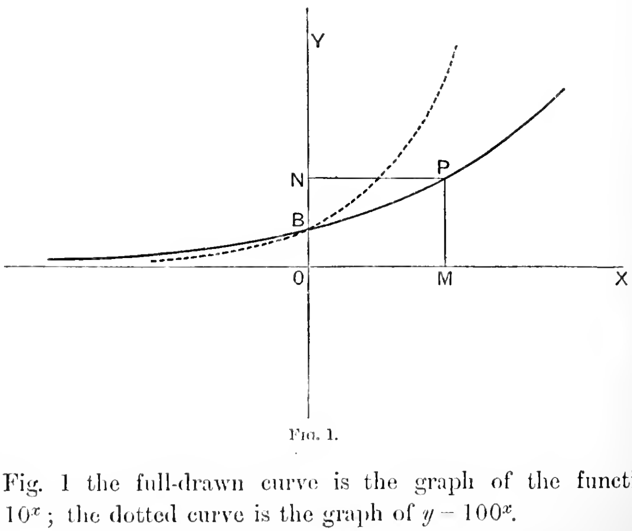


Fig. 1.

In Fig. 1 the full-drawn curve is the graph of the function $y = 10^x$; the dotted curve is the graph of $y = 100^x$.

It will be observed that the two curves cross the axis of y at the same point B, whose ordinate is $+1$; and that for one and the same value of y the abscissa of the one curve is double that of the other.

§ 3.] The reasoning by which we showed that the equation $y=a^x$, certain restrictions being understood, determines y as a continuous function of x, shows that the same equation, under the same restrictions, determines x as a continuous function of y. This point will perhaps be made clearer by graphical considerations. If we obtain the graph of y as a function of x from the equation $y=a^x$, the curve so obtained enables us to calculate x when y is given; that is to say, is the graph of x regarded as a function of y. Thus, if we look at the matter from a graphical point of view, we see that the continuity of the graph means the continuity of y as a function of x, and also the continuity of x as a function of y.

When we determine x as a function of y by means of the equation $y=a^x$, we obviously introduce a new kind of transcendental function into algebra, and some additional nomenclature becomes necessary to enable us to speak of it with brevity and clearness.

The constant quantity a is called the *base*.

y is called the *exponential of x to base a* (and is sometimes written $\exp_a x$).*

x is called the *logarithm of y to base a*, and is usually written $\log_a y$.

The two equations

$$y=a^x \qquad (1),$$
$$x=\log_a y \qquad (2),$$

are thus merely different ways of writing the same functional relation. It follows, therefore, that *all the properties of our new logarithmic function must be derivable from the properties of our old exponential function, that is to say, from the laws of indices.*

---

* This notation is little used in elementary text-books, but it is convenient when in place of x we have some complicated function of x. Thus $\exp_a(1+x+x^2)$ is easier to print than a^{1+x+x^2}.

The student should also notice that it follows from (1) and (2) that *the equation*

$$y = a^{\log_a y} \tag{3}$$

is an identity.

§ 4.] If the same base a be understood throughout, we have the following leading properties of the logarithmic function:—

I. *The logarithm of a product of positive numbers is the sum of the logarithms of the separate factors.*

II. *The logarithm of the quotient of two positive numbers is the excess of the logarithm of the dividend over the logarithm of the divisor.*

III. *The logarithm of any power (positive or negative, integral or fractional) of a positive number is equal to the logarithm of the number multiplied by the power.*

Let $y_1, y_2, \ldots, y_n$ be n positive numbers, $x_1, x_2, \ldots, x_n$ their respective logarithms to base a, so that

$$x_1 = \log_a y_1, \quad x_2 = \log_a y_2, \quad \ldots, \quad x_n = \log_a y_n.$$

By the definition of a logarithm we have

$$y_1 = a^{x_1}, \quad y_2 = a^{x_2}, \quad \ldots, \quad y_n = a^{x_n}.$$

Hence $\quad y_1 y_2 \ldots y_n = a^{x_1} a^{x_2} \ldots a^{x_n} = a^{x_1 + x_2 + \ldots + x_n},$

by the laws of indices.

Hence, by the definition of a logarithm,

$$x_1 + x_2 + \ldots + x_n = \log_a (y_1 y_2 \ldots y_n),$$

that is, $\quad \log_a y_1 + \log_a y_2 + \ldots + \log_a y_n = \log_a (y_1 y_2 \ldots y_n).$

We have thus established I.

Again, $\quad y_1/y_2 = a^{x_1}/a^{x_2} = a^{x_1 - x_2},$

by the laws of indices.

Hence, by the definition of a logarithm,

$$x_1 - x_2 = \log_a (y_1/y_2),$$

that is, $\quad \log_a y_1 - \log_a y_2 = \log_a (y_1/y_2),$

which is the analytical expression of II.

Again, $\quad y_1^r = (a^{x_1})^r = a^{rx_1},$

by the laws of indices.

Hence, by the definition of a logarithm,

$$rx_1 = \log_a y_1^r,$$

that is, $$r \log_a y_1 = \log_a y_1^r,$$

which is the analytical expression of III.

Example 1.

$$\log 21 = \log (7 \times 3) = \log 7 + \log 3.$$

As the equation is true for any base, provided all the logarithms have the same base, it is needless to indicate the base by writing the suffix.

Example 2.

$$\log (113/29) = \log 113 - \log 29.$$

Example 3.

$$\log (540/539) = \log (2^2 . 3^3 . 5/7^2 . 11),$$
$$= 2 \log 2 + 3 \log 3 + \log 5 - 2 \log 7 - \log 11.$$

Example 4.

$$\log \sqrt[3]{(49)} / \sqrt[7]{(21)} = \log (7^{2/3}/3^{1/7} 7^{1/7}),$$
$$= \tfrac{2}{3} \log 7 - \tfrac{1}{7} \log 3 - \tfrac{1}{7} \log 7,$$
$$= \tfrac{11}{21} \log 7 - \tfrac{1}{7} \log 3.$$

COMPUTATION AND TABULATION OF LOGARITHMS.

§ 5.] If the base of a system of logarithms be greater than unity, we have seen that the *logarithm of any positive number greater than unity is positive; and the logarithm of any positive number less than unity is negative.*

The logarithm of unity itself is always zero, whatever the base may be.

The logarithm of the base itself is of course unity, since $a = a^1$.

The logarithm of any power of the base, say a^r, is r; and, in particular, the logarithm of the reciprocal of the base is -1.

The logarithm of $+\infty$ *is* $+\infty$, *and the logarithm of* 0 *is* $-\infty$.

It is further obvious that the logarithm of a negative number could not (with our present understanding) be any real quantity. With such, however, we are not at present concerned.

The logarithm of any number which is not an integral power of the base will be some fractional number, positive or negative, as the case may be. For reasons that will appear presently, it is usual so to arrange a logarithm that it consists of a positive

fractional part less than unity, and an integral part, positive or negative, as the case may be.

The positive fractional part is called the *mantissa.*

The integral part is called the *characteristic.*

For example, the logarithm of ·0451 to base 10 is the negative number $-1·3458235$. In accordance with the above understanding, we should write

$$\begin{aligned}\log_{10}·0451 &= -1·3458235 = -2 + (1 - ·3458235),\\ &= -2 + ·6541765.\end{aligned}$$

For the sake of compactness, and at the same time to prevent confusion, this is usually written

$$\log_{10}·0451 = \bar{2}·6541765.$$

In this case then the characteristic is $\bar{2}$ (that is, -2), and the mantissa is ·6541765.

§ 6.] To find the logarithm of a given number y to a given base a is the same problem as to solve the equation

$$a^x = y,$$

where a and y are given and x is the unknown quantity.

There are various ways in which this may be done; and it will be instructive to describe here some of the more elementary, although at the same time more laborious, approximative methods that might be used.

In the first place, it is always easy to find the characteristic or integral part of the logarithm of any given number y. We have simply to *find by trial two consecutive integral powers of the base between which the given number y lies. The algebraically less of these two is the characteristic.*

Example 1.

To find the characteristic of $\log_3 451$.

We have the following values for consecutive integral powers of 3:—

Power	1	2	3	4	5	6
Value	3	9	27	81	243	729

Hence $3^5 < 451 < 3^6$. Hence $\log_3 451$ lies between 5 and 6. Therefore

$$\log_3 451 = 5 + \text{a proper fraction.}$$

Hence $$\text{char. } \log_3 451 = 5.$$

Example 2.

Find the characteristic of $\log_3 \cdot 0451$. We have

Powers of Base	0	-1	-2	-3
Values	1	·333 . . .	·111 . . .	·037 . . .

Hence $3^{-3} < \cdot 0451 < 3^{-2}$; that is to say,

$$\log_3 \cdot 0451 = -3 + \text{a proper fraction.}$$

Hence $$\text{char. } \log_3 \cdot 0451 = \bar{3}.$$

When the base of the system of logarithms is the radix of the scale of numerical notation, the characteristic can always be obtained by merely counting the digits.

For example, if the radix and base be each 10, then

If the number have an integral part, the characteristic of its logarithm is + (one less than the number of digits in the integral part).

If the number have no integral part, the characteristic is − (one more than the number of zeros that follow the decimal point).

The proof of these rules consists simply in the fact that, if a number lie between 10^n and 10^{n+1}, the number of digits by which it is expressed is $n+1$; and, if a number lie between $10^{-(n+1)}$ and 10^{-n}, the number of zeros after the decimal point is n.

For example, 351 lies between 10^2 and 10^3. Hence char. $\log_{10} 351 = 2 = 3 - 1$, according to the rule.

Again, ·0351 lies between ·01 and ·1, that is, between 10^{-2} and 10^{-1}. Hence char. $\log_{10} \cdot 0351 = -2 = -(1+1)$, which agrees with the rule.

The rule suggests at once that, *if* 10 *be adopted as the base of our system of logarithms, then the characteristic of a logarithm depends merely on the position of the decimal point; and the mantissa is independent of the position of the decimal point, but depends merely on the succession of digits.*

We may formally prove this important proposition as follows :—

Let N be any number formed by a given succession of digits, c the characteristic, and m the mantissa of its logarithm. Then any other number which has the same succession of digits as N, but has the decimal point placed differently, will have the form 10^iN, where i is an integer, positive or negative, as the case may be. But $\log_{10}10^i N = \log_{10}10^i + \log_{10}N$, by § 4, $= i + \log_{10}N = (i+c)+m$. Now, since by hypothesis m is a positive proper fraction, and c and i are integers, the mantissa of $\log_{10}10^i N$ is m, and the characteristic is $i+c$. In other words, the characteristic alone is altered by shifting the decimal point.

§ 7.] The process used in § 6 for finding the characteristic of a logarithm can be extended into a method for finding the mantissa digit by digit.

Example.

To calculate $\log_{10}4{\cdot}217$ to three places of decimals.

The characteristic is obviously 0. Let the three first digits of the mantissa be xyz. Then we have

$$4{\cdot}217=10^{0\cdot xyz}, \text{ hence } (4{\cdot}217)^{10}=10^{x\cdot yz}.$$

We must now calculate the 10th power of 4·217. In so doing, however, there is no need to find all the significant figures—a few of the highest will suffice. We thus find

$$1778400=10^{x\cdot yz}.$$

We now see that x is the characteristic of $\log_{10}1778400$. Hence $x=6$. Dividing by 10^6, and raising both sides of the resulting equation to the 10th power, we find

$$(1{\cdot}778)^{10}=10^{y\cdot z}\text{ ; hence } 315{\cdot}7=10^{y\cdot z}.$$

Hence $y=2$. Dividing by 10^2, and raising to the 10th power, we now find

$$(3{\cdot}16)^{10}=10^z\text{ ; hence } 99280=10^z.$$

Hence $z=5$ very nearly.

We conclude, therefore, that

$$\log_{10}4{\cdot}217=\cdot625 \text{ nearly.}$$

This method of computing logarithms is far too laborious to be of any practical use, even if it were made complete by the addition of a test to ascertain what effect the figures neglected in the calculation of the 10th powers produce on a given decimal place of the logarithm; it has, however, a certain theoretical

interest on account of its direct connection with the definition of a logarithm.

By a somewhat similar process a logarithm can be expressed as a continued fraction.

§ 8.] *If a series of numbers be in geometric progression, their logarithms are in arithmetic progression.*

Let the numbers in question be $y_1, y_2, y_3, \ldots, y_n$. Let the logarithm of the first to base a be α, and the logarithm of the common ratio of the G.P. $y_1, y_2, y_3, \ldots, y_n$ to the same base be β. Then we have the following series of corresponding values :—

$$\begin{array}{ccccc} y_1, & y_2, & y_3, & \ldots, & y_n, \\ \| & \| & \| & & \| \\ a^{\alpha}, & a^{\alpha+\beta}, & a^{\alpha+2\beta}, & & a^{\alpha+(n-1)\beta}, \end{array}$$

from which the truth of the proposition is manifest.

As a matter of history, it was this idea of comparing two series of numbers, one in geometric, the other in arithmetic progression, that led to the invention of logarithms; and it was on this comparison that most of the early methods of computing them were founded.

The following may be taken as an example. Let us suppose that we know the logarithms x_1 and x_9 of two given numbers, y_1 and y_9; then we can find the logarithms of as many numbers lying between y_1 and y_9 as we please. We have

$$y_1 = a^{x_1}, \quad y_9 = a^{x_9}.$$

Let us insert a geometric mean, y_5, between y_1 and y_9, then

$$y_5 = (y_1 y_9)^{\frac{1}{2}} = a^{(x_1+x_9)/2} = a^{x_5}, \text{ say,}$$

where x_5 is the arithmetic mean between x_1 and x_9. We have now the following system :—

$$\begin{array}{llll} \text{Logarithm} & x_1 & x_5 & x_9, \\ \text{Number} & y_1 & y_5 & y_9. \end{array}$$

Next insert geometric means between y_1, y_5 and y_5, y_9. The logarithms of the corresponding numbers will be the arithmetic means between x_1, x_5 and x_5, x_9. We thus have the system—

$$\begin{array}{llllll} \text{Logarithms} & x_1, & x_3, & x_5, & x_7, & x_9; \\ \text{Numbers} & y_1, & y_3, & y_5, & y_7, & y_9. \end{array}$$

Proceeding in like manner, we derive the system—

Logarithms	x_1,	x_2,	x_3,	x_4,	x_5,	x_6,	x_7,	x_8,	x_9;
Numbers	y_1,	y_2,	y_3,	y_4,	y_5,	y_6,	y_7,	y_8,	y_9;

and so on. Each step in this calculation requires merely a multiplication, the extraction of a square root, an addition accompanied by division by 2, and each step furnishes us with a new number and the corresponding logarithm.

Since $x_1, x_2, \ldots, x_n$ form an A.P., the logarithms are spaced out equally, but the same is not true of the corresponding numbers which are in G.P. It is therefore a table of antilogarithms* that we should calculate most readily by this method. It will be observed, however, that by inserting a sufficient number of means we can make the successive numbers differ from each other as little as we please; and by means of the method of interpolation by first differences, explained in the last section of this chapter, we could space out the numbers equally, and thus convert our table of antilogarithms into a table of logarithms of the ordinary kind.

As a numerical example we may put $a=10, y_1=1, y_9=10$; then $x_1=0$, $x_9=1$. Proceeding as above indicated, we should arrive at the following table:—

Number.	Logarithm.	Number.	Logarithm.
1·0000	0·0000	4·2170	0·6250
1·3336	0·1250	5·6235	0·7500
1·7783	0·2500	7·4990	0·8750
2·3714	0·3750	10·0000	1·0000
3·1622	0·5000		

§ 9.] In computing logarithms, by whatever method, it is obvious that it is not necessary to calculate independently the logarithms of composite integers after we have found to a sufficient degree of accuracy the logarithms of all primes up to a certain magnitude. Thus, for example, $\log 4851 = \log 3^2.7^2 11$ $= 2 \log 3 + 2 \log 7 + \log 11$. Hence log 4851 can be found when the logarithms of 3, 7, and 11 are known.

* By the antilogarithm of any number N is meant the number of which N is the logarithm.

Again, *having computed a system of logarithms to any one base* a, *we can without difficulty deduce therefrom a system to any other base* b. *All we have to do is to multiply all the logarithms of the former system by the number* $\mu = 1/\log_a b$.

For, if $x = \log_b y$, then $y = b^x$.

Hence $\log_a y = \log_a b^x$,

$= x \log_a b$, by § 4.

Hence $\log_b y = x = \log_a y/\log_a b$ (1).

The number μ is often called the *modulus* of the system whose base is b with respect to the system whose base is a.

Cor. 1. If in the equation (1) we put $y = a$, we get the following equation, which could easily be deduced more directly from the definition of a logarithm :—

$$\log_b a = 1/\log_a b,$$

or

$$\log_a b \log_b a = 1 \quad (2).$$

Cor. 2. The equation $y = b^x$ may be written

$$y = a^{x \log_a b} \text{ or } y = a^{x/\log_b a}.$$

Hence *the graph of the exponential* b^x *can be deduced from the graph of the exponential* a^x *by shortening or lengthening all the abscissæ of the latter in the same ratio* 1 : $log_a b$.

This is the general theorem corresponding to a remark in § 2.

We may also express this result as follows :—

Given any two exponential graphs A *and* B, *then either* A *is the orthogonal projection of* B, *or* B *is the orthogonal projection of* A, *on a plane passing through the axis of* y.

USE OF LOGARITHMS IN ARITHMETICAL CALCULATIONS.

§ 10.] We have seen that, if we use the ordinary decimal notation, the system of logarithms to base 10 possesses the important advantages that the characteristic can be determined by inspection, and that the mantissa is independent of the position of the decimal point. On account of these advantages this system is used in practical calculations to the exclusion of all others.

No.		0	1	2	3	4	5	6	7	8	9
3050	484	2998	3141	3283	3426	3568	3710	3853	3995	4137	4280
51		4422	4564	4707	4849	4991	5134	5276	5418	5561	5703
52		5845	5988	6130	6272	6414	6557	6699	6841	6984	7126
53		7268	7410	7553	7695	7837	7979	8121	8264	8406	8548
54		8690	8833	8975	9117	9259	9401	9543	9686	9828	9970
55	485	0112	0254	0396	0539	0681	0823	0965	1107	1249	1391
56		1533	1676	1818	1960	2102	2244	2386	2528	2670	2812
57		2954	3096	3239	3381	3523	3665	3807	3949	4091	4233
58		4375	4517	4659	4801	4943	5085	5227	5369	5511	5653
59		5795	5937	6079	6221	6363	6505	6647	6788	6930	7072
60		7214	7356	7498	7640	7782	7924	8066	8208	8350	8491
3061		8633	8775	8917	9059	9201	9343	9484	9626	9768	9910
62	486	0052	0194	0336	0477	0619	0761	0903	1045	1186	1328
63		1470	1612	1754	1895	2037	2179	2321	2462	2604	2746
64		2888	3029	3171	3313	3455	3596	3738	3880	4021	4163
65		4305	4446	4588	4730	4872	5013	5155	5297	5438	5580
66		5722	5863	6005	6146	6288	6430	6571	6713	6855	6996
67		7138	7279	7421	7563	7704	7846	7987	8129	8270	8412
68		8554	8695	8837	8978	9120	9261	9403	9544	9686	9827
69		9969	0110	0252	0393	0535	0676	0818	0959	1101	1242
70	487	1384	1525	1667	1808	1950	2091	2232	2374	2515	2657
3071		2798	2940	3081	3222	3364	3505	3647	3788	3929	4071
72		4212	4353	4495	4636	4778	4919	5060	5202	5343	5484
73		5626	5767	5908	6050	6191	6332	6473	6615	6756	6897
74		7039	7180	7321	7462	7604	7745	7886	8027	8169	8310
75		8451	8592	8734	8875	9016	9157	9299	9440	9581	9722
76		9863	0004	0146	0287	0428	0569	0710	0852	0993	1134
77	488	1275	1416	1557	1698	1839	1981	2122	2263	2404	2545
78		2686	2827	2968	3109	3251	3392	3533	3674	3815	3956
79		4097	4238	4379	4520	4661	4802	4943	5084	5225	5366
80		5507	5648	5789	5930	6071	6212	6353	6494	6635	6776
3081		6917	7058	7199	7340	7481	7622	7763	7904	8045	8185
82		8326	8467	8608	8749	8890	9031	9172	9313	9454	9594
83		9735	9876	0017	0158	0299	0440	0580	0721	0862	1003
84	489	1144	1285	1425	1566	1707	1848	1989	2129	2270	2411
85		2552	2692	2833	2974	3115	3256	3396	3537	3678	3818
86		3959	4100	4241	4381	4522	4663	4804	4944	5085	5226
87		5366	5507	5648	5788	5929	6070	6210	6351	6492	6632
88		6773	6914	7054	7195	7335	7476	7617	7757	7898	8038
89		8179	8320	8460	8601	8741	8882	9023	9163	9304	9444
90		9585	9725	9866	0006	0147	0287	0428	0569	0709	0850
3091	490	0990	1131	1271	1412	1552	1693	1833	1973	2114	2254
92		2395	2535	2676	2816	2957	3097	3238	3378	3518	3659
93		3799	3940	4080	4220	4361	4501	4642	4782	4922	5063
94		5203	5343	5484	5624	5765	5905	6045	6186	6326	6466
95		6607	6747	6887	7027	7168	7308	7448	7589	7729	7869
96		8010	8150	8290	8430	8571	8711	8851	8991	9132	9272
97		9412	9552	9693	9833	9973	0113	0253	0394	0534	0674
98	491	0814	0954	1094	1235	1375	1515	1655	1795	1935	2076
99		2216	2356	2496	2636	2776	2916	3057	3197	3337	3477
3100		3617	3757	3897	4037	4177	4317	4457	4597	4738	4878

Diff.

142	
1	14
2	28
3	43
4	57
5	71
6	85
7	99
8	114
9	128

141	
1	14
2	28
3	42
4	56
5	71
6	85
7	99
8	113
9	127

In printing a table of logarithms to base 10 it is quite unnecessary, even if it were practicable, to print characteristics. The mantissæ alone are given, corresponding to a succession of five digits, ranging usually from 10000 to 99999.*

A glance at p. 520, which is a facsimile of a page of the logarithmic table in Chambers's *Mathematical Tables*, will show the arrangement of such a table. To take out the logarithm of 30715 from the table, we run down the column headed "No." until we come to 3071; the first three figures of the mantissa are 487 (standing over the blank in the first half column); the last four are found by running along the line till we reach the column headed 5, they are 3505. The characteristic is seen by inspection to be 4. Hence $\log 30715 = 4{\cdot}4873505$.

To find the number corresponding to any given logarithm we have of course simply to reverse the process.

To find the logarithm of ·030715 we have to proceed exactly as before, only a different characteristic, namely $\bar{2}$, must be prefixed to the mantissa. We thus find $\log {\cdot}030715 = \bar{2}{\cdot}4873505$.

If we wish to find the logarithm of a number, say 3·083279, where we have more digits than are given in the table, then we must take the nearest number whose logarithm can be found by means of the table, that is to say, 3·0833. We thus find $\log 3{\cdot}0833 = 0{\cdot}4890158$† nearly. Greater accuracy can be attained by using the column headed "Diff.," as will be explained presently.

Conversely, if a logarithm be given which is not exactly coincident with one given in the table, we take the one in the table that is nearest to it, and take the corresponding number as an approximation to the number required. Greater accuracy can be obtained by using the difference column. Thus the number whose logarithm is 1·4872191 has for its first five

---

* For some purposes an extension of the table is required, and such extensions are supplied in various ways, which need not be described here. For rapidity of reference in calculations that require no great exactness a short table for a succession of 3 digits, ranging from 100 to 999, is also usually given.

† The bar over 0158 indicates that these digits follow 489, and not 488.

significant digits 30705; but, if we wish the best approximation with five digits, we ought to take 30706. Since the characteristic is 1, the actual number in question has two integral digits. Hence the required number is 30·706, the error being certainly less than ·0005.

§ 11.] The principle which underlies the application of logarithms to arithmetical calculation is the very simple one that, *since to any number there corresponds one and only one logarithm, a number can be identified by means of its logarithm.*

It is this principle which settles how many digits of the mantissa of a logarithm it is necessary to use in calculations which require a given degree of accuracy.

Suppose, for example, that it is necessary to be accurate down to the fifth significant figure; and let us inquire whether a table of logarithms in which the mantissæ are given to four places would be sufficient. In such a table we should find log 3·0701 = 0·4871, log 3·0702 = 0·4871; the table is therefore not sufficiently extended to distinguish numbers to the degree of accuracy required. Five places in the mantissa would, in the present instance, be sufficient for the purpose; for log 3·0701 = 0·48715, log 3·0702 = 0·48716. Towards the end of the table, however, five places would scarcely be sufficient; for log 9·4910 = 0·97731 and log 9·4911 = 0·97731.

§ 12.] The great advantage of using in any calculation logarithms instead of the actual numbers is that we can, in accordance with the rules of § 4, replace every multiplication by an addition, every division by a subtraction, and every operation of raising to a power or extracting a root by a multiplication or division.

The following examples will illustrate some of the leading cases. We suppose that the student has a table of the logarithms of all numbers from 10000 to 100000, giving mantissæ to seven places.

Example 1.

Calculate the value of $1·6843 \times ·00132 \div ·3692$.

If $A = 1·6843 \times ·00132 \div ·3692$,

$\log A = \log 1·6843 + \log ·00132 - \log ·3692$.

$$\begin{aligned}
\log 1{\cdot}6843 &= \phantom{\bar{3}}{\cdot}2264194 \\
\log {\cdot}00132 &= \bar{3}{\cdot}1205739 \\
&\overline{\bar{3}{\cdot}3469933} \\
\log {\cdot}3692 &= \bar{1}{\cdot}5672617 \\
\log A &= \bar{3}{\cdot}7797316.
\end{aligned}$$

Hence $A = {\cdot}0060219.$

Observe that the negative characteristics must be dealt with according to algebraic rules.

Example 2.

To extract the cube root of ·016843.

Let $A=({\cdot}016843)^{1/3}$, then

$$\begin{aligned}
\log A &= \tfrac{1}{3}\log {\cdot}016843, \\
&= \tfrac{1}{3}(\bar{2}{\cdot}2264194), \\
&= \tfrac{1}{3}(\bar{3}+1{\cdot}2264194), \\
&= \bar{1}{\cdot}4088065. \\
A &= {\cdot}25633.
\end{aligned}$$

Example 3.

Calculate the value of $A=(368)^{7/3}/(439)^{5/9}$.

$\operatorname{Log} A = \frac{7}{3}\log 368 - \frac{5}{9}\log 439.$

$$\begin{aligned}
\tfrac{7}{3}\log 368 &= \tfrac{7}{3}(2{\cdot}5658478) = 5{\cdot}9869782 \\
\tfrac{5}{9}\log 439 &= \tfrac{5}{9}(2{\cdot}6424645) = 1{\cdot}4680358 \\
\log A &= 4{\cdot}5189424 \\
A &= 33033.
\end{aligned}$$

Example 4.

Find how many digits there are in $A=(1{\cdot}01)^{10000}$.

$$\begin{aligned}
\log A &= 10000\log 1{\cdot}01, \\
&= 10000\times{\cdot}0043214, \\
&= 43{\cdot}214.
\end{aligned}$$

Hence the number of digits in A is 44.

Example 5.

To solve the exponential equation $1{\cdot}2^x=1{\cdot}1$ by means of logarithms.

We have $\log 1{\cdot}2^x = \log 1{\cdot}1.$

Therefore $x\log 1{\cdot}2 = \log 1{\cdot}1.$

Hence $$x=\frac{\log 1{\cdot}1}{\log 1{\cdot}2}=\frac{{\cdot}0413927}{{\cdot}0791812}.$$

Hence $$\begin{aligned}\log x &= \log {\cdot}0413927 - \log {\cdot}0791812, \\ &= \bar{1}{\cdot}7183059.\end{aligned}$$

Therefore $x = {\cdot}52276.$

Remark.—It is obvious that we can solve any such equation as $a^{x^2-px+q}=b$, where p, q, a, b are all given. For, taking logarithms of both sides, we have

$$(x^2-px+q)\log a = \log b.$$

We can now obtain the value of x by solving a quadratic equation.

INTERPOLATION BY FIRST DIFFERENCES.

§ 13.] The method by which it is usual to find (or "*interpolate*") the value of the logarithm of a number which does not happen to occur in the table is one which is applicable to any function whose values have been tabulated for a series of equidifferent values of its independent variable (or "*argument*").

The general subject of interpolation belongs to the calculus of finite differences, but the special case where first differences alone are used can be explained in an elementary way by means of graphical considerations.

We have already seen that the increment of an integral function of x of the 1st degree, $y = Ax + B$ say, is proportional to the increment of its argument; or, what comes to the same thing, if we give to the argument x a series of equidifferent values, a, $a + h$, $a + 2h$, $a + 3h$, &c., the function y will assume a series of equidifferent values $Aa + B$, $Aa + B + Ah$, $Aa + B + 2Ah$, $Aa + B + 3Ah$, &c.

If, therefore, we were to tabulate the values of $Ax + B$ for a series of equidifferent values of x, the differences between successive values of y ("*tabular differences*") would be constant, no matter to how many places we calculated y.

Conversely, a function of x which has this property, that the differences between the successive values of y corresponding to equidifferent values of x are absolutely constant, must be an integral function of x of the 1st degree.

If, however, we take the difference, h, of the argument small enough, and do not insist on accuracy in the value of y beyond a certain significant figure, then, for a limited extent of the table of any function, it will be found that the tabular differences are constant.

Referring, for example, to p. 520, it will be seen that the difference of two consecutive logarithms is constant, and equal to ·0000141, from log 30660 up to log 30899, or that there is merely an accidental difference of a unit in the last place; that is to say, the difference remains constant for about 240 entries.

A similar phenomenon will be seen in the following extract from Barlow's Tables, provided we do not go beyond the 7th significant figure:—

Number.	Cube Root.	Diff.
2301	13·2019740	19122
2302	13·2038862	19117
2303	13·2057979	19111
2304	13·2077090	19105
2305	13·2096195	

Let us now look at the matter graphically. Let ACSDQB be a portion of the graph of a function $y=f(x)$; and let us suppose that up to the nth significant figure the differences of y are constant for equidifferent values of x, lying between OE and OH. This means that in calculating (up to the nth significant figure) values of y corresponding to values of x between OE and OH we may replace the graph by the straight line AB. Thus, for example, if x = OM, then f(OM) = MQ; and PM is the value calculated by means of the straight line AB. Our statement then is that PM − QM, that is PQ, is less than a unit in the n significant place.

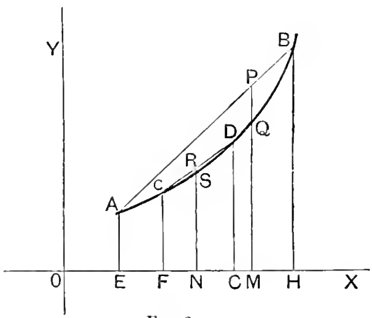


Fig. 2.

If this be so, then, *a fortiori*, it will be so if we replace a portion of the graph, say CD, lying between A and B by a straight line joining C and D.

In other words, *if up to the nth place the increment of the function for equidifferent values of x be constant, between certain limits, then, to that degree of accuracy at least, the increment of the function will be proportional to the increment of the argument for all values within those limits.*

§ 14.] Let us now state the conclusion of last article under

an analytical form, all the limitations before mentioned as to constancy of tabular (or first) difference being supposed fulfilled.

Let h be the difference of the arguments as they are entered in the table, D the tabular difference $f(a+h)-f(a)$, $a+h'$ a value of the argument, which does not occur in the table, but which lies between the values a and $a+h$, which do occur, so that $h'<h$. Then, by last paragraph,

$$\frac{f(a+h')-f(a)}{f(a+h)-f(a)}=\frac{(a+h')-a}{(a+h)-a}.$$

Hence
$$\frac{f(a+h')-f(a)}{\mathrm{D}}=\frac{h'}{h} \qquad (1).$$

Since in (1) $f(a)$, D, h are all known, it gives us a relation between h' and $f(a+h')$. When, therefore, one or other of these is given, we can calculate the other. We have in fact

$$f(a+h')=f(a)+\frac{h'}{h}\mathrm{D} \qquad (2),$$

and
$$a+h'=a+\frac{f(a+h')-f(a)}{\mathrm{D}}h \qquad (3).$$

From (2) we find a value of the function corresponding to a given intermediate value of the argument. From (3) we find an intermediate value of the argument corresponding to a given intermediate value of the function.

The equations (2) and (3) are sometimes called the *Rule of Proportional Parts.*

Example 1.

To find by means of first differences the value of $\sqrt[3]{(2303 \cdot 45)}$ as accurately as Barlow's Table will allow.

By the rule of proportional parts, we have

$$\begin{aligned}\sqrt[3]{(2303 \cdot 45)} &= \sqrt[3]{(2303 \cdot 00)}+\tfrac{45}{100}(\cdot 00191),\\ &=13 \cdot 20580+\cdot 00086,\\ &=13 \cdot 20666,\end{aligned}$$

which will be found correct down to the last figure.

The only labour in the above calculation consists in working out the fraction 45/100 of the tabular difference. In tables of

logarithms even this labour is spared the calculator; for under each difference there is a small table of proportional parts, giving the values of 1/10, 2/10, 3/10, 4/10, 5/10, 6/10, 7/10, 8/10, 9/10 of the difference in question (see the last column on p. 520). It will be observed that, if we strike the last figure off each of the proportional parts (increasing the last of those left if the one removed exceeds 5), we have a table of the various hundredths, and so on. Hence we can use the table twice over (in some cases it might be oftener), as in the following example:—

Example 2.

To find $\log 30{\cdot}81345$.

We may arrange the corresponding contributions as follows:—

$$\begin{array}{rr} 30{\cdot}81300 & 1{\cdot}4887340 \\ 40 & 56 \\ 5 & 7 \\ \hline \log 30{\cdot}81345 = & 1{\cdot}4887403 \end{array}$$

Example 3.

To find the number whose logarithm is $\bar{1}{\cdot}4871763$.

$$\begin{array}{rr} \bar{1}{\cdot}4871763 & \\ \bar{1}{\cdot}4871667 & {\cdot}3070200 \\ 96 & \\ 85 & 60 \\ 11 & 8 \\ \hline \text{antilog}\,\bar{1}{\cdot}4871763 = & {\cdot}3070268 \end{array}$$

Here we set down under the given logarithm the next lowest in the table, and opposite to it the corresponding number ·30702.

Next, we write down ·0000096, the difference of these two logarithms, and look for the greatest number in the table of proportional parts which does not exceed 96—this is 85. We set down 85, and opposite to it the corresponding figure 6.

Lastly, we subtract 85 from 96, the result being ·0000011. We then imagine a figure struck off every number in the table of proportional parts, look for the remaining one which stands nearest to 11, and set down the figure, namely 8, corresponding to it, as the last digit of the number we are seeking.

Exercises XLIII.

(1.) Find the characteristics of $\log_{10}36983$, $\log_{10}5^8$, $\log_{10}5^{-3}$, $\log_{10}{\cdot}00068$.

(2.) Find the characteristics of $\log_5 1067$, $\log_5{\cdot}0138$, $\log_{\sqrt{3}}8$, $\log_{\sqrt{3}}1/8$.

(3.) Find $\log_2 8\sqrt[3]{2}$.

(4.) Calculate $\log_2 36{\cdot}432$ to two places of decimals.

Calculate out the values of tho following as accurately as your tables will allow:—

(5.) $4163 \times 7{\cdot}835$. (6.) $\cdot 3068 \times \cdot 0015 \div \cdot 0579$.

(7.) $(5{\cdot}0063745)^5$. (8.) $\sqrt[5]{(5{\cdot}0063745)}$.

(9.) $(\cdot 01369)^{12}$. (10.) $(\cdot 001369)^{\frac{1}{5}}$.

(11.) $\{15(\cdot 318)^{\frac{1}{7}}/16\}^{\frac{1}{11}}$. (12.) $\{(1{\cdot}035)^7 - 1\}/\{1{\cdot}035 - 1\}$.

(13.) The population of a country increases each year by $\cdot 13$ °/ₒ of its amount at the beginning of the year. By how much °/ₒ will it have increased altogether after 250 years?

(14.) If the number of births and deaths per annum be $3{\cdot}5$ and $1{\cdot}2$ °/ₒ respectively of the population at the beginning of each year, after how many years will the population be trebled?

(15.) Calculate the value of $\sqrt[15]{(32^{6\cdot 8} + 55^{3\cdot 6})}$.

(16.) Calculate the value of $1 + e + e^2 + \ldots + e^{19}$, where $e = 2{\cdot}71828$.

(17.) Find a mean proportional between $3{\cdot}17934$ and $3{\cdot}987636$.

(18.) Insert three mean proportionals between $65{\cdot}342$ and $88{\cdot}63$.

(19.) The 1st and 13th terms of a geometric progression are 3 and 65 respectively: find the common ratio.

(20.) The 4th and 7th terms of a geometric progression are 31 and 52 respectively: find the 5th term.

(21.) How many terms of the series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ must be taken in order that its sum may differ from unity by less than a millionth?

(22.) Given $\log_{10} 5 = \cdot 69897$, find the number of digits in $(\sqrt{5})^{99}$.

(23.) Given $\log 2673 = 3{\cdot}427$, and $\log 3267 = 3{\cdot}51415$, find $\log 11$.

(24.) Find the first four significant figures and the number of digits in $1.2.3.4 \ldots 20$.

(25.) How many terms of $3^1, 3^2, 3^3, \ldots$ must I take in order that the product may just exceed 100000?

(26.) Given $\log_x 36 = 1{\cdot}3678$, find x.

(27.) Given $\log_x 6\frac{1}{4} = 2$, find x.

Solve the following equations:—

(28.) $21^x = 20$. (29.) $2^x 15^x = 5$. (30.) $2^{x^2} = 5 \times 2^x$.

(31.) $6^x = 5y, \quad 7^x = 3y$. (32.) $3^x - 3^{-x} = 5$.

(33.) $2^{3x+2y} = 5, \quad 4^{2x} = 2^{2y+3}$. (34.) $3^{2x} 5^{3x-4} = 7^{x-1} 11^{2-x}$.

(35.) $(a+b)^{2x}(a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}$.

(36.) $x^{x+y} = y^{4a}, \quad y^{x+y} = x^a$.

(37.) Find by means of a table of common logarithms $\log_e 16.345$, where $e = 2{\cdot}71828$.

(38.) Show that $x = a^{1/\log_x a}$; and that $x = a^{\log_a b \log_b c \log_c d \log_d x}$.

(39.) Show that

$$\frac{\log_a(\log_a \mathrm{N})}{\sqrt{(\log_a b)}} - \frac{\log_b(\log_b \mathrm{N})}{\sqrt{(\log_b a)}} = \frac{\log_a \sqrt{(\log_a b)}}{\sqrt{(\log_a b)}} - \frac{\log_b \sqrt{(\log_b a)}}{\sqrt{(\log_b a)}}.$$

(40.) Show that the logarithm of any number to base a^n is a mean proportional between its logarithms to the bases a and a^{n^2}.

(41.) If P, Q, R be the pth, qth, rth terms of a geometric progression, show that $\Sigma(q-r)\log P=0$.

(42.) If ABC be in harmonic progression, show that $\log(A+C)+\log(A+C-2B)=2\log(A-C)$.

(43.) If a, b, c be in G.P., show that $\Sigma\{\log_a(b/c)\}^{-1}=-3\{\log_b(c/a)\}^{-1}$.

(44.) If a, b, c be in G.P., and $\log_c a$, $\log_b c$, $\log_a b$ in A.P., then the common difference of the latter is 3/2.

(45.) If $a^2+b^2=c^2$, then $\log_{b+c}a+\log_{c-b}a=2\log_{b+c}a\log_{c-b}a$.

(46.) If $\dfrac{x(y+z-x)}{\log x}=\dfrac{y(z+x-y)}{\log y}=\dfrac{z(x+y-z)}{\log z}$, then $y^z z^y=z^x x^z=x^y y^x$.

(47.) If $x_2=\log_{x_n}x_1$, $x_3=\log_{x_1}x_2$, $x_4=\log_{x_2}x_3$, . . ., $x_n=\log_{x_{n-2}}x_{n-1}$, $x_1=\log_{x_{n-1}}x_n$, then $x_1x_2\ldots x_n=1$.

Historical Note.—The honour of devising the use of logarithms as a means of abbreviating arithmetical calculations, and of publishing the first logarithmic table, belongs to John Napier (1550-1617) of Merchiston (in Napier's day near, in our day in, Edinburgh). This invention was not the result of a casual inspiration, for we learn from Napier's *Rabdologia* (1617), in which he describes three other methods for facilitating arithmetical calculations, among them his calculating rods, which, under the name of "Napier's Bones," were for long nearly as famous as his logarithms, that he had devoted a great part of his life to the consideration of methods for increasing the power and diminishing the labour of arithmetical calculation. Napier published his invention in a treatise entitled "Mirifici Logarithmorum Canonis Descriptio, ejusque usus, in utraque Trigonometria ut etiam in omni Logistica Mathematica, Amplissimi, facillimi, et expeditissimi explicatio. Authore ac Inventore Ioanne Nepero, Barone Merchistonii, &c., Scoto, Edinburgi (1614)." In this work he explains the use of logarithms; and gives a table of logarithmic sines to 7 figures for every minute of the quadrant. In the *Canon Mirificus* the base used was neither 10 nor what is now called Napier's base (see the chapter on logarithmic series in the second part of this work). Napier himself appears to have been aware of the advantages of 10 as a base, and to have projected the calculation of tables on the improved plan; but his infirm health prevented him from carrying out this idea; and his death three years after the publication of the *Canon Mirificus* prevented him from even publishing a description of his methods for calculating logarithms. This work, entitled *Mirifici Logarithmorum Canonis Constructio*, &c., was edited by one of Napier's sons, assisted by Henry Briggs.

To Henry Briggs (1556-1630), Professor of Geometry at Gresham College, and afterwards Savilian Professor at Oxford, belongs the place of honour next to Napier in the development of logarithms. He recognised at once the merit and seized the spirit of Napier's invention. The idea of the superior advantages of a decimal base occurred to him independently; and he visited Napier in Scotland in order to consult with him regarding a scheme for the calculation of a logarithmic table of ordinary numbers on the improved plan. Finding Napier in possession of the same idea in a slightly better form, he adopted his suggestions, and the result of the visit was that Briggs undertook the work which Napier's declining health had interrupted. Briggs published the first thousand of his logarithms in 1617; and, in his *Arithmetica Logarithmica*, gave to 14 places of decimals the logarithms of all integers from 1 to 20,000, and from 90,000 to 100,000. In the preface to the last-mentioned work he explains the methods used for calculating the logarithms themselves, and the rules for using them in arithmetical calculation.

While Briggs was engaged in filling up the gap left in his table, the work of calculating logarithms was taken up in Holland by Adrian Vlacq, a bookseller of Gouda. He calculated the 70,000 logarithms which were wanting in Briggs' Table; and published, in 1628, a table containing the logarithms to 10 places of decimals of all numbers from 1 to 100,000. The work of Briggs and Vlacq has been the basis of all the tables published since their day (with the exception of the tables of Sang, 1871); so that it forms for its authors a monument both lasting and great.

In order fully to appreciate the brilliancy of Napier's invention and the merit of the work of Briggs and Vlacq, the reader must bear in mind that even the exponential notation and the idea of an exponential function, not to speak of the inverse exponential function, did not form a part of the stock-in-trade of mathematicians till long afterwards. The fundamental idea of the correspondence of two series of numbers, one in arithmetic, the other in geometric progression, which is so easily represented by means of indices, was explained by Napier through the conception of two points moving on separate straight lines, the one with uniform, the other with accelerated velocity. If the reader, with all his acquired modern knowledge of the results to be arrived at, will attempt to obtain for himself in this way a demonstration of the fundamental rules of logarithmic calculation, he will rise from the exercise with an adequate conception of the penetrating genius of the inventor of logarithms.

For the full details of this interesting part of mathematical history, and in particular for a statement of the claims of Jost Bürgi, a Swiss contemporary of Napier's, to credit as an independent inventor of logarithms, we refer the student to the admirable articles "Logarithms" and "Napier," by J. W. L. Glaisher, in the *Encyclopædia Britannica* (9th ed.). An English translation of the *Constructio*, with valuable bibliographical notes, has been published by Mr. W. R. Macdonald, F.F.A. (Edinb. 1889).

CHAPTER XXII.

Theory of Interest and Annuities Certain.

§ 1.] Since the mathematical theory of interest and annuities affords the best illustration of the principles we have been discussing in the last two chapters, we devote the present chapter to a few of the more elementary propositions of this important practical subject. What we shall give will be sufficient to enable the reader to form a general idea of the principles involved. Those whose business requires a detailed knowledge of the matter must consult special text-books, such as the *Text-Book of the Institute of Actuaries*, Part I., by Sutton.*

SIMPLE AND COMPOUND INTEREST.

§ 2.] When a sum of money is lent for a time, the borrower pays to the lender a certain sum for the use of it. The sum lent is spoken of as the *capital* or *principal*; the payment for the privilege of using it as *interest*. There are various ways of arranging such a transaction; one of the commonest is that the borrower repays after a certain time the capital lent, and pays also at regular intervals during the time a stated sum by way of interest. This is called paying *simple interest* on the borrowed capital. The amount to be paid by way of interest is usually stated as so much per cent per annum. Thus 5 per cent (5 %) per annum means £5 to be paid on every £100 of capital, for

* Full references to the various sources of information will be found in the article "Annuities" (by Sprague), *Encyclopædia Britannica*, 9th edition, vol. ii.

every year that the capital is lent. In the case of simple interest, the interest payable is sometimes reckoned strictly in proportion to the time; that is to say, allowance is made not only for whole years or other periods, but also for fractions of a period. Sometimes interest is allowed only for integral multiples of a period mutually agreed on. We shall suppose that the former is the understanding. If then r denote the interest on £1 for one year, that is to say, one-hundredth of the named rate per cent, n the time reckoned in years and fractions of a year, P the *principal*, I the whole interest paid, A the *amount*, that is, the sum of the principal and interest, both reckoned in pounds, we have

$$I = nrP \qquad (1); \qquad\qquad A = I + P = P(1 + nr) \qquad (2).$$

These formulæ enable us to solve all the ordinary problems of simple interest.

If any three of the four I, n, r, P, or of the four A, n, r, P, be given, (1) or (2) enables us to find the fourth.

Of the various problems that thus arise, that of finding P when A, n, r are given is the most interesting. We suppose that a sum of money A is due n years hence, and it is required to find what sum paid down at once would be an equitable equivalent for this debt. If simple interest is allowed, the answer is, such a sum P as would at simple interest amount in n years to A. In this case $P = A/(1 + nr)$ is called the *present value* of A, and the difference $A - P = A\{1 - 1/(1 + nr)\}$ $= Anr/(1 + nr)$ is called the *discount*. Discount is therefore the deduction allowed for immediate payment of a sum due at some future time. The discount is less than the simple interest (namely Anr) on the sum for the period in question. When n is not large, this difference is slight.

Example.

Find the difference between the interest and the discount on £1525 due nine months hence, reckoning simple interest at $3\frac{1}{2}$ %. The difference in question is given by

$$Anr - Anr/(1 + nr) = An^2r^2/(1 + nr).$$

In the present case $A=1525$, $n=9/12=3/4$, $r=3\cdot5/100=\cdot035$. Hence

$$\text{Difference}=1525\times(\cdot02625)^2\div(1\cdot02625),$$
$$=£1:0:5\tfrac{1}{2}.$$

§ 3.] In last paragraph we supposed that the borrower paid up the interest at the end of each period as it became due. In many cases that occur in practice this is not done; but, instead, the borrower pays at the end of the whole time for which the money was lent a single sum to cover both principal and interest. In this case, since the lender loses for a time the use of the sums accruing as interest, it is clearly equitable that the borrower should pay interest on the interest; in other words, that the interest should be added to the principal as it becomes due. In this case the principal or interest-bearing capital periodically increases, and the borrower is said to pay *compound interest*. It is important to attend carefully here to the understanding as to the period at which the interest is supposed to become due, or, as it is put technically, to be *convertible* (into principal); for it is clear that £100 will mount up more rapidly at 5 % compound interest convertible half-yearly than it will at 5 % compound interest payable annually. In one year, for instance, the amount on the latter hypothesis will be £105, on the former £105 plus the interest on £2 : 10s. for a half-year, that is, £105 : 1 : 3.

In what follows we shall suppose that no interest is allowed for fractions of the interval (*conversion-period*) between the terms at which the interest is convertible, and we shall take the conversion-period as unit of time. Let P denote the principal, A the accumulated value of P, that is, the principal together with the compound interest, in n periods; r the interest on £1 for one period; $1+r=R$ the amount of £1 at the end of one period.

At the end of the first period P will have accumulated to $P+Pr$, that is, to PR. The interest-bearing capital or principal during the second period is PR; and this at the end of the second period will have accumulated to $PR+PRr$, that is, to PR^2. The principal during the third period is PR^2, and the amount at the end of that period PR^3, and so on. In short, *the*

amount increases in a geometrical progression whose common ratio is R; *and at the end of* n *periods we shall have*

$$A = PR^n \tag{1}$$

By means of this equation we can solve all the ordinary problems of compound interest; for it enables us, when any three of the four quantities A, P, R, n are given, to determine the fourth. In most cases the calculation is greatly facilitated by the use of logarithms. See the examples worked below.

Cor. 1. *If* I *denote the whole compound interest on* P *during the* n *periods, we have*

$$I = A - P = P(R^n - 1) \tag{2}$$

Cor. 2. *If* P *denote the present value of a sum* A *due* n *periods hence, compound interest being allowed, then, since* P *must in* n *periods amount to* A, *we have*

$$A = PR^n,$$

so that

$$P = A/R^n \tag{3}$$

The discount on the present understanding is therefore

$$A(1 - 1/R^n) \tag{4}$$

Example 1.

Find the amount in two years of £2350 : 5 : 9 at $3\frac{1}{2}$ °/。 compound interest, convertible quarterly.

Here $P = 2350{\cdot}2875$, $n = 8$, $r = 3{\cdot}5/400 = {\cdot}00875$, $R = 1{\cdot}00875$.

$$\begin{aligned} \log A &= \log P + n \log R, \\ \log P &= 3{\cdot}3711210 \\ n \log R^* &= {\cdot}0302684 \\ \hline & 3{\cdot}4013894 \end{aligned}$$

$$A = £2519{\cdot}936 = £2519 : 18 : 8.$$

Example 2.

How long will it take £186 : 14 : 9 to amount to £216 : 9 : 7 at 6 °/。 compound interest, convertible half-yearly.

---

* When n is very large, the seven figures given in ordinary tables hardly afford the necessary accuracy in the product $n \log R$. To remedy this defect, supplementary tables are usually given, which enable the computer to find very readily to 9 or 10 places the logarithms of numbers (such as R) which differ little from unity.

Here $P=186{\cdot}7375$, $A=216{\cdot}4792$, $R=1{\cdot}03$.

$$n=(\log A-\log P)/\log R$$

$$=\frac{{\cdot}0641847}{{\cdot}0128372}=5{\cdot}00\ldots\ldots$$

Hence the required time is five half-years, that is, $2\frac{1}{2}$ years.

Example 3.

To find the present value of £1000 due 50 years hence, allowing compound interest at 4 %, convertible half-yearly.

Here $A=1000$, $n=100$, $R=1{\cdot}02$. We have $P=A/R^n$.

$$\log P=\log 1000-100\log 1{\cdot}02,$$
$$=3-100\times{\cdot}0086002,$$
$$=2{\cdot}1399800.$$
$$P=£138{\cdot}032=£138:0:8.$$

§ 4.] In reckoning compound interest it is very usual in practice to reckon by the year instead of by the conversion-period, as we have done above, the reason being that different rates of interest are thus more readily comparable. It must be noticed, however, that when this is done the rate of interest to be used must not be the *nominal rate* at which the interest due at each period is reckoned, but such a rate (commonly called the *effective rate*) as would, if convertible annually, be equivalent to the nominal rate convertible as given.

Let r_n denote the effective rate of interest per pound which is equivalent to the nominal rate r convertible every $1/n$th part of a year; then, since the amount of £1 in one year at the two rates must be the same, we have

$$(1+r)^n=1+r_n,$$

that is,
$$r_n=(1+r)^n-1 \qquad (1),$$
and
$$r=(1+r_n)^{1/n}-1 \qquad (2).$$

The equations (1) *and* (2) *enable us to deduce the effective rate from the nominal rate, and* vice versa.

Example.

The nominal rate of interest is 5 %, convertible monthly, to find the effective rate.

Here $\quad r={\cdot}05/12={\cdot}004166.$

Hence $\quad r_{12}=(1{\cdot}004166)^{12}-1,$

$$=1{\cdot}05114-1.$$
$$r_{12}={\cdot}05114.$$

Hence the effective rate is $5{\cdot}114$ %.

ANNUITIES CERTAIN.

§ 5.] When a person has the right to receive every year a certain sum of money, say £A, he is said to possess an *annuity* of £A. This right may continue for a fixed number of years and then lapse, or it may be vested in the individual and his heirs for ever; in the former case the annuity is said to be *terminable*, in the latter *perpetual*. A good example of a terminable annuity is the not uncommon arrangement in lending money where B lends C a certain sum, and C repays by a certain number of equal annual instalments, which are so adjusted as to cover both principal and interest. The simplest example of a perpetual annuity is the case of a freehold estate which brings its owner a fixed income of £A per annum.

Although in valuing annuities it is usual to speak of the whole sum which is paid yearly, yet, as a matter of practice, the payment may be by half-yearly, quarterly, &c. instalments; and this must be attended to in annuity calculations. Just as in compound interest, the simplest plan is to take the interval between two consecutive payments, or the conversion-period, as the unit of time, and adjust the rate of interest accordingly.

In many cases an annuity lasts only during the life of a certain named individual, called the nominee, who may or may not be the annuitant. In this and in similar cases an estimate of the probable duration of human life enters into the calculations, and the annuity is said to be *contingent*. In the second part of this work we shall discuss this kind of annuities. For the present we confine ourselves to cases where the annual payment is *certainly* due either for a definite succession of years or in perpetuity.

§ 6.] One very commonly occurring annuity problem is *to find the accumulated value of a* FORBORNE *annuity*. An annuitant B, who had the right to receive n successive payments at n successive equidistant terms, has for some reason or other not received these payments. The question is, What sum should he receive in compensation?

To make the question general, let us suppose that the last of the n instalments was due m periods ago.

It is clear that the whole accumulated value of the annuity is the sum of the accumulated values of the n instalments, and that compound interest must in equity be allowed on each instalment.

Now the nth instalment, due for m periods, amounts to AR^m, the $n-1$th to AR^{m+1}, the $n-2$th to AR^{m+2}, and so on. Hence, if V denote the whole accumulated value, we have

$$V = AR^m + AR^{m+1} + \ldots + AR^{m+n-1} \qquad (1).$$

Summing the geometric series, we have

$$V = AR^m(R^n - 1)/(R - 1) \qquad (2).$$

Cor. *If the last instalment be only just due,* $m = 0$, *and the accumulated value of the forborne annuity is given by*

$$V = A(R^n - 1)/(R - 1) \qquad (3).$$

Example.

A farmer's rent is £156 per annum, payable half-yearly. He was unable to pay for five successive years, the last half-year's rent having been due three years ago. Find how much he owes his landlord, allowing compound interest at 3 %.

Here $A=78$, $R=1{\cdot}015$, $m=6$, $n=10$.

$$V=78\times1{\cdot}015^6(1{\cdot}015^{10}-1)/{\cdot}015.$$

$$10\log 1{\cdot}015 = {\cdot}0646600,$$

$$1{\cdot}015^{10} = 1{\cdot}16054.$$

$$V=78\times1{\cdot}015^6\times{\cdot}16054/{\cdot}015.$$

$$\begin{array}{ll}
\log 78 & = 1{\cdot}8920946 \\
6\log 1{\cdot}015 & = {\cdot}0387960 \\
\log {\cdot}16054 & = \bar{1}{\cdot}2055833 \\
\hline
 & 1{\cdot}1364739 \\
\log {\cdot}015 & = \bar{2}{\cdot}1760913 \\
\hline
\log V & = 2{\cdot}9603826
\end{array}$$

$$V = £912{\cdot}814 = £912 : 16 : 3.$$

§ 7.] Another fundamental problem is *to calculate the purchase price of a given annuity.* Let us suppose that B wishes, by paying down at once a sum £P, to acquire for himself and his heirs the right of receiving n periodic payments of £A each, the first payment to be made m periods hence. We have to find P.

P is obviously the sum of the present values of the n pay-

ments. Now the first of these is due m years hence; its present value is therefore A/R^m. The second is due $m+1$ years hence; A's present value is therefore A/R^{m+1}, and so on. Hence

$$P = \frac{A}{R^m} + \frac{A}{R^{m+1}} + \ldots + \frac{A}{R^{m+n-1}} \qquad (1).$$

Hence

$$P = \frac{A}{R^m}\left(1 - \frac{1}{R^n}\right) \Big/ \left(1 - \frac{1}{R}\right),$$

$$= \frac{A}{R^{m+n-1}}(R^n - 1)/(R - 1) \qquad (2).$$

Cor. 1. The ratio of the purchase price of an annuity to the annual payment is often spoken of as the *number of years' purchase* which the annuity is worth. *If the "period" understood in the above investigation be a year, and p be the number of years' purchase, then we have from* (2)

$$p = (R^n - 1)/R^{m+n-1}(R - 1) \qquad (3).$$

If the period be $1/q$th of a year, since the annual payment is then qA, we have

$$p = (R^n - 1)/qR^{m+n-1}(R - 1) \qquad (4).$$

Cor. 2. If the annuity be *not "deferred,"* as it is called, *but begin to run at once, that is to say, if the first payment be due one period hence,* then $m = 1$, and we have*

$$P = A(R^n - 1)/R^n(R - 1),$$

$$= A(1 - R^{-n})/(R - 1) \qquad (5).$$

Also

$$p = (R^n - 1)/R^n(R - 1),$$

$$= (1 - R^{-n})/(R - 1) \qquad (6);$$

or

$$p = (1 - R^{-n})/q(R - 1) \qquad (7),$$

according as the period of conversion is a year or the qth part of a year.

Cor. 3. To obtain the *present value of a deferred perpetual annuity*, or, as it is often put, *the present value of the reversion of a perpetual annuity, we have merely to make n infinitely great in the equation* (2). *We thus obtain*

* This is the usual meaning of "beginning to run at once." In some cases the first payment is made at once. In that case, of course, $m = 0$.

$$P = \frac{A}{R^m} \bigg/ \left(1 - \frac{1}{R}\right),$$

$$= A/R^{m-1}(R-1) \qquad (8).$$

Hence, for the number of years' purchase, we have

$$p = 1/R^{m-1}(R-1) \qquad (9),$$

or

$$p = 1/qR^{m-1}(R-1) \qquad (10),$$

according as the period of conversion is a year or $1/q$*th of a year.*

Cor. 4. *When the perpetual annuity begins to run at once the formulæ* (8), (9), (10) *become very simple. Putting* $m = 1$ *we have*

$$P = A/(R-1),$$

$$= A/(1+r-1),$$

$$= A/r \qquad (11).$$

For the number of years' purchase

$$p = 1/r \qquad (12);$$

or

$$p = 1/qr \qquad (13),$$

according as the period of conversion is a year or $1/q$*th of a year.*

If the period be a year, remembering that, if s be the rate per cent of interest allowed, then $r = s/100$, we see that

$$p = 100/s \qquad (14).$$

Hence the following very simple rule for the value of a perpetual annuity. *To find the number of years' purchase, divide* 100 *by the rate per cent of interest corresponding to the kind of investment in question.* This rule is much used by practical men. The following table will illustrate its application :—

Rate %	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
No. of years' purchase .	33	28	25	22	20	18	17

Example.

A sum of £30,000 is borrowed, to be repaid in 30 equal yearly instalments which are to cover both principal and interest. To find the yearly payment, allowing compound interest at $4\frac{1}{2}$ %.

Let A be the annual payment, then £30,000 is the present value of an

annuity of £A payable yearly, the annuity to begin at once and run for 30 years. Hence, by (5) above,

$$\begin{aligned} 30{,}000 &= A(1 - 1{\cdot}045^{-30})/{\cdot}045, \\ A &= 1350/(1 - 1{\cdot}045^{-30}), \\ -30 \log 1{\cdot}045 &= \bar{1}{\cdot}4265110, \\ 1{\cdot}045^{-30} &= {\cdot}267000. \\ A &= 1350/{\cdot}733, \\ &= £1841 : 14 : 11. \end{aligned}$$

§ 8.] It would be easy, by assuming the periodic instalments or the periods of an annuity to vary according to given laws, to complicate the details of annuity calculations very seriously; but, as we should in this way illustrate no general principle of any importance, it will be sufficient to refer the student to one or two instances of this kind given among the examples at the end of this chapter.

It only remains to mention that in practice the calculation of interest and annuities is much facilitated by the use of tables (such as those of Turnbull, for example), in which the values of the functions $(1+r)^n$, $(1+r)^{-n}$, $\{(1+r)^n - 1\}/r$, $\{1-(1+r)^{-n}\}/r$, $r/\{1-(1+r)^{-n}\}$, &c., are tabulated for various values of $100r$ and n. For further information on this subject see the *Text-Book of the Institute of Actuaries*, Part I., p. 151.

Exercises XLIV.

(1.) The difference between the true discount and the interest on £40,400 for a period x is £4, simple interest being allowed at 4 %; find x.

(2.) Find the present value of £15,000 due 50 years hence, allowing $4\frac{1}{2}$ % compound interest, convertible yearly.

(3.) Find the amount of £150 at the end of 14 years, allowing 3 % compound interest, convertible half-yearly, and deducting 6d. per £ for income-tax.

(4.) How long will it take for a sum to double itself at 6 % compound interest, convertible annually?

(5.) How long will it take for one penny to amount to £1000 at 5 % compound interest, convertible annually?

(6.) On a salary of £100, what difference does it make whether it is paid quarterly or monthly? Work out the result both for simple and for compound interest at the rate of 4·2 %.

(7.) A sum £A is laid out at 10 % compound interest, convertible annually, and a sum £2A at 5 % compound interest, convertible half-yearly. After how many years will the amounts be equal?

(8.) Show that the difference between bankers' discount and true discount, simple interest being supposed, is

$$An^2r^2\{1 - nr + n^2r^2 - n^3r^3 + . . . \text{ ad } \infty\}.$$

(9.) If $r \ngtr 5/100$, $n \ngtr 10$, find an upper limit for the error in taking $100(1 + {}_nC_1r + {}_nC_2r^2 + {}_nC_3r^3)$ as the amount of £100 in n years at $100r$ °/ₒ compound interest, convertible annually.

(10.) If £I_c and £I_s denote the whole compound and the whole simple interest on £P for n years at $100r$ °/ₒ, convertible annually, show that

$$I_c - I_s = P({}_nC_2r^2 + {}_nC_3r^3 + . . . + r^n).$$

(11.) A man owes £P, on which he pays $100r$ °/ₒ annually, the principal to be paid up after n years. What sum must he invest, at $100r'$ °/ₒ, so as to be just able to pay the interest annually, and the principal £P when it falls due?

(12.) B has a debenture bond of £500 on a railway. When the bond has still five years to run, the company lower the interest from 5 °/ₒ, which was the rate agreed upon, to 4 °/ₒ, and, in compensation, increase the amount of B's bond by x °/ₒ. Find x, supposing that B can always invest his interest at 5 °/ₒ.

(13.) A person owes £20,122 payable 12 years hence, and offers £10,000 down to liquidate the debt. What rate of compound interest, convertible annually, does he demand?

(14.) A testator directed that his trustees, in arranging his affairs, should set apart such sums for each of his three sons that each might receive the same amount when he came of age. When he died his estate was worth £150,000, and the ages of his sons were 8, 12, and 17 respectively. Find what sum was set apart for each, reckoning 4 °/ₒ compound interest for accumulations.

(15.) B owes to C the sums $A_1, A_2, . . ., A_r$ at dates $n_1, n_2, . . ., n_r$ years hence. Find at what date B may equitably discharge his debt to C by paying all the sums together, supposing that they all bear the same rate of interest; and

1st. Allowing interest and interest in lieu of discount where discount is due.
2nd. Allowing compound interest, and true discount at compound interest.

(16.) Required the accumulated value at the end of 15 years of an annuity of £50, payable in quarterly instalments. Allow compound interest at 5 °/ₒ.

(17.) A loan of £100 is to be paid off in 10 equal monthly instalments. Find the monthly payment, reckoning compound interest at 6 °/ₒ.

(18.) I borrow £1000, and repay £10 at the end of every month for 10 years. Find an equation for the rate of interest I pay. What kind of interest table would help you in practically solving such a question as this?

(19.) The reversion after 2 years of a freehold worth £168 : 2s. a year is to be sold: find its present value, allowing interest at 2 °/ₒ, convertible annually.

(20.) Find the present value of a freehold of £365 a year, reckoning compound interest at 3¼ °/ₒ, convertible half-yearly, and deducting 6d. per £ of income-tax.

(21.) If a perpetual annuity be worth 25 years' purchase, what annuity to

continue for 3 years can be bought for £5000 so as to bring the same rate of interest?

(22.) If 20 years' purchase be paid for an annuity to continue for a certain number of years, and 24 years' purchase for one to continue twice as long, find the rate of interest (convertible annually).

(23.) Two proprietors have equal shares in an estate of £500 a year. One buys the other out by assigning him a terminable annuity to last for 20 years. Find the annuity, reckoning $3\frac{3}{4}$ °/$_\circ$ compound interest, convertible annually.

(24.) The reversion of an estate of £150 a year is sold for £2000. How long ought the entry to be deferred if the rate of interest on the investment is to be $4\frac{3}{4}$ °/$_\circ$, convertible annually?

(25.) If a lease of 19 years at a nominal rent be purchased for £1000, what ought the real rent to be in order that the purchaser may get 4 °/$_\circ$ on his investment (interest convertible half-yearly)?

(26.) B and C have equal interests in an annuity of £A for $2n$ years (payable annually). They agree to take the payments alternately, B taking the first. What ought B to pay to C for the privilege he thus receives?

(27.) A farmer bought a lease for 20 years of his farm at a rent of £50, payable half-yearly. After 10 years had run he determined to buy the freehold of the farm. What ought he to pay the landlord if the full rent of the farm be £100 payable half-yearly, and 3 °/$_\circ$ be the rate of interest on investments in land?

(28.) What annuity beginning n years hence and lasting for n years is equivalent to an annuity of £A, beginning now and lasting for n years?

(29.) A testator left £100,000 to be shared equally between two institutions B and C; B to enjoy the interest for a certain number of years, C to have the reversion. How many years ought B to receive the interest if the rate be $3\frac{1}{2}$ °/$_\circ$, convertible annually?

(30.) If a man live m years, for how many years must he pay an annuity of £A in order that he may receive an annuity of the same amount for the rest of his life? Show that, if the annuity to be acquired is to continue for ever, then the number of years is that in which a sum of money would double itself at the supposed rate of interest.

(31.) A gentleman's estate was subject to an annual burden of £100. His expenses in any year varied as the number of years he had lived, and his income as the square of that number. In his 21st year he spent £10,458, and his income, before deducting the annual burden, was £4410. Show that he ran in debt every year till he was 50.

(32.) A feu is sold for £1500, with a feu-duty of £18 payable annually, and a casualty of £100 payable every 50 years. What would have been the price of the feu if it had been bought outright? Reckon interest at $4\frac{1}{2}$ °/$_\circ$.

(33.) Find the accumulation and also the present value of an annuity when the annual payments increase in A.P.

(34.) Solve the same problem when the increase is in G.P.

(35.) The rental of an estate is £mA to begin with; but at the end of every q years the rental is diminished by £A, owing to the incidence of fresh taxation. Find the present value of the estate.

APPENDIX

ON THE GENERAL SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS; AND ON THE CASES WHERE SUCH EQUATIONS CAN BE SOLVED BY MEANS OF QUADRATIC EQUATIONS.

§ 1.] Since cubic and biquadratic equations are of frequent occurrence in elementary mathematics, and many interesting geometric problems can be made to depend on their solution, a brief account of their leading properties may be useful to readers of this book. Incidentally, we shall meet with some principles of importance in the General Theory of Equations.

COMMENSURABLE ROOTS AND REDUCIBILITY.

§ 2.] We shall suppose in all that follows that the coefficients $p_0 \ldots, p_n$ of any equation,

$$p_0x^n + p_1x^{n-1} + \ldots + p_n = 0 \qquad (1),$$

are all real commensurable numbers. If, as in chap. xv., § 21, we put $x = \xi/m$, we derive from (1) the equivalent equation

$$p_0\xi^n + p_1m\xi^{n-1} + \ldots + p_{n-1}m^{n-1}\xi + p_nm^n = 0 \qquad (2),$$

each of whose roots is m times a corresponding root of (1). If we then choose m so that $mp_1/p_0, \ldots, m^{n-1}p_{n-1}/p_0, m^np_n/p_0$ are all integral—for example, by taking for m the L.C.M. of the denominators of the fractions $p_1/p_0, \ldots, p_{n-1}/p_0, p_n/p_0$—we shall reduce (2) to *the form*

$$\xi^n + q_1\xi^{n-1} + \ldots + q_n = 0 \qquad (3),$$

in which all the coefficients are positive or negative integers, and the

coefficients of the highest term unity. We may call this the *Special Integral Form.*

§ 3.] If, as in chap. xv., § 22, we put $x = \xi + a$, we transform the equation (1) into

$$p_0\xi^n + q_1\xi^{n-1} + . . . + q_n = 0,$$

where $q_1 = np_0a + p_1$. Hence, if we take $a = -p_1/np_0$, the transformed equation becomes

$$p_0\xi^n + q'_2\xi^{n-2} + . . . + q'_n = 0 \qquad (4),$$

wherein $q'_2, . . ., q'_n$ have now determinate values. It follows that

By a proper linear transformation, we can always deprive an equation of the nth degree of its highest term but one.

We can, of course, combine the transformations of §§ 2, 3, and reduce an equation to a special integral form wanting the highest term but one.

§ 4.] *If an equation of the special integral form has commensurable roots, these roots must be integral, and can only be exact divisors of its absolute term.* For, suppose that the equation (3) has the fractional commensurable root a/b, where a may be supposed to be prime to b. Then we have the identity

$$(a/b)^n + q_1(a/b)^{n-1} + . . . + q_n \equiv 0,$$

whence

$$a^n/b \equiv -q_1a^{n-1} - q_2a^{n-2}b - . . . - q_nb^{n-1},$$

which is impossible, since the left-hand side is a fraction and the right-hand side an integer.

Also, if $x = a$ be any integral root, we must have

$$q_n/a \equiv -a^{n-1} - q_1a^{n-2} - . . . - q_{n-1}.$$

Hence, since the right-hand side is obviously an integral number, a must be an exact divisor of q_n.

The commensurable roots of an equation, if any exist, can therefore always be found by a limited number of arithmetical operations.

We have merely to reduce the equation to an equivalent special integral form, and substitute the divisors of its absolute term one after the other in the characteristic. The number of trials may in most cases be reduced by obtaining upper and

lower limits for the roots by means of the theorem of chap. xii., § 21, by graphical methods, or otherwise.

Example. $x^3-10x^2+31x-30=0$. This equation is already in the special integral form. Hence the only possible commensurable roots are ± 1, ± 2, ± 3, ± 5, ± 6, ± 10, ± 15, ± 30. It is obvious that the equation has no negative roots; and, since $x(x^2+31)=10(x^2+3)$, it is useless to try $x=10$ or any larger number. Of the remaining possible numbers, $+1$, $+2$, $+3$, $+5$, it is immediately found that $+2$, $+3$, and $+5$ are roots.

§ 5.] An equation is said to be *reducible* when it has a root or roots in common with an equation of lower degree (having of course commensurable coefficients); *irreducible* if it has no root in common with any equation of lower degree.

If an equation $f(x)=0$ have roots in common with an equation of lower degree $\phi(x)=0$, then the product of the linear factors corresponding to all such common roots, say $g(x)$, is the G.C.M. of the characteristics $f(x)$ and $\phi(x)$, which can be deduced from these functions by purely rational operations. It follows that $f(x)\equiv g(x)h(x)$, where $g(x)$ and $h(x)$ are integral functions of x having commensurable coefficients. The roots of $f(x)=0$ are therefore the aggregate of the roots of the two equations $g(x)=0$, $h(x)=0$, each of lower degree than the original equation. Each of these new equations may be reducible or irreducible; but it is obvious that at last we must arrive at a series of irreducible equations the aggregate of whose roots are the roots of $f(x)=0$; and the characteristics of these equations are the irreducible factors of $f(x)$, irreducible in the sense that they cannot be decomposed into integral factors of lower degree having commensurable coefficients.

§ 6.] The following theorem, often appealed to in the theory of equations, is an immediate consequence of the notion of irreducibility explained in the last paragraph.

If an irreducible equation A *have a root in common with an equation* B (*reducible or irreducible*), *then every root of* A *is a root of* B.

For, if only some of the roots of A were roots of B, then a common commensurable factor of the characteristics of A and B could be found of less degree than A itself; and A would not be irreducible.

§ 7.] *When an equation is reducible it can be reduced by a finite number of arithmetical operations.*

Consider the equation

$$f(x) \equiv x^n + p_1 x^{n-1} + \ldots + p_n = 0 \qquad (5),$$

whose roots are $a_1, a_2, \ldots a_n$.
If (5) be reducible, then $f(x)$ must have a commensurable factor of the 1st, or 2nd, . . ., or fth degree, where (f) is the greatest integer in $n/2$.

If $f(x)$ has a commensurable factor of the 1st degree, (5) has a commensurable root, which can be found as explained in § 4.

If $f(x)$ has a commensurable factor of the sth degree, let $a_1, a_2, \ldots, a_s$ be the corresponding roots; and let

$$(x - a_1)(x - a_2) \ldots (x - a_s) \equiv x^s - q_1 x^{s-1} + \ldots \pm q_s \qquad (6),$$

where $\quad q_1 = \sum_1^s a_1,\ q_2 = \sum_1^s a_1 a_2,\ \ldots,\ q_s = a_1 a_2 \ldots a_s,$

are by hypothesis all commensurable. If we form the equations

$$\prod_1^n (x - \sum_1^s a_1) = 0,\ \prod_1^n (x - \sum_1^s a_1 a_2) = 0,\ \ldots\ \prod_1^n (x - a_1 a_2 \ldots a_s) = 0 \quad (7),$$

the roots of which are respectively all the different values of

$$\sum_1^s a_1,\quad \sum_1^s a_1 a_2,\quad \ldots,\quad a_1 a_2 \ldots a_s$$

obtained by taking every possible selection of s of the roots of (5), then the coefficients of these equations are symmetric functions of the roots of (5), therefore rational functions of the coefficients of (5), and therefore commensurable. But $q_1, q_2, \ldots, q_s$ are obviously roots of the respective equations (7), and, being commensurable, can be found by a finite number of trials.

In point of fact, it is in general sufficient to determine q as a commensurable root of the first of (7), and then determine the function (6) as the G.C.M. of $f(x)$ and $\prod_1^n (x - q_1 + \sum_1^{s-1} a_1)$. The proof, and the discussion of possible exceptions, may be left to the reader.

EQUATIONS SOLUBLE BY MEANS OF SQUARE ROOTS.

§ 8.] Reference to the theory of irrational operations laid down in chap. x. will show that any function which is constructed by means of a finite number of the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$ may without loss of generality be supposed to be constructed as follows:—

Starting with the series of rational numbers, we construct a number of quadratic irrationals, say $\sqrt{a_1}$, $\sqrt{a_2}$, . . . Any rational functions of these may be reduced to linear forms, such as $A + B\sqrt{a_1} + \ldots + C\sqrt{a_1}\sqrt{a_2} + \ldots$, which we may describe as quadratic irrationals of the *first order*. We suppose that $\sqrt{a_1}$, $\sqrt{a_2}$, . . . are independent, in the sense that there is no linear relation, such as $P + Q\sqrt{a_1} + \ldots + R\sqrt{a_1}\sqrt{a_2} + \ldots = 0$, connecting them; for, if there were any such relation, we could use it to get rid of $\sqrt{a_1}$ by expressing it as a linear function of $\sqrt{a_2}$ . . ., and thus reduce the number of independent square roots.

Next consider $\sqrt{b_1}$, $\sqrt{b_2}$, . . ., where b_1, b_2, . . . are quadratic irrationals of the first order reduced as above. Then take rational functions of these, and reduce them to linear forms containing only independent square roots as before. These we may call quadratic irrationals of the second order.

Proceeding in this way, we can build up quadratic irrational functions of any order; and it is obvious that *every function which is constructed by means of a finite number of the operations* $+$, $-$, $\times$, $\div$, $\sqrt{}$ *is a quadratic irrational of a finite order in the above scale.*

For example, $3 + \sqrt{2} + 2\sqrt{2}\sqrt{3}$, $\sqrt{(\sqrt{2} + \sqrt{3})} + \sqrt{(\sqrt{2} - \sqrt{3})}$, $\sqrt{2} + \sqrt{\{3 + \sqrt{(\sqrt{2} + \sqrt{3})}\}}$ are quadratic irrationals of the first, second, and third orders respectively.

Def. If we give every possible arrangement of signs to the square roots in a quadratic irrational function which contains p square roots, we get 2^p values, which are said to be *conjugate* to one another. It should be noticed that the conjugates may not be all different; *e.g.*, $\sqrt{(\sqrt{2} + \sqrt{3})} + \sqrt{(\sqrt{2} - \sqrt{3})}$ and $\sqrt{(\sqrt{2} - \sqrt{3})} + \sqrt{(\sqrt{2} + \sqrt{3})}$ are conjugate with respect to $\sqrt{3}$, but they are not distinct.

§ 9.] *If any quadratic irrational x_1, containing p square roots, is a root of an irreducible equation $f(x) = 0$, then all its conjugates, x_2, . . ., x_μ, ($\mu = 2^p$) are also roots of the same equation.*

For, if we substitute x_1 in the equation, we shall in the first place get an identity the left-hand side of which will be a linear quadratic irrational function of say the kth order. Since our square roots are independent, such a relation cannot exist unless each term is zero independently. We are thus led to a series of identities involving quadratic irrational functions of order $k - 1$, or lower. Applying the same reasoning again and again, we are

at last led to a series of identities in which there are no square roots at all. These must hold whatever the signs of the square roots originally involved may have been. Hence, if x_1 nullify $f(x)$, so must also $x_2, \ldots, x_\mu$.

It is easy to see, by reasoning similar to that employed in chap. x., § 16, that the function

$$\psi(x) \equiv (x - x_1)(x - x_2) \;.\;.\;.\; (x - x_\mu)$$

is an integral function of degree $\mu = 2^p$ in x, whose coefficients are commensurable.

If $x_1, \ldots, x_\mu$ are all unequal, then, since these are all roots of $f(x) = 0$, $\psi(x)$ and $f(x)$ can only differ by a constant factor. Therefore the degree of $f(x)$ is 2^p.

If $x_1, \ldots, x_\mu$ are not all unequal, then each of them must be repeated the same number of times; otherwise, by dividing $\psi(x)$ by a proper power of $f(x)$ we should obtain the characteristic function of an equation which has some but not all of the roots of $f(x) = 0$, which, by § 6, is impossible, since $f(x) = 0$ is irreducible. It follows that the degree of $f(x)$ is a factor of 2^p, that is to say, is some power of 2 lower than the pth. Hence the following important result:—

The degree of an irreducible equation which can be solved by means of square roots must be a power of 2; and its roots can all be deduced from any one by varying the signs of the square roots involved in the expression for that root.

By a process of continually pairing factors which are conjugate with respect to any particular square root, we can reduce the characteristic of any equation soluble by means of square roots to the product of two conjugate factors, each of which contains only the square root of a commensurable number. Hence *any such equation can be put into the form*

$$(x^n + p_1 x^{n-1} + \;.\;.\;. + p_n)^2 - \rho(x^{n-1} + q_1 x^{n-2} + \;.\;.\;. + q_{n-1})^2 = 0,$$

where $p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}$, and ρ are all commensurable. This transformation may be regarded as the first step in the solution of the equation.*

* For further discussion of this matter, see Petersen's *Théorie des Équations Algébriques* (trans. by Laurent, 1897), chap. vii., from which the substance of §§ 8, 9 are taken.

THE CUBIC EQUATION.

§ 10.] We have seen (§ 3) that every cubic equation can be reduced to the form

$$x^3 + qx + r = 0 \qquad (1).$$

If ω be an imaginary cube root of unity, we have the identity

$$x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

It follows that the roots of the equation

$$x^3 - (3yz)x + (y^3 + z^3) = 0 \qquad (2)$$

are $-y - z,\ \ -\omega y - \omega^2 z,\ \ -\omega^2 y - \omega z$.

Now, if

$$yz = -q/3, \quad y^3 + z^3 = r \qquad (3),$$

the equations (1) and (2) are identical.

From (3) we see that y^3 and z^3 are the roots of the quadratic

$$\xi^2 - r\xi - q^3/27 = 0 \qquad (4).$$

Hence, if

$$L = \frac{r}{2} + \sqrt{\left(\frac{r^2}{4} + \frac{q^3}{27}\right)},\ M = \frac{r}{2} - \sqrt{\left(\frac{r^2}{4} + \frac{q^3}{27}\right)} \qquad (5),$$

then the roots of (1) are

$$-\sqrt[3]{L} - \sqrt[3]{M},\ \ -\omega\sqrt[3]{L} - \omega^2\sqrt[3]{M},\ \ -\omega^2\sqrt[3]{L} - \omega\sqrt[3]{M} \qquad (6).$$

When $r^2/4 + q^3/27$ is positive, (6) gives the roots of the cubic in a convenient form for computation; and in this case one root is real and two imaginary. $-\sqrt[3]{L} - \sqrt[3]{M}$ is the famous expression of Cardan for the root of a cubic equation, which formed the subject of a notorious controversy between him and his countryman Tartaglia.

If $r^2/4 + q^3/27$ is negative, so that q must be a negative number, the expressions (6) in their algebraic form are useless for the purposes of numerical calculation. We may, however, use the circular functions, as in chap. xii., § 17, for finding the principal values of $\sqrt[3]{L}$ and $\sqrt[3]{M}$. We thus find that the roots are $-2\rho^{\frac{1}{3}}\cos\theta/3$, $-2\rho^{\frac{1}{3}}\cos(2\pi+\theta)/3$, $-2\rho^{\frac{1}{3}}\cos(4\pi+\theta)/3$, where $\rho = (-q)^{\frac{3}{2}}/27^{\frac{1}{2}}$, and $\cos\theta = r27^{\frac{1}{2}}/2(-q)^{\frac{3}{2}}$. The three roots are therefore all real (Cardan's Irreducible Case).*

If $r^2/4 + q^3/27$ be zero, two of the roots are equal, and all three real.

---

* See also Pressland and Tweedie's *Trigonometry*, § 124.

§ 11.] Since 3 is not a power of 2, it follows from § 8 that a cubic cannot be solved by means of square roots unless it is reducible: in other words, unless it has a commensurable root. In that case it may happen that the other two roots are also commensurable, so that no irrational operations at all are required for the solution.

It follows from the result of §§ 9 and 10, that an irreducible cubic requires both square and cubic roots for its formal solution.

THE BIQUADRATIC EQUATION.

§ 12.] The solution of the biquadratic

$$x^4 + px^3 + qx^2 + rx + s = 0 \qquad (1)$$

can always be made to depend on the solution of a cubic and two quadratic equations.

For we may conjoin with (1) the equation

$$x^2 - y = 0 \qquad (2),$$

and regard (1) as the x-eliminant of the system (1) (2). Now, using (2) to transform (1), we see that the system (1) (2) is equivalent to the system

$$qx^2 + pxy + y^2 + rx + s = 0, \quad x^2 - y = 0 ;$$

and this system again to

$$(q - \lambda)x^2 + pxy + y^2 + rx + \lambda y + s = 0 \qquad (3),$$

and

$$x^2 - y = 0 \qquad (4).$$

Now (see chap. vii., § 13) we can always determine λ so that (3) shall break up into two linear equations. We have merely to choose λ so that

$$(q - \lambda)\cdot 1 \cdot s + 2\left(\frac{\lambda}{2}\right)\left(\frac{r}{2}\right)\left(\frac{p}{2}\right) - (q - \lambda)\left(\frac{\lambda}{2}\right)^2 - 1\cdot\left(\frac{r}{2}\right)^2 - s\left(\frac{p}{2}\right)^2 = 0.$$

In other words, λ must be a root of the cubic equation

$$\lambda^3 - q\lambda^2 + (pr - 4s)\lambda + 4qs - r^2 - p^2s = 0 \qquad (5),$$

called *Lagrange's Resolvent*.

If we substitute in (3) for λ any one of the three roots of (5), then (3) takes the form

$$(y + \alpha x + \beta)(y + \gamma x + \delta) = 0 ;$$

and the system (3) (4) leads to the two quadratics

$$x^2 + \alpha x + \beta = 0, \; x^2 + \gamma x + \delta = 0 \qquad (6),$$

the roots of which are the four roots of the biquadratic (1).

If x_1, x_2, x_3, x_4 be the roots of the biquadratic taken in a certain order, then we see from (6) that $\beta + \delta = x_1x_2 + x_3x_4$. Also from the identity $(q - \lambda)x^2 + pxy + y^2 + rx + \lambda y + s \equiv (y + \alpha x + \beta)(y + \gamma x + \delta)$ we see, by comparing coefficients of y, that $\lambda = \beta + \delta = x_1x_2 + x_3x_4$. Hence the following interesting result :—

Cor. *The roots of Lagrange's cubic resolvent are* $x_1x_2 + x_3x_4$, $x_1x_3 + x_2x_4$, $x_1x_4 + x_2x_3$.

The above method for solving a biquadratic is merely an analytical translation of the geometrically obvious fact that the four points of intersection of two conic sections can be determined by finding the intersections with either of the conics of any one of the three pairs of straight lines which contain all the four points.

Example.

$$x^4 - 5x^3 + 5x^2 + 5x - 6 = 0 \qquad (1).$$

Conjoin

$$x^2 - y = 0 \qquad (2).$$

Then we have the system

$$5x^2 - 5xy + y^2 + 5x - 6 = 0, \quad x^2 - y = 0,$$

equivalent to

$$(5 - \lambda)x^2 - 5xy + y^2 + 5x + \lambda y - 6 = 0 \qquad (3).$$

$$x^2 - y = 0 \qquad (4).$$

The cubic resolvent is

$$\lambda^3 - 5\lambda^2 - \lambda + 5 = 0 \, ;$$

that is

$$(\lambda - 5)(\lambda^2 - 1) = 0 \qquad (5),$$

which happens to have three commensurable roots. We take the simplest, $\lambda = 1$, and find that (3) becomes

$$(y - 4x + 3)(y - x - 2) = 0 \, ;$$

so that the roots of (1) are given by

$$x^2 - 4x + 3 = 0, \; x^2 - x - 2 = 0 \qquad (6).$$

They are in fact 1, 3, 2, -1.

§ 13.] If the cubic resolvent is irreducible, λ will involve an incommensurable cube root, and the roots of the biquadratic will not be expressible by means of square roots alone. Hence, in order that the roots of the biquadratic may be expressible by means of square roots alone, it is necessary that the cubic resolvent should have a commensurable root, which can always be found as explained in § 4.*

---

* See also Ex. xlv. (21).

This condition is obviously sufficient; for, if we use the value of λ, then α, β, γ, δ will all be expressible by means of a single square root (see chap. xii., § 13), viz. $\sqrt{\{p^2 - 4(q - \lambda)\}}$; and each of the two quadratics (6) will be soluble by means of an additional square root. The expression for a root will in general be a quadratic irrational of the second order of the form $A + \sqrt{(B + \sqrt{C})}$, where A, B, C are rational functions of the coefficients; but in particular cases it may reduce to the simpler form $A + \sqrt{B} + \sqrt{C} + D\sqrt{B}\sqrt{C}$; or it may happen that two or four of the roots may be commensurable.

§ 14.] If a biquadratic be reducible, it may reduce (i.) to a linear equation and a cubic; (ii.) to two quadratics; (iii.) to two linear equations and a quadratic; (iv.) to four linear equations.

It should be noticed as regards a biquadratic that reducibility and solubility by means of square roots (*i.e.* by means of quadratic equations) alone are not the same thing. For example, a biquadratic may have one commensurable root, and its other roots may be the roots of an irreducible cubic. Again, $(x^2 + 3x + 1)^2 - 2(2x^2 - 2x + 1)^2 = 0$ is evidently soluble by means of square roots; but it is not reducible, for we cannot factorise the characteristic without introducing the surd $\sqrt{2}$.

§ 15.] There is another way of solving a biquadratic which is often convenient in practice. Suppose the biquadratic reduced to the form

$$x^4 + qx^2 + rx + s = 0 \qquad (1).$$

Then $x_1 + x_2 + x_3 + x_4 = 0$; and we can reduce the characteristic to the form

$$\{x^2 - (x_1 + x_2)x + x_1x_2\}\{x^2 + (x_1 + x_2)x + x_3x_4\} \qquad (2).$$

Hence, if we put

$$x_1 + x_2 = \sqrt{\rho},\ x_1x_2 = \beta - \sqrt{\gamma\rho},\ x_3x_4 = \beta + \sqrt{\gamma\rho} \qquad (3),$$

we have

$$(x^2 + \beta)^2 - \rho(x + \gamma)^2 \equiv x^4 + qx^2 + rx + s \qquad (4).*$$

Therefore

$$2\beta - \rho = q,\ -2\rho\gamma = r,\ \beta^2 - \rho\gamma^2 = s \qquad (5),$$

which are equivalent to

$$\beta = (q + \rho)/2, \quad \gamma = -r/2\rho \qquad (6);$$

* The whole of the present process is a natural application of the last remark in § 9.

together with

$$\rho^3 + 2q\rho^2 + (q^2 - 4s)\rho - r^2 = 0 \qquad (7),$$

which is *Descartes' Cubic Resolvent.*

When any root of (7) is known, the values of β and γ are given by (6) and the roots of the biquadratic are given by

$$x^2 - \sqrt{\rho}\, x + \beta - \sqrt{\gamma\rho} = 0,\ x^2 + \sqrt{\rho}\, x + \beta + \sqrt{\gamma\rho} = 0 \qquad (8).$$

As before, the necessary and sufficient condition for solubility by means of square roots is that the resolvent shall have at least one commensurable root.

Cor. 1. *The roots of Descartes' resolvent are three of the six quantities*

$$(x_1 + x_2)^2,\ (x_1 + x_3)^2,\ (x_1 + x_4)^2,\ (x_3 + x_4)^2,\ (x_2 + x_4)^2,\ (x_2 + x_3)^2,$$

which are equal in pairs.

Cor. 2. *If the biquadratic be reducible to two quadratics, one of the roots of Descartes' resolvent must be a perfect square; and this condition is sufficient.*

§ 16.] If the solution of a geometric problem be expressed by a series of equations, the necessary and sufficient condition for solubility by means of the ruler and compass alone is that these equations either are, or are replaceable by a series of linear and quadratic equations (see Introduction, § 240). The foregoing considerations often enable us to settle the possibility or impossibility of such a solution. For example, the abscissa, or ordinate, of the intersection of two conics is in general the root of a biquadratic equation: hence the intersections of two conics cannot be constructed by the ruler and compass alone, unless the cubic resolvent of this biquadratic have a commensurable root.

Exercises XLV.

(1.) Prove that the biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$ is soluble by square roots if $p^3 - 4pq + 8r = 0$.

(2.) Discuss the Lagrangian resolvent of $x^4 + px^3 + qx^2 \pm px + 1 = 0$.

Solve the following biquadratics :—

(3.) $x^4 + 10x^3 + 22x^2 - 15x + 2 = 0$.

(4.) $x^4 + 10x^3 + 35x^2 + 50x + 4 = 0$.

(5.) $x^4 + 2(p-q)x^3 + (p^2+q^2)x^2 + 2pq(p-q)x + pq(p^2+pq+q^2) = 0$.

(6.) $2x^4 - x^3 - 9x^2 + 4x + 3 = 0$.

(7.) $2x^4+x^3-3x^2-8x-12=0$.

(8.) $2x^4+5x^3+6x^2-x-6=0$.

(9.) $2x^4+3x^3+16x+6=0$.

(10.) $x^4-4x^3-4x^2+16x-8=0$.

(11.) $x^4-6x^2+8x-3=0$.

(12.) If an equation of the special integral form of § 2 reduce to equations of lower degree, prove that each of these equations is also of the special integral form (see Weber's *Algebra*, § 2).

(13.) Show that, without solving equations of higher degree than the second, we can determine α, β, γ so that the substitution $y=\alpha+\beta x+\gamma x^2$ shall transform any cubic equation into the form $y^3+A=0$.

(14.) Show that, without solving equations of higher degree than the third, we can determine α, β, γ, δ so that the substitution $y=\alpha+\beta x+\gamma x^2+\delta x^3$ shall transform any biquadratic into the form $y^4+Ay^2+B=0$.

(15.) Show that $\sqrt[3]{\{r/2+\sqrt{(r^2/4+q^3/27)}\}}$ is expressible in the form $x+\sqrt{y}$ where x and y are rational, when, and only when, the cubic $x^3+qx+r=0$ has a commensurable root. What bearing has this on the solution of a cubic equation?

(16.) Show that one root of Descartes' resolvent of the biquadratic $x^4+qx^2+rx+s=0$ is always real and positive; and that the roots of the biquadratic are 1° all real, if the other two roots (ρ_2, ρ_3) of the resolvent are both real and positive; 2° all imaginary, if ρ_2 and ρ_3 are both real, negative, and unequal; 3° two real and two imaginary, if ρ_2 and ρ_3 are both real, negative, and equal, or if ρ_2 and ρ_3 are both imaginary.

(17.) If the roots of Lagrange's resolvent be all real and unequal, show that the roots of the biquadratic are either all real, or else all imaginary; and that, if only one root of the resolvent is real, then two roots of the biquadratic are real and two imaginary.

(18.) Show that a regular heptagon cannot be inscribed in a circle by means of the ruler and compasses only (see Ex. xxxii., 33).

(19.) Show that the inscription in a circle of a regular polygon of 11 sides depends on an irreducible quintic equation.

(20.) Show that, if A, B be two given points in a straight line, we can by the ruler and compass alone find a point on the line such that $AP+BP:AP-BP=AP^3:BP^3$.

(21.) Starting from the remark at the end of § 8, prove that, if a biquadratic equation be soluble by square roots alone, then its Lagrangian resolvent must have at least one commensurable root.

(22.) Show that, if the roots of the biquadratic $x^4+qx^2+rx+s=0$ be rational functions of two quadratic surds, then the cubic $y^3-4qy^2+4(q^2-4s)y+8r^2=0$ must have a commensurable root, say $y=m$; and $(q/2-m/8)^2+r^2/2m$ must be the square of a rational number. Are these conditions sufficient?

RESULTS OF EXERCISES.

I.

(5.) 1st. The number of digits is 34 ; for the best approximation the first three digits are 126. 2nd. The number of digits is 20 ; the first three 184. **(6.)** -1, $+1$. **(7.)** $a-b$. **(8.)** 1707 ; 30521/415800. **(9.)** 6. **(10.)** $aa+2ab+bb$; $aa-bb$; $9aa-36bb$; $\frac{1}{9}aa-\frac{1}{36}bb$. **(11.)** $2(mm-1)aa+2(nn-1)bb$. **(12.)** $2xy+2/xy$. **(13.)** $2yz/x+2zx/y+2xy/z+2/xyz$. **(14.)** $\frac{1}{4}xx+\frac{1}{8}zx-\frac{1}{16}yy+\frac{1}{64}zz$.

II.

(1.) $1/2^4.3^{12}.5=1/42515280$. **(2.)** The second is greater by 65280. **(3.)** $1/2$. **(4.)** $16a^3b^2d^2$. **(5.)** $c^7/a^{14}b^5x^8y^{14}$. **(6.)** $(81/16)a^6b^6c^{12}x^2$. **(7.)** y^9z^2/x^5. **(8.)** $(xyz)^{49}$. **(9.)** 1. **(10.)** a^{p+q-2r}. **(11.)** $x^{(a-c)(1-c)}$. **(12.)** $1/x^{p^2+q^2}$. **(13.)** 1. **(15.)** $1+x^{2a-b-c}-x^{-2a+b+c}+x^{-a+2b-c}+x^{a-2b+c}-x^{-a-b+2c}+x^{a+b-2c}$. **(16.)** $a^{p+2q}+2a^{p+q}/b^p+a^p/b^{2p}+a^{2q}/b^q+2a^q/b^{p+q}+1/b^{2p+q}$.

III.

(1.) $x+y$. **(2.)** 1. **(3.)** $(x^2-y^2)/xy$. **(4.)** y. **(5.)** $1/bc(-a+b-c)$. **(6.)** $(a^4-a^2b^2+2ab^3-b^4)/(a^2-b^2)$. **(7.)** $4xy^2/(x^4-y^4)$. **(8.)** $2(b^2x^2+a^2y^2)/abxy$. **(9.)** $ab/(a^2+b^2)$. **(10.)** a^2-b^2. **(11.)** $-(4x+2x^3)/(1+x^2+x^4)$. **(12.)** 1. **(13.)** 1. **(14.)** The function is $=1$. **(15.)** $(adf-ac)/(bdf-be-cf)$. **(16.)** $(a^2-4ab+4b^2-1)/(a^3-6a^2b+12ab^2-8b^3-2a+4b)$. **(17.)** $(a^2-b^2+1)/(a^2-b^2+2)$.

IV.

(7.) $2.3^3.7.11^2$; $3^5.5^2.7^2$. **(8.)** 53.

V.

(1.) 120. **(2.)** $x^8-2x^4y^4+y^8$. **(3.)** x^8-y^8. **(4.)** $x^6-3x^4y^2+3x^2y^4-y^6$. **(5.)** $x^8-16x^6y^2+96x^4y^4-256x^2y^6+256y^8$. **(6.)** $b^2c^4-b^4c^2+c^2a^4-c^4a^2+a^2b^4-a^4b^2$. **(7.)** $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$. **(8.)** $27a^3+8b^3-1+54a^2b+36ab^2-12b^2+6b+9a-27a^2-36ab$. **(9.)** $x^4+2x^3+3x^2+4x+5+4/x+3/x^2+2/x^3+1/x^4$.

(10.)
$$+a^4+b^4+c^4+4a^3b+4ab^3+4b^3c+4bc^3+4ca^3+4c^3a+6a^2b^2$$
$$+6b^2c^2+6c^2a^2+12a^2bc+12ab^2c+12abc^2\ ;$$
$$+\quad+\quad+\quad-\quad-\quad+\quad+\quad-\quad-\quad+$$
$$+\quad+\quad+\quad-\quad-.$$

(11.) Four types ; 3 like x^4 ; 6 like x^3y ; 3 like x^2yz ; 3 like x^2y^2. **(12.)**

Three types; 4 like a^3; 12 like a^2b; 4 like abc. **(13.)** $(2+3+4)^3=729$. **(14.)** $x^2/(b^2-c^2)+$&c. $+$&c. $+2(ca-ab)yz/(a^2-b^2)(c^2-a^2)+$&c. $+$&c. **(15.)** $2(x^2+y^2+z^2)$. **(16.)** 0. **(17.)** $ax+by+cz$. **(18.)** $-\Sigma a^6+2\Sigma a^5b+\Sigma a^4b^2-10\Sigma a^4bc-4\Sigma a^3b^3+8\Sigma a^3b^2c-18a^2b^2c^2$. **(22.)** abc. **(23.)** $3abc$. **(24.)** $2(b^3c-bc^3+c^3a-ca^3+a^3b-ab^3)$. **(25.)** $(a^2-b^2)x^4+2ab^2x^3y+(2a^2+2b^2-a^2b^2)x^2y^2-2ab^2xy^3+(a^2-b^2)y^4$.

VI.

(1.) (α) 2nd. (β) Fractional. (γ) 4th. (δ) 2nd. **(2.)** $2x^2+10x+14$. **(3.)** $\frac{5}{6}x^2+\frac{5}{6}x$. **(4.)** $9x^4-78x^2+121$. **(5.)** $x^6-x^4\Sigma a^2+x^2\Sigma a^2b^2-a^2b^2c^2$. **(6.)** $x^6-(p^2+q^2+1)x^4+(p^2q^2+p^2+q^2)x^2-p^2q^2$. **(7.)** $x^{10}-15x^8y^2+85x^6y^4-225x^4y^6+274x^2y^8-120y^{10}$. **(8.)** $abcx^3-(b-c)(c-a)(a-b)x^2y+(\Sigma a^2b-\Sigma a^3-3abc)xy^2+(b-c)(c-a)(a-b)y^3$. **(11.)** $(b^2-c^2)x^4+2c(b-a)x^3+(c-a)(c+a-2b)x^2+2a(c-b)x+(a^2-b^2)$. **(12.)** $x^8-x^6-x^2+1$. **(13.)** $4x^3+10x^2y+8xy^2+3y^3$. **(14.)** $4x^4-x^2y^2+4y^4$. **(15.)** $x^{12}-2x^6+1$. **(16.)** $x^{12}+2x^{10}-x^8-4x^6-x^4+2x^2+1$. **(17.)** $\frac{9}{16}x^6-\frac{1}{4}x^4+\frac{7}{24}x^3-\frac{1}{64}x^2+\frac{1}{81}$. **(18.)** $ax^6-a(a+b)x^5y+b(2a^2+1)x^4y^2-a^2b^2x^3y^3+ax^2y^4-b(a-b)xy^5+by^6$. **(19.)** $4x^6+12a^2x^4+12b^4x^2$. **(20.)** $x^{24}-12a^2x^{22}+66a^4x^{20}-220a^6x^{18}+495a^8x^{16}-792a^{10}x^{14}+924a^{12}x^{12}-$&c. **(21.)** $x^{12}-3x^8a^5+3x^4a^{10}-a^{15}$. **(22.)** $2187x^7+1701x^6+567x^5+105x^4+\frac{35}{3}x^3+\frac{7}{9}x^2+\frac{7}{243}x+\frac{1}{2187}$. **(23.)** $a^8+8a^7bx^2+28a^6b^2x^4+56a^5b^3x^6+70a^4b^4x^8+$&c. **(24.)** $x^{108}-9x^{96}y^6+36x^{84}y^{12}-84x^{72}y^{18}+126x^{60}y^{24}-$&c. **(25.)** $1+3x+6x^2+10x^3+15x^4+18x^5+19x^6+$&c. **(26.)** 266. **(27.)** -1975. **(30.)** 320.

VII.

(1.) $360x^4+1782x^3+3305x^2+2722x+840$. **(2.)** $pqrx^3-(q-r)(r-p)(p-q)x^2+(-\Sigma p^3+\Sigma p^2q-3pqr)x+(q-r)(r-p)(p-q)$. **(3.)** $x^8-30a^2x^6+273a^4x^4-820a^6x^2+576a^8$. **(4.)** $x^6-3x^4+3x^2-1$. **(5.)** $\frac{1}{6}x^6+\frac{11}{60}x^5+\frac{37}{40}x^4+\frac{133}{120}x^3+\frac{23}{40}x^2+\frac{7}{4}x+\frac{2}{3}$. **(6.)** $x^6-\frac{1}{64}$. **(7.)** $x^6+(\Sigma l^2/mn)x^5+2\Sigma(mn/l^2)x^4+(4+l^3/m^3+m^3/n^3+n^3/l^3)x^3+2(\Sigma l^2/mn)x^2+(\Sigma mn/l^2)x+1$. **(8.)** $2048x^{11}-33792x^{10}+253440x^9-1140480x^8+3421440x^7-7185024x^6+10777536x^5-11547360x^4+8660520x^3-4330260x^2+1299078x-177147$. **(9.)** $x^{24}+8x^{21}y^3+28x^{18}y^6+56x^{15}y^9+70x^{12}y^{12}+$&c. **(10.)** $x^{18}+10x^{17}+41x^{16}+80x^{15}+36x^{14}+168x^{13}-364x^{12}-208x^{11}+286x^{10}+572x^9+$&c.

VIII.

(1.) $A+B\Sigma x+C\Sigma x^2+D\Sigma xy+E\Sigma x^3+F\Sigma x^2y+G\Sigma xyz$. **(2.)** $\Sigma x^3y^3+\Sigma x^4yz+3x^2y^2z^2$. Three types present, four missing, viz., x^6, x^5y, x^4y^2, x^3y^2z. **(3.)** $\frac{1}{3}x-\frac{1}{3}y$. **(4.)** $P\{(y''-y')(x-x')-(x''-x')(y-y')\}$, where P is any constant. This may also be written $P\{(y''-y')x-(x''-x')y+x''y'-x'y''\}$. **(5.)** $(y'x-x'y)(y''x-x''y)/(x'-y')(x''-y'')$. **(6.)** $A=-8$, $B=-12$, $C=20$. **(7.)** $l=21$, $m=-76$, $n=60$. **(8.)** $l=6$, $m=-15$, $n=10$. **(9.)** $\Sigma P(x-b)(x-c)(x-d)/(a-b)(a-c)(a-d)$. **(10.)** $b^3c^3+b^4c^2+b^5c$, $b^3c^3+b^2c^4+bc^5$. **(11.)** $3\Sigma x^3-\Sigma x^2y$.

IX.

(1.) $Q=x^3-3x^2+3x-1$, $R=0$. **(2.)** $Q=3x^4+\frac{11}{2}x^3-\frac{11}{4}x^2+\frac{69}{8}x-\frac{67}{16}$, $R=\frac{93}{16}x-\frac{45}{16}$. **(3.)** $Q=4x^3+6x^2+11x+16$, $R=20x-15$. **(4.)** x^2-9. **(5.)** The function $=x^3-2x^2+17x+80-40/(x-7)$. **(6.)** $Q=x^3-5x+3$, $R=0$. **(7.)** Q

$=9x^4+6x^3+x^2+2x+1$, $R=0$. **(8.)** $Q=x^2-8x+15$, $R=0$. **(9.)** $Q=x^3+\frac{1}{2}x^2+\frac{1}{3}x+\frac{1}{4}$, $R=0$. **(10.)** $Q=x^2-\frac{3}{2}x+\frac{11}{5}$, $R=-\frac{193}{70}x-\frac{83}{40}$. **(11.)** $Q=\frac{1}{2}x^6-\frac{1}{4}x^5+\frac{1}{8}x^4-\frac{1}{16}x^3+\frac{1}{32}x^2-\frac{1}{64}x+\frac{1}{128}$, $R=0$. **(12.)** $x-1$. **(13.)** $x^8y^8(x^6+x^5y+x^4y^2+\ldots+y^6)$. **(14.)** $3a^2-2ab+b^2$. **(15.)** $a^6-a^5b+a^4b^2-\ldots+b^6$. **(16.)** $x^2-3xy+y^2$. **(17.)** $x^2-2xy+4y^2$. **(18.)** $x^3+x^2y+3xy^2+3y^3$. **(19.)** $1+x+x^2+\ldots+x^9$. **(20.)** $x^4-x^3-x^2-2x+4$. **(21.)** bx^2+cx-f. **(22.)** $ab+ac-bc$. **(23.)** $1+b+c$. **(24.)** $2(a+b)x$. **(25.)** $a^4-a^3b+3a^2b^2-ab^3+b^4$. **(26.)** $8xy(x^2+y^2)$. **(27.)** $7xy(x+y)$. **(28.)** $Q=6x^3+9x^2+5x+1$, $R=-1$. **(29.)** $(bx+ay)/(bx-ay)=1+2ay/(bx-ay)=-1+2bx/(bx-ay)$. **(30.)** If a be variable, the transformed result is $a^3+2a^2b+5ab^2+10b^3+(15ab^4-13b^5)/(a^2-2ab+b^2)$. **(31.)** $Q=x^3-2x^2+x-4$, $R=12$. **(32.)** $x^3+3x^2-13x-15$. **(33.)** $-303/8$. **(34.)** $2(p+q)$; $p+q=0$. **(35.)** $AQ^2-BPQ+CP^2=0$. **(36.)** $p^2-ap-q+b=0$, $pq-aq+c=0$. **(38.)** $\lambda=1$, $\mu=-3$, $\nu=-2$. **(39.)** $p=2$, $q=3$, $r=3$. **(40.)** The remainder in each case is $rx+s$. **(41.)** $m+1$ must be a multiple of $n+1$. **(42.)** $1-3x+9x^2-27x^3+\ldots+(-3)^nx^n-(-3)^nx^{n+1}/(3x+1)$; $\frac{1}{3x}-\frac{1}{9x^2}+\frac{1}{27x^3}+\ldots-(-1)^n\frac{1}{3^nx^n}+\frac{(-1)^n}{3^nx^n}/(3x+1)$. **(43.)** $\frac{1}{a^2}+\frac{x}{a^3}-\frac{x^3}{a^5}+\left(-\frac{x^4}{a^4}+\frac{x^5}{a^5}\right)/(a^2-ax+x^2)$. **(44.)** $1+2x+5x^2/1.2+16x^3/1.2.3+65x^4/1.2.3.4+\ldots$ **(45.)** $1-ny$; $1+ny$. **(50.)** $x^5-70x^3-377x^2-778x-585$. **(51.)** $P^8-4P^7+2P^6+8P^5-5P^4-8P^3+2P^2+4P+1$, where $P=x+2$; $Q^4+(8x+24)Q^3+(8x-40)Q^2+(-32x+16)Q+16x$, where $Q=x^2+x+1$.

X.

(1.) x^2-1. **(2.)** x^2-x+1. **(3.)** No C.M. **(4.)** $x+1$. **(5.)** x^2+x-6. **(6.)** $x^2-12x+35$. **(7.)** $x^2-16x-15$, use § 7. **(8.)** x^4+x^2-6; compare with Example 5. **(9.)** x^3-3x-2. **(10.)** x^2-1. **(11.)** $4x^2+3x+1$. **(12.)** $x-5$. **(13.)** $4x^2-24x+35$. **(14.)** x^2+2x+1. **(15.)** $x^2+4ax+8a^2$. **(16.)** $3x^2-\sqrt{2}x+1$. **(17.)** $x-1$. **(18.)** $x^2-ax+2a^2$. **(19.)** $(x-1)^2$. **(20.)** The G.C.M. would be a measure of $(p-q)x(x-1)$, neither of the factors of which is in general a measure of either of the given functions. If, however, $p+q+2=0$, then $x-1$ is a measure of both. **(22.)** $a=8$, there is then a factor x^2-4x+3 common to numerator and denominator. **(23.)** Use § 7. **(25.)** Use §§ 6 and 7; the first gives the conditions in the first form, the second gives the single condition. **(26.)** $P=\frac{4}{21}x+\frac{5}{21}$, $Q=-\frac{4}{21}x+\frac{11}{21}$. **(27.)** $P=\frac{4}{21}x+\frac{5}{21}$, $Q=-\frac{4}{21}x+\frac{11}{21}$. **(28.)** $a^8(a-b)(a+b)(a^2+b^2)(a^4+b^4)$. **(29.)** $(x-1)(x-2)(x-3)(x+2)(x+4)$. **(30.)** $(x-1)(x+1)(x+2)(3x-2)(3x+2)$. **(31.)** $(x^2-1)(x-2)^2(x+4)(x+5)(x^2-5)$. **(32.)** The product of the given functions.

XI.

(1.) $2(a-d)(a+b+c+d)$. **(2.)** $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$. **(3.)** $(a^2-3c^2)(a^2-4b^2+c^2)$. **(4.)** $\frac{1}{9}(x+2+\sqrt{2}i)(x+2-\sqrt{2}i)(9x-13+\sqrt{7})(9x-13-\sqrt{7})$. **(5.)** $(\alpha-\beta)(2x-\alpha-\beta)(x-\gamma)^2$. **(6.)** $(x+y)(x-y)\left\{x-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)y\right\}\left\{x-\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)y\right\}\left\{x-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)y\right\}\left\{x-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)y\right\}$. **(7.)**

$(x+y)(x-y)(x+iy)(x-iy)\{x+y(1+i)/\sqrt{2}\}\{x+y(1-i)/\sqrt{2}\}\{x-y(1+i)/\sqrt{2}\}$ $\{x-y(1-i)/\sqrt{2}\}$. **(8.)** $(x+3y+2)(x+3y-2)$. **(9.)** $2(x+2)(x-\frac{1}{2})$. **(10.)** $(x+8)(x-2)$. **(11.)** $(x-5+\sqrt{7})(x-5-\sqrt{7})$. **(12.)** $(x+6)(x-5)$. **(13.)** $(x+7+i\sqrt{7})(x+7-i\sqrt{7})$. **(14.)** $(x+2+i\sqrt{3})(x+2-i\sqrt{3})$. **(15.)** $2(x+4)(x-\frac{3}{2})$. **(16.)** $(x+\sqrt{p+q}+\sqrt{p-q})(x+\sqrt{p+q}-\sqrt{p-q})$. **(17.)** $(x-1)\{x-(b-c)/(b+c)\}$. **(18.)** $(x+p)(x+q)(x-p)(x-q)$. **(19.)** $(ax-by)(bx+ay)$. **(20.)** $\{(1-p)x-(1+p)y\}\{(1-q)x-(1+q)y\}$. **(21.)** $(x-3)(x-5)(x-7)$. **(22.)** $x(x-7+3i\sqrt{11})(x-7-3i\sqrt{11})$. **(23.)** $(x-3)(x-4)(x-6)$. **(24.)** $(x-8)(x+i)(x-i)$. **(25.)** $(x+p)(x+p+q)(x+p-q)$. **(26.)** $(x+1)(x-1)\{(p+q)x+(p-q)\}$. **(27.)** $(x-1)(x-p)(x-p^2)$. **(28.)** $(x-a)(x-b)(x+\sqrt{ab})(x-\sqrt{ab})$. **(29.)** $(x^4-a^4)(x^2+xa+a^2)=$&c. **(30.)** $2(x-y)(1-xy)$. **(31.)** $(x^2+xy+y^2)(x^2-xy+y^2)=$&c. **(32.)** $p=+\sqrt{2}$, $q=-\sqrt{2}$. **(33.)** $(x^2+\sqrt{2}xy+y^2-1)(x^2-\sqrt{2}xy+y^2-1)$. **(34.)** $r=-pa^2$, $s=-qa^2-a^4$. **(35.)** $(x^m+a^m)(x^m-a^m)(x^n+a^n)(x^n-a^n)$. **(36.)** $3a^2x^2(x^2+ax+a^2)(x^2-ax+a^2)=$&c. **(37.)** $(x-1)(y-1)^2$. **(38.)** $(x+3)(2x+y+1)$. **(39.)** $(2x+3y+1)(x-y-1)$. **(40.)** $(x+3)(y+7)$. **(41.)** $(x+2y-z)(x-y+3z)$. **(42.)** Equate the discriminant of the function to zero and thus obtain a cubic equation for λ. **(43.)** When $c=0$, $\lambda=(af^2+bg^2-2fgh)/fg$. **(44.)** $\alpha(\beta'\gamma''-\beta''\gamma')+\beta(\gamma'\alpha''-\gamma''\alpha')+\gamma(\alpha'\beta''-\alpha''\beta')=0$. **(47.)** $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$. **(48.)** $(x+y-a)(x^2+y^2+a^2-xy+ax+ay)$. **(49.)** $3x(x+1)(x-1)^3$. **(50.)** $-(\Sigma x^2+\Sigma yz)(y-z)(z-x)(x-y)$. **(51.)** $(x+y+z)(y-z)(z-x)(x-y)$. **(52.)** $-(y+z)(z+x)(x+y)(y-z)(z-x)(x-y)$. **(53.)** $3(y+z)(z+x)(x+y)$. **(54.)** -1.

(60.) If $$p_2=-\Sigma(y-z)(z-x)=2(\Sigma x^2-\Sigma xy),$$
and $$p_3=(y-z)(z-x)(x-y),\quad s_n=\Sigma(y-z)^n,$$
then it may be shown (see chap. xviii., § 4) that

$$s_{6m+1}=\text{A}p_3p_2^{3m-1}+\text{B}p_3^3p_2^{3(m-1)-1}+\ldots+\text{L}p_3^{2m-1}p_2^2,$$
$$s_{6m-1}=\text{A}p_3p_2^{3m-2}+\text{B}p_3^3p_2^{3(m-1)-2}+\ldots+\text{L}p_3^{2m-1}p_2,$$

where A, B, . . ., L are numerical coefficients. Hence the theorem follows.

XII.

(1.) $(x+3)/(x^2+x-2)$. **(2.)** $(9x^2-x-3)/4(x+5)$. **(3.)** $(2x-3)/(x^2-3x+2)$. **(4.)** $2(x^2+1)/(x^2-1)$. **(5.)** $2x^4+6x^2+2$. **(6.)** $(2x+5a)/(3x+5a)$. **(7.)** $1/(1-x^2)$. **(8.)** $(w+x-y)/(w+z-y)$. **(9.)** $1/(1-x^3)$. **(10.)** $(l^2+m^2)/(p^2+q^2)$. **(11.)** $-s/t$. **(12.)** $y/2(x-y)$. **(13.)** $2(a+bx)/(a^2-b^2)(1-x^2)$. **(14.)** $(a^2-4ab+4b^2-1)/(a^3-6a^2b+12ab^2-8b^3-2a+4b)$. **(15.)** $(6+x)/3(1-x^2)$. **(16.)** 0. **(17.)** $1/(2x^2-1)$. **(18.)** $(240x^2+32x)/(81x^4-16)$. **(19.)** $(3x+2)/x(x+1)^3$. **(20.)** $x^2/(x-1)(x+1)^3(x^2+x+1)$. **(21.)** $1/(x+1)^2$. **(22.)** $4ax(-bx+a^2)/(x^4-a^4)$. **(23.)** $(3x^2+y^2)(x-y)/(x+y)^3$. **(24.)** $(x+2)(x^2-1)/(x^2+x+1)(x^2+x-4)$. **(25.)** $x/8a$. **(26.)** $4x^2y^2/(x^6-y^6)$. **(27.)** -1. **(28.)** $(2cx+c^2+a^2)/(a^2-c^2)(x^2-a^2)(x+c)$. **(29.)** 1. **(30.)** $(x-1)(y-1)/(x+1)(y+1)$. **(31.)** $1+y^2+z^2-y-z-yz$. **(32.)** $(a+b+c)/(a-b-c)$. **(33.)** -3. **(34.)** $1/(x^2-a^2)$. **(35.)** $(a-bx)/(a+bx)$. **(36.)** $2(x^3-1)/\{(x-1)^2-y^2\}$. **(37.)** $(x^2+y^2)^2$. **(38.)** 1. **(39.)** -2. **(40.)** $\Sigma x^3/xyz$. **(41.)** 0. **(42.)** h. **(43.)** 0. **(44.)** $1-\Sigma x^2+2xyz$. **(45.)** 2. **(46.)** 0. **(47.)** 0. **(48.)** 0. **(49.)** $(h-p)(h-q)/\Pi(h+a)$. **(50.)** $h^2/\Pi(h-a)$. **(51.)** $h^2/\Pi(h^2+a^2)$. **(52.)** $-2\Sigma x/xyz$. **(53.)** $1-a-b-c$. **(54.)** 1. **(62.)**

$1-3/(x-2)+8/(x-3)$. **(63.)** $1/2(x-1)-4/(x-2)+9/2(x-3)$. **(64.)** $30x-5/(x+1)-5/(x-1)+80/(x+2)+80/(x-2)$. **(65.)** $17/36(x+1)-5/6(x+1)^2+8/45(x-2)-13/20(x+3)$. **(66.)** $(4x+5)/3(x^2+x+1)-1/3(x-1)$. **(67.)** $1/2(x+1)+(x+1)/2(x^2+1)$. **(68.)** $-1/4(x-1)+(x+1)/4(x^2+1)+(x+5)/2(x^2+1)^2$. **(69.)** $1/(a-b)(a^2-2pa+q)(x-a)-1/(a-b)(b^2-2pb+q)(x-b)+\{(a+b-2p)x+(2p-a)(2p-b)-q\}/(a^2-2pa+q)(b^2-2pb+q)(x^2-2px+q)$. **(70.)** $3/4(x-1)^2-3/8(x-1)+1/8(x+1)+(x-1)/4(x^2+1)$. **(71.)** $2/(x+1)+3/(x+1)^2-(2x-3)/(x^2-2x+3)$. **(72.)** $1/(x-1)-1/2(x+1)-(x+3)/2(x^2+1)$. **(73.)** $-1/x+1/x^2-1/x^3+1/8(x-1)+9/8(x+1)+1/4(x+1)^2-(x+1)/4(x^2+1)$. **(74.)** $(3x^2+x+1)/2(x^4-1)-(3x^2+x+1)/2(x^4+1)$. **(75.)** $1/6+\Sigma\{1/15.3^n+(-1)^n/10.2^n\}x^n$. **(76.)** $1+\frac{1}{2}\Sigma\{(n+2)x^n+(-1)^n x^{2n+1}\}$.

XIII.

(1.) 100242. **(2.)** $22{\cdot}\dot{6}35\dot{4}$. **(3.)** $267{\cdot}3\dot{8}6124\dot{9}$. **(4.)** 2653919. **(5.)** 788001. **(6.)** 20200·1122212 . . . **(7.)** $\cdot 20\dot{4}$. **(8.)** $1+1\times 3+1\times 3.5+3\times 3.5.7+3\times 3.5.7.9$. **(9.)** $1/2!+0/3!+3/4!+1/5!+2/6!+1/7!+2/8!+1/9!+6/10!+4/11!$.* **(10.)** $1/3+0/3.5+1/3^2.5+3/3^2.5^2+1/3^3.5^2+4/3^3.5^3+1/3^4.5^3+\ldots$ **(11.)** 2466411243. **(12.)** 100·1431. **(13.)** 12·74t50. **(14.)** 18·355. **(15.)** $1-\frac{1}{5}+\frac{1}{5.47}-\frac{1}{5.47.50}+\frac{1}{5.47.50.367}-\frac{1}{5.47.50.367.551}+\frac{1}{5.47.50.367.551.1103}$. **(16.)** $1+2+2^5$. **(17.)** $4.1+1.5+1.5^2+1.5^3+1.5^4$; $-1+2.5+1.5^2+1.5^3+1.5^4$. **(18.)** $831'80\frac{5}{8}''$. **(19.)** $300'64''$. **(20.)** 53·617 cubic ft. **(21.)** $11'9\frac{1}{12}''$. **(22.)** $r=7$. **(23.)** $r=4$, &c. **(24.)** 503. **(25.)** $x(x+1)(x+2)(x+3)+1\equiv(x^2+3x+1)^2$. Since $x=10m+p$ where $p=0, 1, 2, \ldots, 9$, we have only 10 different cases to consider. It will be found that the last digit is 5 when $p=1$ or $=6$; in all other cases the last digit is 1. **(28.)** Since $\Sigma p(10^r\sim 10^s)\equiv\Sigma p10^s(10^{r-s}-1)$ is always divisible by $10-1=9$. **(31.)** Since 10^n and all higher powers of 10 are divisible by 2^n, it follows that $p_0+p_1 10+\ldots+p_{n-1}10^{n-1}$ must be divisible by 2^n. **(32.)** $p_0+p_1 10+p_2 10^2=p_0+2p_1+4p_2+8(p_1+12p_2)$ must be divisible by 8, therefore, &c. **(34.)** If the digits in the period of p/n be $q_1, q_2, \ldots, q_s$, those in the period of $1-p/n$ are $9-q_1, 9-q_2, \ldots, 9-q_s$. **(36.)** Since any number may be written $11m+p$ where $p=0, 1, \ldots, 10$, we have merely to show that $0^5, 1^5, 2^5, \ldots, \tau^5$ all end in one or other of the digits 0, 1, τ. We proceed to test thus: 9^2 ends in 4; therefore 9^4 ends in 5; therefore 9^5 ends in 1; and so on. **(38.)** $pqrpqr=(10^3+1)pqr=7.11.13.pqr$. **(39.)** 0^3 ends in 0, 1^3 in 1, 2^3 in 8, 3^3 in 7, 4^3 in 4, 5^3 in 5. Hence the theorem; for every number can be written $10m\pm p$, $p \ngtr 5$. **(40.)** $\mathrm{I}^2=(12m+p)^2$, hence $p=0$ or 6. The latter only is admissible. Hence $\mathrm{I}^2=(12m+6)^2=0+3.12+\mu.12^2$; $\mathrm{I}^3=(12m+6)^3=0+6.12+\nu.12^2$. **(41.)** $\mathrm{N}-p_0-p_1 a-\ldots-p_{n-1}a^{n-1}-(p_m a^m+p_{m+1}a^{m+1}+\ldots)=p_1(r-a)+p_2(r^2-a^2)+\ldots=\mu(r-a)=\mu a^m$; therefore, &c. **(42.)** $\phi(a)$ is simply the result of casting out the nines in the sum of the digits of a, that is, $\phi(a)$ is the remainder when a is divided by 9. Hence, &c. **(43.)** We must have $2xr^2+2yr+2z=zr^2+yr+x$; whence $(2x-z)r+y+(2z-x)/$

* 3 ! stands for 1.2.3, 4 ! for 1.2.3.4, &c.

$r=0$; therefore $(2z-x)/r$ must be an integer. Now $2z-x<2r$, hence either $2z-x=0$ or $=r$. The former can be shown to be inadmissible. The latter leads finally to $z=2t+1$, $y=3t+1$, $x=t$, $r=3t+2$, where t is any positive integer. Hence the theorem.

XIV.

(1.) $18\sqrt[6]{3}$. **(2.)** 1. **(3.)** $7^{1/60}$. **(4.)** $\sqrt[14]{(5^7.3^{31})}$. **(5.)** $(a^{4-3n(n-1)}b^{4+3n(n-1)})^{1/4n(n-1)}$. **(6.)** 1. **(7.)** Each $=x^{2/n(n+1)(n+2)}$. **(8.)** $x^{-2/3}+x^{-1/3}y^{-1/3}+y^{-2/3}$. **(9.)** $x^2+2x+3+2x^{-1}+x^{-2}$. **(10.)** $x^{7/2}-3x^{5/2}+5x^{3/2}-7x^{1/2}+7x^{-1/2}-5x^{-3/2}+3x^{-5/2}-x^{-7/2}$. **(11.)** x^2+x+1. **(12.)** $x^{7/10}-2x^{1/5}y^{1/2}+3x^{9/10}y^{1/5}+x^{-3/10}y-3x^{2/5}y^{7/10}$. **(13.)** $x^{2/3}-2x^{1/3}y^{1/2}+2y$. **(14.)** $2(2^{1/3}+2^{2/3})$. **(15.)** $x^{4/n}+x^{7/2n}y^{1/2n}-2x^{3/n}y^{1/n}-3x^{5/2n}y^{3/2n}+3x^{3/2n}y^{5/2n}+2x^{1/n}y^{3/n}-x^{1/2n}y^{7/2n}-y^{4/n}$. **(20.)** $(a+b)/(a-b)$. **(21.)** $39\frac{1}{2}-6\sqrt{3}-12\sqrt{10}+2\sqrt{30}$. **(22.)** $x^3+(3-\sqrt{2}-\sqrt{3})x^2+(-2-2\sqrt{2}-2\sqrt{3}+2\sqrt{6})x+(-4-2\sqrt{2}+2\sqrt{6})$. **(23.)** $(x-a)/4ax$. **(24.)** $(13m^2+10m+13+12(m-1)\sqrt{m^2+7m+1})/(-5m^2+46m-5)$. **(25.)** $1/q$. **(26.)** 0. **(27.)** $(q\sqrt{p}+q\sqrt{q}+q\sqrt{p-q}-\sqrt{pq(p-q)})/q(p-q)$. **(28.)** $2\{(1-x)/(1-4x)\}\sqrt{(1-4x)}$. **(29.)** $\{2(a+c)^2-b^2+2(a+c)\sqrt{(a+c)^2-b^2}\}/b^2$. **(30.)** $\{-q(p-q)+(p-q)\sqrt{pq}+p\sqrt{q(p-q)}-q\sqrt{p(p-q)}\}/q(q-p)$. **(31.)** $2^{2/3}x^2-(2^{1/3}+2)x+(2^{4/3}-2^{2/3}+1)$. **(32.)** $x^2-(2^{1/3}-1)x+(2^{2/3}+2^{1/3}+1)$. **(33.)** $2b-2\sqrt[3]{\{(a+b)^2(a-b)\}}+2\sqrt[3]{\{(a-b)^2(a+b)\}}$. **(37.)** $(x-2x^{3/4}y^{1/4}+2x^{1/2}y^{1/2}-2x^{1/4}y^{3/4}+y)/(x-y)$. **(38.)** $-\frac{1}{4}\{11+6\sqrt{3}+5\sqrt{5}+4\sqrt{7}+4\sqrt{15}+3\sqrt{21}+2\sqrt{35}+\sqrt{105}\}$. **(39.)** $2\Sigma a(\Sigma a+2\Sigma\sqrt{bc})/(\Sigma a^2-2\Sigma bc)$. **(40.)** The rationalised product is $3^{15}5^5-4^3$. **(41.)** A rationalising factor is $\Sigma(3a-b-c)\sqrt{(b+c-a)}-2\sqrt{\{\Pi(b+c-a)\}}$; the result is $-5\Sigma a^2+6\Sigma ab$. **(42.)** A rationalising factor is $(2\sqrt{5}-3\sqrt{2})(\sqrt{5}+2-\sqrt{3}+\sqrt{6})$; the result 4. **(43.)** A rationalising factor is $19.2^{2/3}+22.2^{1/3}-23$; the result 307. **(44.)** The result $\Sigma a^3+3\Sigma a^2b-21abc$. **(45.)** A rationalising factor is $2^{3/4}+2.2^{2/4}-3.2^{1/4}+1$; the result 7.

XV.

(1.) $\frac{1}{66}(11\sqrt{3}+3\sqrt{11}-\sqrt{462})$. **(2.)** 2. **(3.)** $-2\sqrt{2}$. **(4.)** $12+5\sqrt{3}$. **(5.)** $\frac{1}{6}\sqrt{6}$. **(6.)** $14\frac{1}{2}$. **(7.)** $\pm(\sqrt{10}+\sqrt{15})$. **(8.)** $\pm(1+1/\sqrt{2})$. **(9.)** $\pm(11-\sqrt{2})$. **(10.)** $2^{3/4}(3+13^{1/2})$. **(11.)** $\pm i(\sqrt{6}+\sqrt{15})/3$. **(12.)** $2-\sqrt{2}+2\sqrt{3}$. **(13.)** $4\sqrt{2}-2\sqrt{7}$. **(14.)** 3. **(15.)** $(3+6\sqrt{3}-\sqrt{7}-2\sqrt{21})/2\sqrt[4]{3}$. **(16.)** $\frac{1}{3}\sqrt{6}=\cdot81649$. **(17.)** $3\cdot2518293$. **(18.)** $\sqrt{(6+2p)}$. **(19.)** $\sqrt{(a-c)}+\sqrt{b}$. **(20.)** $\{a+\sqrt{(a^2-4)}\}/\sqrt{2}$. **(21.)** $x/(1-x^2)$. **(23.)** $1+\sqrt{(3/2)}+\sqrt{(5/2)}$. **(24.)** $2-\sqrt{3}-3\sqrt{2}$. **(27.)** $3-2\sqrt{2}$. **(28.)** $5+\sqrt{18}$. **(29.)** $2^{1/6}(3+10^{1/2})$. **(33.)** $3\cdot162277660$; $\cdot0632455532$. **(34.)** $\pm(yz-zx+xy)$. **(35.)** $\pm(5x-3y-z)$. **(36.)** $\pm(3x^2+4x-1)$. **(37.)** $\pm(x^2-2x-1)$. **(38.)** $\pm(2x^2-3xy+4y^2)$. **(39.)** $\pm(x^3-3x+2)$. **(40.)** $\pm(2x^3-3x^2-x+4)$. **(41.)** $\pm(5p^3+3p^2q-3pq^2-5q^3)$. **(42.)** $\pm(x-\sqrt{x}+1)$. **(43.)** $(2x^3-x^2-3)$. **(44.)** $\{(3\pm\sqrt{3})p+(3\mp\sqrt{3})q\}/\sqrt[3]{3}$. **(45.)** $\lambda=1$; the square root is $\pm(x^2+3x-1)$. **(46.)** -6, 92, 105; or 38, -92, 137. **(47.)** 3, 4, 12; or 27, 108, 108. **(49.)** $7x^2-2x+1$. **(50.)** The cube root is x^2+dx+k; $e=3k+3d^2$, $f=6dk+d^3$, $g=3k^2+3kd^2$, $h=3k^2d$.

(**51.**) It is $\{b^2-b(c+a)\}^3$. (**52.**) $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4 \ldots$ (**53.**) $1-\frac{1}{4}x-\frac{3}{32}x^2-\frac{7}{128}x^3-\frac{77}{2048}x^4-\ldots$ (**54.**) $\sqrt{x}\{1+1/2x-1/8x^2+1/16x^3-5/128x^4+7/256x^5 \ldots\}$.

XVI.

(**1.**) $2(a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8)$. (**2.**) $6/5$. (**3.**) 8. (**4.**) $\{8pq(p^2-q^2)/(p^2+q^2)^2\}i$. (**5.**) $2+(\sqrt{3}-4\sqrt{5})i$. (**7.**) $x^4-6x^3+18x^2-26x+21$. (**14.**) $\frac{5}{34}$. (**15.**) $x^2+\sqrt{2}x\sqrt{\{\sqrt{(x^4+y^4)}+x^2\}}+\sqrt{(x^4+y^4)}$. (**16.**) $\sqrt{(\Sigma a^4b^2+2abc\Sigma a^2b)}$. (**18.**) $(x^2+y^2)^{n/2}$; 1. (**19.**) $\pm(3+4i)$. (**20.**) $\pm(\sqrt{13}+i)/\sqrt{2}$. (**21.**) $\pm\frac{1}{6}(3+4i)$. (**22.**) $\pm\{(a+b)+(a-b)i\}$. (**23.**) $\pm\{x+\sqrt{(x^2-1)}i\}$. (**24.**) $\pm[\sqrt{\{(x^2+1)/2\}}+i\sqrt{\{(x^2-1)/2\}}]$. (**25.**) $\pm(3+2i)$, $\pm(2-3i)$. (**26.**) $(x+a)(x-a)(x-\omega a)(x-\omega' a)(x-\omega^2 a)(x-\omega'^3 a)$, where $\omega=(-1+\sqrt{3}i)/2$, $\omega'=(1+\sqrt{3}i)/2$. (**27.**) $(x+1)(x^2-x(\sqrt{5}+1)/2+1)(x^2+x(\sqrt{5}-1)/2+1)$. (**28.**) $\{x^2-2x\cos. 2\pi/7+1\}\{x^2-2x\cos. 4\pi/7+1\}\{x^2-2x\cos. 6\pi/7+1\}$.

(**29.**)
$$\prod_{k=0}^{k=m-1}\left[x^2-2ax\cos\frac{\theta+2k\pi}{m}+a^2\right].$$

XVII.

(**1.**) $151/208 > 5/7$. (**3.**) $(ad-bc)/(c-d)$. (**4.**) $10\frac{12}{23}$; $7+5\sqrt{2}$. (**5.**) $14{\cdot}456\ldots$, $13{\cdot}198\ldots$, $15{\cdot}835\ldots$ (**6.**) $\sqrt{7}+\sqrt{5}$. (**9.**) $(ad-bc)/(b+c-a-d)$; -1; 0.

XVIII.

(**1.**) $145/416$. (**2.**) The real values of x are ± 4. (**3.**) 0, 6. (**4.**) $25xy=12(x^2+y^2)$. (**10.**) $19:16$. (**11.**) $29\frac{17}{23}$ min. past 10. (**13.**) $r>2h$. (**14.**) ·01875 in. (**15.**) 6·373 ft.

XIX.

(**1.**) -2, 5. (**2.**) $a+2b$. (**3.**) $a+b$. (**4.**) 1, 1. (**5.**) a, b, c. (**6.**) $a=5$, $b=-17$. (**7.**) $11x^2-87x+160=0$. (**8.**) $x^2+1=0$. (**9.**) $x^2+ac=0$. (**10.**) $(x-a)\{x^2+(a-b)x-ab+a^2\}=0$. (**11.**) $x(x^2-107)=0$. (**12.**) $x(x+2p-r)=0$. (**13.**) $3x-(a+b+c)=0$. (**14.**) $x^3-\frac{1}{2}(s+t)\{s+t-s^2+st-t^2\}+2st+\mathrm{U}s-\mathrm{U}t=0$. (**15.**) $(\overline{b+d}-\overline{b'+d'})x+c(b-b')+a(d-d')=0$. (**16.**) $\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2-2\mathrm{YZ}-2\mathrm{ZX}-2\mathrm{XY}=0$. (**17.**) $3x^2+2(a+b+c)x-(a^2+b^2+c^2-2ab-2bc-2ca)=0$. (**18.**) $3x^2+28x=0$. (**19.**) $4cm(1-m)x+(1-m)^2y^2=4mc^2$. (**20.**) $x^2(2x^2-4ax+3a^2)=0$. (**21.**) $x-16=0$. (**22.**) $49x-1936=0$. (**23.**) $x^4-3x^2=0$. (**24.**) $5x^4-4bx^3-2a^2x^2-4a^2bx+a^4+4a^2b^2=0$. (**25.**) $x^6=0$. (**26.**) $\Sigma\{x^2(y-z)^4-2yz(z-x)^2(x-y)^2\}=0$. (**27.**) $x^2+y^2+z^2-xy-yz-zx=0$. (**28.**) $x(x^2-1)=0$. (**29.**) $625x^2-24641x+234256=0$. (**30.**) $c^n(x+a)^{n+1}=a^nx^{n+1}$. (**31.**) $27b(a^2-x)=(b-2a)^3$. (**32.**) $xy(x+y)(x^2+xy+y^2)=0$.

XXI.

(**1.**) $z=y^5-5y^3+5y$. (**7.**) $(ab'+a'b)^2=(ac'\mp a'c)^2+(bc'\pm b'c)^2$.

XXII.

(7.) 1·3543, 6·6457. **(8.)** 2·0508. **(11.)** $-6h^2$. **(13.)** +·55826 . . ., −·35826 . . ., and 1·5 exactly. **(14.)** One between −2 and −3, namely, −2·8025 ; the rest imaginary. **(25.)** 1·259921049894873. **(26.)** 2·094551481542326. **(27.)** 9·96666679. **(28.)** 46·7616301847, ·3471623192. **(29.)** 18·64482373095. **(30.)** 123. **(31.)** 4·5195507. **(32.)** 1·475773161. **(33.)** 4·581400362. **(34.)** 2·0520421768796. **(35.)** −1·4142135623730950488.

XXIII.

(1.) $\frac{457}{102}$. **(2.)** 21. **(3.)** $-\frac{317}{426}$. **(4.)** 2. **(5.)** −1·455. **(6.)** $1/(a+b)$. **(7.)** ac/b. **(8.)** b. **(9.)** 0. **(10.)** $\{(a^2+b^2)(a^2-b^2+ab)-a^3+b^3\}/(a^3+b^3)$. **(11.)** $\pm b$. **(12.)** $-2, \frac{5}{8}$. **(13.)** $0, -\frac{5}{2}$. **(14.)** $\frac{7}{2}$. **(15.)** $-\frac{7}{8}$. **(16.)** $-\frac{7}{4}$. **(17.)** 4. **(18.)** $-\frac{1}{2}$. **(19.)** $\frac{7}{4}$. **(20.)** $ab/(a+b)$.

(21.) $$\frac{4abfd-(a^2-b^2)(cf-dc)}{(a^2-b^2)\{(c+f)(a+b)-(c+d)(a-b)\}-4ab\{d(a+b)+f(a-b)\}}.$$

(22.) $(a^2+b^2)/(a+b)$. **(23.)** $a+b$. **(24.)** $2(a-b)(b-c)/(c-a)$. **(25.)** $0, \pm\sqrt{(-ab-bc-ca)}$.

XXIV.

(1.) 6, 9. **(2.)** $\frac{63}{13}, \frac{36}{13}$. **(3.)** 6·60485, 3·68993. **(4.)** 16, 4. **(5.)** $\frac{2}{7}, \frac{3}{7}$. **(6.)** $-\frac{165}{59}, -\frac{127}{59}$. **(7.)** $2b-a, 2a-b$. **(8.)** $2lm(m-3l)a/(2m^2-3l^2), 3lm(2m-l)b/(2m^2-3l^2)$. **(9.)** $-2bc/(a^2+b^2), 2ac/(a^2+b^2)$. **(10.)** $ac/(a^2+b^2), bc/(a^2+b^2)$. **(11.)** $a-b, 2ab/(a-b)$. **(12.)** $a^4-b^4, a^4+a^2b^2+b^4$. **(13.)** $-pq, p+q$. **(14.)** $\lambda=a-b-c, \mu=-ab+bc-ca$. **(17.)** $abc(b-c)(c-a)(a-b)=0$. **(18.)** $\frac{1}{5}x+\frac{48}{5}$. **(19.)** x^2+x-6. **(20.)** $-\frac{1}{12}x^2+\frac{11}{12}x$.

XXV.

(1.) 60, 40, 36. **(2.)** 2, 3, 4. **(3.)** ·7293, ·8039, −·0269. **(4.)** 12, 8, 6. **(5.)** $A=\frac{1}{3}, B=\frac{2}{3}, C=\frac{1}{3}$. **(6.)** $\{(y''-y')x-(x''-x')y-x'y''+x''y'\}/\{(y''-y')x'''-(x''-x')y'''-x'y''+x''y'\}$. **(7.)** $-\frac{1}{20}x^2+\frac{1}{4}x-\frac{3}{10}$. **(8.)** $x=\frac{1}{2}(b+c-a)$, &c. **(9.)** $x=2bc(c^2-a^2)(a+b)/(bc^2+ca^2+ab^2+abc)$, &c. **(10.)** $\Sigma A(x-b)(x-c)/(a-b)(a-c)$. **(11.)** a, b, c. **(12.)** $x=(a+\alpha)(a+\beta)(a+\gamma)/(a-b)(a-c)$, &c. **(13.)** $x=a(a+b+c)/(a-b)(a-c)$, &c. **(14.)** $x=y=z=\Sigma a^2-\Sigma bc$. **(15.)** $x=(m^3+n^3-2lmn-lm^2+l^2m-ln^2+l^2n)/2(\Sigma l^2-\Sigma mn)=(m+n)/2$, &c. **(16.)** $x=(p^2+mn)/(l-m)(l-n)$, &c. **(17.)** $x:y:z=b+c-a:c+a-b:a+b-c$. **(18.)** Put $x+y+z=\rho$, then $x=\rho a/(a+b+c)-(g-h)/(a+b+c)$. **(19.)** $-\frac{112}{61}, -\frac{24}{61}, \frac{88}{61}, \frac{96}{61}$. **(20.)** Express p, q, r in terms of x, y, z, s, then eliminate p, q, r, and there results a system of three equations in x, y, z ; $z=g(ad+bc)/\{bce+bde-bcf-bdf-bef+cde+cef-cdf-abc-abd-abe-acd-ade-adf-acf-aef\}$. **(21.)** 1, 3, 5, 9, 11. **(22.)** By means of the first four equations express all the variables in terms of z ; the last equation then gives

$$z=(afh+bcg-bcf-cfh-bfh-bch)\div(adfh+bceg+bcfh-bfdh-bcch).$$

(**24.**) $x:y:z:u=1/(1+a):1/(1+b):1/(1+c):1/(1+d)$. (**25.**) The required equation is $(B''C')x+(A'C'')y+(A''B')z=A(B''C)+A'(BC'')+A''(B'C)$, where $(B''C')=B''C'-B'C''$, &c.

XXVI.

(**1.**) $b(a+4b+4\sqrt{ab})/a^2$. (**2.**) $m^2n^2/(m-n)^2$. (**3.**) 14. (**4.**) $\frac{27}{7}$. (**5.**) $\frac{13}{6}$. (**6.**) $\alpha-\beta$. (**7.**) $x=\pm\sqrt{\{\frac{4}{3}(\Sigma p^2-\Sigma pq)\}}$. This solution is extraneous if all the radicals be taken positively. (**8.**) p, $2p+2$. (**9.**) $(5a^2-b^2)/4a$. (**10.**) $a+b$. (**11.**) 0, $\frac{25}{12}$. (**12.**) $\frac{111}{25}$, 11. (**13.**) $ab/(a+b)$, $(a+b)/4$. (**14.**) $\frac{25}{36}$, $\frac{49}{144}$.

(**15.**) $\begin{cases} x=a-b, & a+b, & a\pm\sqrt{a(a-2b)}; \\ y=\phantom{a-{}}0, & 2b, & b\mp\sqrt{a(a-2b)}. \end{cases}$

(**16.**) 9, 6. (**17.**) $\begin{cases} x=a\sqrt[3]{\{d/(a^3-b^3)\}}, & \omega \text{ \&c.}, & \omega^2 \text{ \&c.}; \\ y=b\sqrt[3]{\{d/(a^3-b^3)\}}, & \omega \text{ \&c.}, & \omega^2 \text{ \&c.} \end{cases}$

(**18.**) $x=abcd$, $y=-\Sigma abc$, $z=\Sigma ab$, $u=-\Sigma a$. (**19.**) $x=-(b^2-c^2)(a+b)(a+c)$, &c. (**20.**) $x=a/mn$, &c. (**21.**) $x=\frac{1}{2}\Sigma a(\Sigma a+b)$, &c. (**22.**) $-a$, $-b$, $-c$. (**23.**) $x=(b-c)(b^2-c^2)/(2\Sigma a^3-\Sigma a^2b)$, &c. (**24.**) In order that the system be consistent we must have $1/k=1/(k-a)+1/(k-b)+1/(k-c)+1/(k-d)$; then $x:y:z:u=1/(k-a):1/(k-b):1/(k-c):1/(k-d)$. (**25.**) $x=\frac{1}{3}(a+c+d-2b)$, &c. (**26.**) 6, 8. (**27.**) 0, 0; and $\frac{97}{93}$, $-\frac{97}{61}$. (**28.**) $\frac{1}{63}$, 18. (**29.**) $x=\pm\sqrt{\{\frac{1}{2}(\lambda^4+\mu^4)(1/\lambda^2+1/\mu^2)\}}$ $y=\pm\sqrt{\{\frac{1}{2}(\lambda^4+\mu^4)(1/\lambda^2-1/\mu^2)\}}$. (**30.**) a, b; and $a-a(a-b)/c(c-b)$, $b-b(b-a)/c(c-a)$. (**31.**) $-b-c$, $-a-c$. (**32.**) $x=\pm\sqrt{2a}$, &c. (**33.**) $x=\pm\sqrt{2bc/a}$, &c. (**34.**) $-a$, $-b$, $-c$.

XXVIII.

(**1.**) 0, -1. (**2.**) $\frac{1}{2}$, $\frac{2}{3}$. (**3.**) 0, 0. (**4.**) 1, -2. (**5.**) $\frac{1}{2}(3\pm i)$. (**6.**) $-1\pm\sqrt{6}$. (**7.**) $\{-p\alpha+q\beta\pm\sqrt{pq}(\beta-\alpha)\}/(p-q)$. (**8.**) $\{-2pq\pm(p^2-q^2)i\}/(p^2+q^2)$. (**9.**) $\frac{1}{2}(-1\pm\sqrt{17})$. (**10.**) -4, -7. (**11.**) $\frac{14}{17}$, $\frac{13}{15}$. (**12.**) $\frac{1}{2}(10\pm\sqrt{7}i)$. (**13.**) $11\pm 7i$. (**14.**) 200, 1. (**15.**) -53, -49. (**16.**) 53, -49. (**17.**) $-3\sqrt{7}\pm 2\sqrt{2}$. (**18.**) $1+\sqrt{2}\pm\sqrt{3}$. (**19.**) $-11-5i$, $-12-7i$. (**20.**) $7+4i$, $1-6i$. (**21.**) $2\pm\frac{1}{3}\sqrt{3}$. (**22.**) $(1\pm 8i)/13$. (**23.**) $\frac{3}{4}$, $\frac{5}{8}$. (**24.**) 0, $a+b$. (**25.**) $-2a\pm(b+c)$. (**26.**) $a+c$, $-a-b$. (**27.**) $\sqrt{(m/n)}$, $\sqrt{(n/m)}$. (**28.**) $(a+b)/ab$, $-2/(a+b)$. (**29.**) a, $b(2a+b)/(a-b)$. (**30.**) 1, $(b+c-2a)/(c+a-2b)$. (**31.**) c, $-c$. (**32.**) $\Sigma a^2\pm\Sigma ab$. (**33.**) $(a+b+c)/3$. (**34.**) $\frac{1}{2}(1+\sqrt{29})$.

XXIX.

(**1.**) ± 1, $\pm\sqrt{\frac{5}{8}}i$. (**2.**) $\pm\sqrt{(\Sigma a^2-\Sigma ab)}$, Σa. (**3.**) $-2\pm 3i$, 1, 3.

(**4.**) $$\tfrac{1}{2}[-(a-2)\pm a\sqrt{2}+\sqrt{\{4-4a-a^2\mp 2\sqrt{2}a(a-2)\}}],$$ $$\tfrac{1}{2}[-(a-2)\pm a\sqrt{2}-\sqrt{\{4-4a-a^2\mp 2\sqrt{2}a(a-2)\}}].$$

(**5.**) 2, $\frac{1}{4}(-3\pm i\sqrt{23})$. (**6.**) -1, $(\sqrt{a}\pm i\sqrt{(3a+4)})/2\sqrt{a}$. (**7.**) $-\omega$, $-\omega^2$, 1, 1. (**8.**) ω/p, ω^2/p, p^2, $-p^2$.

(**9.**) $$\tfrac{1}{2}\{3\pm\sqrt{(\sqrt{41}+4)}\}+\tfrac{1}{2}\{-1\pm\sqrt{(\sqrt{41}-4)}\}i,$$ $$\tfrac{1}{2}\{3\pm\sqrt{(\sqrt{41}+4)}\}-\tfrac{1}{2}\{-1\pm\sqrt{(\sqrt{41}-4)}\}i.$$

(**10.**) 2, 3, 1, -1. (**11.**) $-\frac{24}{11}$, 0, $-\frac{3}{2}$.

XXX.

(1.) $1/a+1/b,\ -2/a$. **(2.)** $\pm\sqrt{(a^2-ab+b^2)}$. **(3.)** $(1\pm\sqrt{19})/2$. **(4.)** 3. **(5.)** $3,\ -\frac{1}{3}$. **(6.)** $(bd-2bc)/(2ca-cd)$. **(7.)** $0,\ \{c(a^2+b^2)-(a^3+b^3)\}/\{c(a^2+b^2)-c^2(a+b)\}$. **(8.)** $\pm\sqrt{\{(2a^2b^2-c^2a^2-b^2c^2)/(a^2+b^2-2c^2)\}},\ 0$. **(9.)** $(a^2+b^2)/(a+b),\ a+b,\ 0$. **(10.)** $-2(a+b)+2c$. **(11.)** $0,\ 0$. **(12.)** $\{ab(c+d)-cd(a+b)\}/(ab-cd)$. **(13.)** $\pm\sqrt{2ai},\ 0$. **(14.)** 2. **(15.)** $5,\ -3\frac{1}{2}$. **(16.)** $\pm\frac{3}{2}a^2/b$. **(17.)** $\frac{1}{3}\Sigma a$.

XXXI.*

(1.) $4,\ \frac{1}{2}(-3\pm\sqrt{7}i)$. **(2.)** $\frac{1}{3}(-6+4\sqrt{3})$. **(3.)** $\frac{1}{2}(1-\sqrt{5})a$. **(4.)** $c,\ (2a^2b+b^2c-a^2c)/(a^2+2bc-b^2)$. **(5.)** $[\frac{1}{6}(5\pm\sqrt{52}),\ -\frac{4}{3}],3$. **(6.)** $(a^2-2a+2)^2/4(a-1)^2$. **(7.)** $0, 1, [\frac{1}{2}(-3\pm\sqrt{7}i)]$. **(8.)** $[\frac{24}{7}], 0$. **(9.)** $[ab/(a+b)], 0$. **(10.)** $(3\pm\sqrt{22})/2$. **(11.)** $-3\pm\sqrt{\frac{25}{3}+\frac{2}{3}\sqrt{37}}$. **(12.)** $4(a+b),\ [8(a+b)(2a+b)(a+2b)/(a-b)^2]$. **(13.)** $[(a^3+b^3)/ab], (a+b)(a^2+3ab+b^2)/ab$. **(14.)** $[-(9a^2+14ab+9b^2)/8(a+b)]$; if $b=a$, $x=-2a$, which does not satisfy the equation. **(15.)** $5,\ [-\frac{13}{3}]$. **(16.)** $+(a^2-b^2)/2\sqrt{2(a^2+b^2)}$. **(17.)** $-\frac{23}{24},\ -1$. **(18.)** $+2a/\sqrt{3}$. **(19.)** $0,\ \infty$. **(20.)** 42, 15. **(21.)** Reduces to $x=0$, along with a reciprocal biquadratic whose roots a, a, $(9\pm4\sqrt{5})a$ are all extraneous. **(22.)** $4a(m-2)/(m^2-4m+8)$. **(23.)** $0,\ [\pm4\sqrt{5a/q}]$. **(24.)** $a[-(m+n)/2(m-n)\pm\frac{1}{2}\sqrt{\{1-4mn/(m^2+n^2)\}}]$. **(25.)** $(\Sigma ab)^2/4abc$. **(26.)** $\pm(a^2-n^2b^2)/2\sqrt{\{n(n-1)(a^2-nb^2)\}}$. **(27.)** $-\sqrt{(ab)},\ [+\sqrt{(ab)},\ 0]$.

XXXII.

(1.) $b,\ c$, &c. **(2.)** $\frac{2}{3}\log_e\frac{1}{2}\{p\pm\sqrt{(p^2-4q)}\}$. **(3.)** $\{a+b)/(a-b)\}^{2pq/(p-q)}$, $\{(a-b)/(a+b)\}^{2pq/(p-q)}$, &c. **(4.)** $2,\ -1$. **(5.)** $2+\frac{1}{2}\log\frac{13}{7}/\log 3$. **(6.)** $-\lambda+\frac{1}{2}\{\mu\pm\sqrt{(\mu^2-4)}\}$. **(7.)** $\pm a,\ \pm\sqrt{(a^2+2)}i$. **(8.)** $2,\ 1/2,\ -3,\ -1/3$. **(9.)** $\frac{2}{3},\ \frac{3}{2},\ \frac{1}{2}(3\pm\sqrt{5})$. **(10.)** $\frac{1}{2}(3\pm\sqrt{5}),\ \frac{1}{4}(1\pm\sqrt{15}i)$. **(11.)** $4,\ -1/4,\ 2,\ -1/2$. **(12.)** $\{a-b\pm\sqrt{(b^2-2ab-3a^2)}\}/2a,\ -1$. **(13.)** $\{-(a+b)\pm\sqrt{(b^2+2ab-3a^2)}\}/2a,\ 1$. **(14.)** $\pm\sqrt{\{(-b\pm\sqrt{b^2-4ac})/2a\}}$. **(15.)** $\{-b\pm\sqrt{(b^2-4a^2)}\}/2a,\ \pm1$. **(16.)** $\{-b\pm\sqrt{(b^2\pm4ac)}\}/2a$ (4 solutions). **(17.)** $\frac{1}{4}\{1\pm\sqrt{5}+\sqrt{(10\mp2\sqrt{5})}i\}$, $\frac{1}{4}\{1\pm\sqrt{5}-\sqrt{(10\mp2\sqrt{5})}i\},\ -1$. **(18.)** $\frac{1}{2}(-3\pm\sqrt{5}),\ \frac{1}{2}(-5\pm\sqrt{21}),\ 1$. **(19.)** $\frac{1}{48}\{-13\pm\sqrt{73}+\sqrt{(-2062\mp26\sqrt{73})}\}$, $\frac{1}{48}\{$&c. $-$ &c.$\}$. **(21.)** $\frac{1}{2}(-5\pm\sqrt{33})$, $\frac{1}{2}(-5\pm\sqrt{29})$. **(22.)** The equation is equivalent to $x^2+(p-q)x+pq=\pm i\sqrt{\{pq(p^2+q^2)\}}$. **(23.)** $\pm\sqrt{\{\frac{1}{10}(221\pm\sqrt{48241})\}}$. **(24.)** $-\frac{10}{3},\ 3,\ \frac{1}{6}(-1\pm\sqrt{251}i)$. **(25.)** Reduces to a reciprocal biquadratic, the roots of which are extraneous. **(26.)** $0,\ [\pm\sqrt{24}]$. **(27.)** -4. **(28.)** Put $\xi=\{(x-a)/(x+a)\}^{\frac{1}{2}}$; the equation then becomes a reciprocal cubic. **(29.)** $-2\pm\frac{1}{2}\{\sqrt{(\sqrt{45}+4)}\pm\sqrt{(\sqrt{45}-4)}i\}$. **(30.)** $\frac{1}{2}(-7\pm\sqrt{77}),\ [\frac{1}{2}(-7\pm\sqrt{53})]$. **(31.)** $16,\ [\frac{6493}{324}]$. **(32.)** $\frac{1}{2}\{-p\pm\sqrt{(p^2+4q)}\}$, where $q=\frac{1}{3}\{-\Sigma a\pm2\sqrt{(\Sigma a^2-\Sigma ab)}\}$. **(34.)** Reduces to a reciprocal biquadratic, all the roots of which are extraneous except $\frac{1}{2}\sqrt{(2+2\sqrt{2})}$. **(35.)** $\pm1,\ 2\pm\sqrt{3},\ \frac{1}{4}(1\pm\sqrt{15}i)$. **(36.)** $\frac{1}{2}\{a+b\pm\sqrt{3}(a-b)i\},\ a,\ b,\ \infty$. **(37.)** $\pm2/\sqrt{(20\sqrt{6}-45)},\ \pm2i/\sqrt{(20\sqrt{6}+45)}$. **(38.)** $-\frac{21}{2}\pm\frac{1}{2}\sqrt{229},\ -\frac{21}{2}\pm\frac{1}{2}\sqrt{21}$.

---

* When *extraneous* solutions are given at all, they are in most cases distinguished by enclosing them in square brackets, thus $[-\frac{13}{3}]$.

XXXIII.

(1.) $12, 18$; $18, 12$. **(2.)** $7, -4$; $4, -7$. **(3.)** $7, 3, -7, -3$; $3, 7, -3, -7$. **(4.)** $\frac{1}{3}, 5$; ${\tfrac{2}{3}}^{3}, 3$. **(5.)** $0, 3, \frac{1}{2}(-1 \pm \sqrt{7}i)$; $0, 3, \frac{1}{2}(-1 \mp \sqrt{7}i)$.

(6.) $1,\ 2,\ \frac{1}{6}(-11 \pm \sqrt{209})$; $2,\ 1,\ \frac{1}{6}(-11 \mp \sqrt{209})$.

(7.) $\frac{1}{2}\{b \pm a^{\frac{1}{2}}(2b-a)^{\frac{1}{2}}\}$; $\frac{1}{2}\{b \mp a^{\frac{1}{2}}(2b-a)^{\frac{1}{2}}\}$.

(8.) $0,\ (bq-ap)(q^2-p^2)/\{(a^2+b^2)(p^2+q^2)-4abpq\}$;
$0,\ (bp-aq)(q^2-p^2)/\{(a^2+b^2)(p^2+q^2)-4abpq\}$.

(9.) $\{1+ab \pm \sqrt{(a^2-1)(b^2-1)}\}/(a+b)$;
$\{1-ab \pm \sqrt{(a^2-1)(b^2-1)}\}/(a-b)$; two solutions.

(10.) $\{c \pm \sqrt{c(cd-4ab)/d}\}/2a$;
$\{c \mp \qquad \&c. \qquad \}/2b$.

(12.) $x = \pm\sqrt{\{(325 \pm 3\sqrt{11721})/68\}}$, $y/x = (-107 \pm \sqrt{11721})/2$.

(13.) $7, -7$; $5, -5$. **(14.)** $5, -3$; $-3, 5$. **(15.)** $\pm a\sqrt{(a^2+b^2)}/b$; $\pm b\sqrt{(a^2+b^2)}/a$; two solutions.

(16.) $\frac{1}{2}(\sqrt{5} \pm 1)p$; $\frac{1}{2}(\sqrt{5} \mp 1)q$. **(17.)** $\pm\frac{1}{3}\sqrt{3}(2b-a)$, $\pm ai$; $\pm\frac{1}{3}\sqrt{3}(2a-b)$, $\mp bi$.

(18.) $0,\ \frac{1}{2}ab(b\omega + a\omega^2)$;
$0,\ \frac{1}{2}ab(b\omega - a\omega^2)$; where $\omega^3 = 1$.

(19.) $\pm\{2p^2(a^2p^2+b^2q^2)/(p^4-q^4)\}^{\frac{1}{2}}$;
$\pm\{2q^2(a^2p^2+b^2q^2)/(p^4-q^4)\}^{\frac{1}{2}}$; 4 solutions. **(20.)** $\frac{3}{2}, -\frac{3}{2}$. **(21.)** $3, -1$; $1, -3$.

(22.) $4, 3, -6, -2$; $3, 4, -2, -6$. **(23.)** $\pm(\sqrt{3} \pm \sqrt{2})$, $\pm(9\sqrt{3} \pm 11\sqrt{2})$; 4 solutions.

(24.) $\pm 3, \pm 2$; $\pm 2, \pm 3$; 4 solutions. **(25.)** $6, -2$; $2, -6$. **(26.)** $3, 6, 3\omega, 6\omega, 3\omega^2, 6\omega^2$; $6, 3, 6\omega, 3\omega, 6\omega^2, 3\omega^2$.

(27.) $2, 3, 2\omega, 3\omega, 2\omega^2, 3\omega^2$; $3, 2, 3\omega, 2\omega, 3\omega^2, 2\omega^2$. **(28.)** $\pm a^{\frac{1}{3}}(a^{\frac{1}{3}} - \omega b^{\frac{1}{3}})^{\frac{1}{2}}$, $\pm \omega b^{\frac{1}{3}}(a^{\frac{1}{3}} - \omega b^{\frac{1}{3}})^{\frac{1}{2}}$.

(29.) $5, 2, -5, -2$; $2, 5, -2, -5$. **(30.)** $2, 4, 2\omega, 4\omega, 2\omega^2, 4\omega^2$; $4, 2, 4\omega, 2\omega, 4\omega^2, 2\omega^2$.

(31.) $\frac{1}{2}[b \pm \sqrt{\{-3b^2 \pm 2\sqrt{(2a^4+2b^4)}\}}]$;
$\frac{1}{2}[b \mp \sqrt{\{-3b^2 \pm 2\sqrt{(2a^4+2b^4)}\}}]$.

(32.) $3, 2, \frac{1}{2}(5 \pm \sqrt{151}i)$; $2, 3, \frac{1}{2}(5 \mp \sqrt{151}i)$. **(33.)** $x = \dfrac{2a}{2 \pm p \pm \sqrt{(p^2+4)}}$, &c.

(34.) $1, 2, \frac{1}{2}(3 \pm \sqrt{19}i)$; $2, 1, \frac{1}{2}(3 \mp \sqrt{19}i)$. **(35.)** $(2^{\frac{1}{3}} + 2^{\frac{2}{3}}\omega + \omega^2)/2^{\frac{1}{3}}$; $\omega^2/2^{\frac{1}{3}}$.

(36.) $5v, 2v'$; $2v, 5v'$; where $v^4 = +1$, $v'^4 = -1$. **(37.)** $\frac{5}{34}(17 \pm \sqrt{51}i)$; $\frac{5}{34}(17 \mp \sqrt{51}i)$.

(38.) If $v = y/x$, then $v(1+v^2) = a^3(1+v^4)$, a reciprocal biquadratic.

(39.) $\pm 3, \pm 2$; $\pm 2, \pm 3$; 8 solutions; and 8 more given by $x+y = \pm\sqrt{(-14 \pm 6\sqrt{7})}$; $x-y = \pm\sqrt{(-14 \mp 6\sqrt{7})}$.

(40.) $\pm a\sqrt{ab/(a^2+b^2)}$, $\pm ia\sqrt{ab/(a^2+b^2)}$; $\pm b\sqrt{ab/(a^2+b^2)}$, $\mp ib\sqrt{ab/(a^2+b^2)}$. **(41.)** $2, 8$; $8, 2$.

(42.) Rationalise the first equation, using the second in the process, and thus find a quadratic for xy. **(43.)** $(a^2-b^2)/2a$; $\{a^2+b^2 \pm \sqrt{(a^4-6a^2b^2+b^4)}\}/4a$.

(**44.**) 10, 13 ; 13, 10. (**45.**) 2, 8 ; 8, 2. (**46.**) $\frac{5}{4}a$; $2a$.

(**47.**) We can derive $(x-y)^2 - 2a(x+y) + a^2 = 0$;
$$(x-y)^2 - \sqrt{2}b(x+y) + \sqrt{2}ab - b^2 = 0.$$

(**48.**) $\pm bi$, $\sqrt{2a+b}$ (bis) ; $\mp ai$, $\sqrt{2b+a}$ (bis).

(**49.**) If $u = xy/ab$, $v = x^2/a^2 + y^2/b^2$, we can derive
$$(m-2)u^2 + 2(m-n)u + (m-2) + nv = 0 ;$$
$$mu^2 + 2(m+n)u + m - (n+2)v = 0.$$

(**50.**) $\pm a/\sqrt{(a+b)}$; $\mp b/\sqrt{(a+b)}$. (**51.**) $\frac{3}{2}$; $\frac{2}{3}$.

(**52.**) One real solution is $\frac{1}{2}\{4a+1\pm\sqrt{(8a+1)}\}$; $\frac{1}{2}\{\quad -1\mp\sqrt{(8a+1)}\}$.
Another is given by $y^3 + 2ay^2 + 1 = 0$, $x = y^{-2}$.

XXXIV.

(**1.**) $x = +(b-c)/(abc)^{\frac{1}{3}}$, $y = +(c-a)/(abc)^{\frac{1}{3}}$, $z = \&c.$;
$x = \omega(b-c)/(abc)^{\frac{1}{3}}$, $y = \omega(c-a)/(abc)^{\frac{1}{3}}$, $z = \&c.$;
$x = \omega^2(b-c)/(abc)^{\frac{1}{3}}$, $y = \&c.$ $z = \&c.$;
where $\omega^3 = +1$. (**2.**) Eliminate z between the first two equations, and put $\xi = x - c$, $\eta = y - c$. The following are solutions :—
$x = b + c$, $y = c + a$, $z = a + b$,
$x = \{b^2 + c^2 - a(b+c)\}/(b+c-a)$, $y = \&c.$, $z = \&c.$
(**3.**) $x = 2$, $y = 3$, $z = 1$; or $x = -6$, $y = -7$, $z = -5$. (**4.**) $x = 3$, $y = 2$, $z = 1$. (**5.**) $x = \pm\frac{1}{7}\sqrt{(1001)}$, $y = \pm\frac{1}{13}\sqrt{(1001)}$, $z = \pm\frac{1}{11}\sqrt{(1001)}$, two solutions. (**6.**) $x = y = z = \pm\sqrt{2}/2$. (**7.**) $x = \pm(a^2b^2 + a^2c^2 - b^2c^2)/2abc$, $y = \pm\&c.$, $z = \pm\&c.$, two solutions. (**8.**) We derive by subtraction from the first two equations $(x-y)(a-z) = 0$, and from the first and third $(x-z)(a-y) = 0$. Combining these two with one of the original equations, we obtain the following five solutions (the last three twice over):—
$$\begin{array}{lllll} x = \alpha, & \beta, & (p^2-a^2)/a, & a, & a ; \\ y = \alpha, & \beta, & a, & (p^2-a^2)/a, & a ; \\ z = \alpha, & \beta, & a, & a, & (p^2-a^2)/a ; \end{array}$$
where α and β are the roots of $x^2 + ax - p^2 = 0$. (**9.**) Eight solutions, as follows :—
$x = z = (\pm 5 + \sqrt{409})/3$, $y = (\mp 10 + \sqrt{409})/3$;
$x = z = (\pm 5 - \sqrt{409})/3$, $y = (\pm 10 - \sqrt{409})/3$;
$x = (\pm\sqrt{1635} + i3\sqrt{33})/6$, $z = (\pm\sqrt{1635} - i3\sqrt{33})/6$,
$y = \pm\sqrt{(1635)}/6$;
$x = (\pm\sqrt{1635} - i3\sqrt{33})/6$, $z = (\pm\sqrt{1635} + i3\sqrt{33})/6$,
$y = \pm\sqrt{(1635)}/6$;
upper signs together and lower together throughout.

(**10.**) $x = \pm\left(\frac{1}{b^2} + \frac{1}{c^2}\right)\Big/2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^{\frac{1}{2}}$, $y = \&c.$, $z = \&c.$ (**11.**) If we add the

three equations, we obtain the equation $(a+b+c)(x+y+z)=3(x+y+z)^2$. Hence $x+y+z=0$, or $=(a+b+c)/3$. The three equations can therefore be replaced by three linear equations: $x=0$, $y=0$, $z=0$, and $x=(3bc+2ca+ab+b^2-c^2)/2(bc+ca+ab)$, &c., are solutions. (**12.**) The equations are linear in x^2-yz, y^2-zx, z^2-xy. Solving, we obtain, $x^2-yz=p$, $y^2-zx=q$, $z^2-xy=r$, say. If we now put $x=uz$, $y=vz$, we obtain the following biquadratics for u and v:—

$$(r^2-pq)u^4-(p^2-qr)u^3-(r^2-pq)u+(p^2-qr)=0,$$
$$(r^2-pq)v^4-(q^2-rp)v^3-(r^2-pq)v+(q^2-rp)=0.$$

We thus find the following values for u and v:—

$$u=1,\quad \omega,\quad \omega^2,\quad (p^2-qr)/(r^2-pq),$$
$$v=1,\quad \omega^2,\quad \omega,\quad (q^2-rp)/(r^2-pq).$$

The first three pairs give $x=\infty$, $y=\infty$, $z=\infty$. The last pair gives $x=\pm(p^2-qr)/\sqrt{(p^3+q^3+r^3-3pqr)}$, $y=\pm$&c., $z=\pm$&c. (**13.**) From the first equation we see that $x=\rho a(b-c)(\sigma+a)$, $y=\rho b(c-a)(\sigma+b)$, $z=$&c., where ρ and σ are arbitrary. The second equation gives the following quadratic for σ, $\{\Sigma a^2-\Sigma bc\}\sigma^2+\{\Sigma a^2(b+c)-6abc\}\sigma+\{\Sigma b^2c^2-abc\Sigma a\}=0$. When σ is known, the third equation gives $\rho=\pm 1/\sqrt{\Sigma a^2(b-c)^2(\sigma+a)^2}$. Hence we obtain four sets of values for x, y, z. (**14.**) From the first three equations we have $x+yz=\rho/a$, &c. From these, squaring and using the last equation, we deduce $(1-y^2)(1-z^2)=\rho^2/a^2$, &c. From these last we deduce $x=\pm\sqrt{(1\pm\rho a/bc)}$, $\pm$&c. $\pm$&c. Substituting these values in the last equation we find $\rho=0$ and $\rho=\pm(\Sigma a^4-2\Sigma b^2c^2)/4abc$. Hence $x=y=z=-1$; and $x=-(a^2-b^2-c^2)/2bc$, $y=-$&c., $z=-$&c.

(**15.**)
$$\begin{aligned} x&=0,\ 4a,\quad \tfrac{3}{2}a,\quad -a,\quad -a;\\ y&=0,\ 4a,\quad -a,\quad \tfrac{3}{2}a,\quad -a;\\ z&=0,\ 4a,\quad -a,\quad -a,\quad \tfrac{3}{2}a. \end{aligned}$$

(**16.**) The given equations may be written $(x-y)^2+(y-z)^2=a^2$, &c. Hence we have $(y-z)^2=(b^2+c^2-a^2)/2$, &c. Hence $y-z=\pm\sqrt{(b^2+c^2-a^2)/2}$, &c. The system is therefore insufficient to determine the three variables; in fact it will not be a consistent system unless $\Sigma a^4-\Sigma b^2c^2=0$. (**17.**) If $\rho=xyz$, $x=uz$, $y=vz$, we may write the equations $a\rho=(v+1)/u$, $b\rho=(u-1)/v$, $c\rho=u+v$. Eliminating u and v we find $\rho^2=(b+c-a)/abc$; and so on. (**18.**) If x be eliminated, the resulting equations may be written

$$\begin{aligned} 2\xi\eta+\eta^2-14\eta-28\xi-81&=0,\\ \xi^2-(\eta-7)^2\qquad\qquad&=0, \end{aligned}$$

where $\xi=yz$, $\eta=y^2+z^2$; one set of solutions is $x=3$, $y=1$, $z=2$; another $x=3$, $y=2$, $z=1$; &c. (**19.**) From the given equations we can deduce $(b^4y-c^4z)/(y-z)=$&c.$=$&c.$=\sigma$, say. Whence $(a^4-\sigma)x=(b^4-\sigma)y=(c^4-\sigma)z=\tau$, say. We can then determine σ and τ by means of the given equations. Result, $x=\rho\{\Pi(a^8-b^4c^4)\}^{\frac{1}{2}}/(a^8-b^4c^4)(\Sigma a^{12}-3a^4b^4c^4)^{\frac{1}{4}}$, where ρ is any one of the 4th roots of $+1$. (**20.**) The equations can be written $xy+xz-yz=\rho/a^2$, &c., where $\rho=x^2y^2z^2/(yz+zx+xy)$. Result, $x=\pm 2bc(\Sigma b^2c^2)^{\frac{1}{2}}/a(b^2+c^2)$, $y=\pm$&c., $z=\pm$&c., two solutions. (**21.**) $x=\pm\sqrt{2}a^{\frac{3}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}(b+c)/\{\Sigma a^3(b+c)^3\}^{\frac{1}{2}}$,

$y = \pm$&c., $z = \pm$&c. **(22.)** $x = y = z = 8$; and $x = -6$, $y = -4$, $z = -2$. **(23.)** $x = d(b-a)/(c-d)$, $y = c(b-a)/(c-d)$, $z = b(c-d)/(b-a)$, $u = a(c-d)/(b-a)$.

(24.) The real solutions are

$$x = 3,\ 1,\ 2;$$
$$y = 2,\ 3,\ 1;$$
$$z = 1,\ 2,\ 3.$$

(25.) We have $(x^2 - yz)^2 - (y^2 - zx)(z^2 - xy) = a^4 - b^2c^2$.

Hence $x = \rho(a^4 - b^2c^2)$, $y = \rho(b^4 - c^2a^2)$, &c.

$$x = \pm(a^4 - b^2c^2)/\sqrt{(\Sigma a^6 - 3a^2b^2c^2)}, \text{ \&c.}$$

XXXV.

(1.) $[(ac')\{(ad'') + (bc'')\} - (ac'')\{(ad') + (bc')\}] \times [(bd'')\{(ad') + (bc')\} - (bd')\{(ad'') + (bc'')\}] = [(ac')(bd'') - (ac'')(bd')]^2$. **(2.)** $(l^2 + m^2 - 1)^2(a+b)^2 - (l^2 + m^2 - 1)\{(l^2 - m^2)^2 - l^2 - m^2\}(a^2 - b^2)^2 + l^2m^2(a-b)^2 = 0$. **(3.)** $a^4 + b^4 = c^2(a^2 + b^2)$. **(4.)** $8d^6 = \Pi(b^2 + c^2 - a^2)$. **(5.)** $p + q + r = 0$. **(8.)** $c(a-b)^4 - 2c(p+q)(a-b)^2 + (cp - q)(p - cq) = 0$. **(9.)** Eliminate z, and put $\xi = x + y$, $\eta = xy$, in the three resulting equations, then eliminate ξ, and there results two quadratics in η, &c. **(10.)** $\Sigma a^3b^3 = 5a^2b^2c^2$. **(11.)** Put $u = \Sigma x$, $v = \Sigma xy$, $w = xyz$, eliminate v and w, and reduce the resulting equations to two quadratics in u. **(12.)** Let $\xi = x + a$, $\eta = y + b$, $\zeta = z + c$, then $\xi\eta\zeta = abc$. We have $\eta\zeta - (b+c)(\eta + \zeta) = a^2 - (b+c)^2$, &c. These give ξ, η, ζ in terms of $\eta\zeta$, $\zeta\xi$, $\xi\eta$. Again multiplying the last three equations by ξ, η, ζ, we have $abc - (b+c)(\xi\eta + \zeta\xi)$. These give ξ, η, ζ again in terms of $\eta\zeta$, $\zeta\xi$, $\xi\eta$. We thus get three linear equations for ξ, η, ζ, &c. **(13.)** We can deduce $x_2x_n + y_2y_n = \text{const.} = {}_2k_n^2$ say; $x_3x_n + y_3y_n = {}_3k_n^2$, . . .; and finally $x_{n-1}x_n + y_{n-1}y_n = {}_{n-1}k_n^2$. Now either ${}_{n-1}k_n^2 = k_n^2$, in which case the system is indeterminate, or ${}_{n-1}k_n^2 \neq k_n^2$, in which case the system is inconsistent.

XXXVI.

(1.) $-p^5 + 5p^3q - 5pq^2$. **(2.)** $(p^6 - 6p^4q + 9p^2q^2 - 2q^3)/(p^2 - 4q)$. **(3.)** $(-p^5 + 5p^3q - 5pq^2)/q^5$. **(4.)** $\pm(p^4 - 3p^2q + q^2)\sqrt{(p^2 - 4q)}/q^5$. **(5.)** $\{-p(p^2 - 3q) \pm (p^2 - q)\sqrt{(p^2 - 4q)}\}/\mp p(p^2 - q)(p^2 - 3q)\sqrt{(p^2 - 4q)}$. **(6.)** $p^2 - 2q + 2pq + 2q^2$. **(7.)** $3\text{A}x^2 - 3\text{A}^2x + \text{A}^3 - \text{B}^3 = 0$. **(9.)** $x^2 - (2h^2 + 2ph + p^2 - 2q)x + h^4 + 2ph^3 + (p^2 + 2q)h^2 + 2pqh + q^2 = 0$. **(12.)** $4aa'(c'a) + (a'^2b^2) = 0$. **(14.)** $(p_1^3 - 3p_1p_2 + 3p_3)/p_3$. **(15.)** $(p_2^5 - 5p_1p_2^3p_3 + 5p_1^2p_2p_3^2 + 5p_2^2p_3^2 - 5p_1p_3^3)/p_3^5$. **(16.)** $p_1^2p_2^2 - 2p_2^3 - 2p_1^3p_3 + 4p_1p_2p_3 - p_3^2$. **(17.)** $(6p_1p_2p_3 - p_2^3 - p_1^3p_3)/(p_1^3p_3 - p_2^3)$. **(18.)** $2p_1^2 - 6p_2$. **(19.)** $p_1^4 - 6p_1^2p_2 + 9p_2^2$. **(20.)** $2p_1^3 - 3p_1p_2 - 3p_3$. **(21.)** $p_1^4 - 4p_1^2p_2 + 2p_2^2 + 4p_1p_3 - 4p_4$. **(22.)** $(3p_1p_4^2 - 3p_2p_3p_4 + p_3^3)/p_4^3$. **(23.)** $p_2^2 - 2p_1p_3 + 2p_4$. **(24.)** $p_1p_3 - 4p_4$. **(25.)** $3p_1^4 - 8p_1^2p_2 + 4p_2^2 - 4p_1p_3 + 16p_4$. **(26.)** $x^3 - p_2x^2 + (p_1p_3 - 4p_4)x - p_1^2p_4 + 4p_2p_4 - p_3^2 = 0$. **(30.)** $2p_2' + 2p_2 - p_1p_1'$. **(31.)** $= 2(p_1^2 + p_1^{2\prime}) - 4(p_2 + p_2') - 2p_1p_1'$. **(32.)** $(p_2 - p_2')^2 + p_1p_1'(p_2 + p_2') + p_1^2p_2' + p_1^{2\prime}p_2$. **(36.)** $x^2 - 4acx + 4a^2d^2 + 4b^2c^2 - 4b^2d^2 = 0$. **(37.)** $a^2b' + a'b^2 = 2aa'c$. **(38.)** The roots are $\frac{3}{2}$, $\frac{3}{2}$, $\frac{5}{3}$. **(40.)** The roots are -7, $8 \pm \sqrt{15}$. **(46.)** From the first two equations we deduce $p_2 = p_1^2/\sqrt{6}$, $3p_3 = (3/\sqrt{6} - 1)p_1^3$, &c. **(49.)** $l^4 + 3m^4 - 6l^2m^2 + 8ln^3 - 6p^4 = 0$. **(50.)** $(2a^4 - b^4 + 2a^2b^2 - c^4)^2 = 4d^6(a^2 + 2b^2)$.

XXXVII.

(1.) Roots real, $+$, $+$. **(2.)** Roots real, $+$(num. greater), $-$. **(3.)** Roots imaginary. **(4.)** Roots equal, $+$. **(5.)** Roots real, $+$(num. greater), $-$. **(6.)** Roots real, $-$, $-$. **(7.)** Roots real $+$, $-$ (num. greater). **(8.)** 1° b^2-4ac, $-$; $a+c$, $+$. 2° b^2-4ac, $+$. 3° b^2-4ac, $-$; $a+c$, $-$. 4° $b=0$, $a=c$. 5° $a+c=0$. **(9.)** D (the discriminant) $=-4(p-q)^2$. **(10.)** $\mathrm{D}=-4(c-b)^2(a-b)^2$. **(11.)** $\mathrm{D}=2\Sigma(b-c)^2$. **(12.)** $\mathrm{D}=2\Sigma a^2(b-c)^2$. **(13.)** $4q^3+27r^2=0$. **(14.)** The roots are $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{3}$. **(16.)** $256ae^3-27d^4=0$. **(18.)** $x^2-x=0$. **(19.)** $6x^2+x-2=0$. **(20.)** $x^2-6x+7=0$. **(21.)** $x^2-2(2a^2-1)x+1=0$. **(22.)** $x^2-2ax+a^2+\beta^2=0$. **(23.)** $x^4-4x^3-4x^2+16x-8=0$. **(24.)** $x^4+2x^2+25=0$.

(27.) $\begin{vmatrix} -x & 1 & 1 & 1 \\ qr & -x & q & r \\ rp & p & -x & r \\ pq & p & q & -x \end{vmatrix}=0.$ **(28.)** $x^6-2x^5-x^4+x^2-2x-1=0$. **(29.)** $\begin{vmatrix} c-x & b & a \\ ap & c-x & b \\ bp & ap & c-x \end{vmatrix}=0.$

(30.) $(x-p)^{2n+1}-q^{2n+1}=0$. **(31.)** $4x^2-77$. **(32.)** $-12x^3+118x^2-372x+386$.

XXXIX.

(1.) 3 seconds. **(2.)** 84. **(3.)** 13s. 9d., 7. **(4.)** 1·72 %. **(5.)** $2\frac{6}{11}$ hours. **(6.)** 9. **(7.)** 54 gallons. **(8.)** 8 h. 27 m. 16 sec.+. **(9.)** 100 yards per minute; 150 yards per minute. **(10.)** 5s. 10d. **(11.)** 15, 12. **(12.)** pq/h, $pq/h-p$. **(13.)** $a\sqrt{m}/(1+\sqrt{m})$, $a/(1+\sqrt{m})$. **(14.)** 1021 pence. **(15.)** He was 43 years of age in the year 1849. **(16.)** $(ac-bd)/(a-b+c-d)$. **(17.)** 7 miles. **(18.)** A at 10 A.M., B at 9.30 A.M. **(19.)** $(q-p)/2r+d/2$. **(20.)** £14,800. **(21.)** $x=(m+n+2)/(mn-1)$. **(22.)** 9 : 1. **(23.)** 35, 25. **(24.)** $25\frac{1}{2}$, 18, $67\frac{1}{2}$. **(25.)** 10·088″, 10·288″. **(26.)** 106 yards, very nearly. **(27.)** $ct/(b+c)$, $bt/(b+c)$, $a(b^2-c^2)/2bct$. **(28.)** 10815 : 10827. **(29.)** $aba'b'(a-a')(b-b')/(ab'-a'b)^2$. **(30.)** In A $100\,m(n+1)/(2mn+m+n)$, in B $100\,n(m+1)/(2mn+m+n)$. **(31.)** 20, 30. **(32.)** The distances from XII are given by $x=60p/11$, where $p=0$, 1, 2, . . ., 11. **(33.)** If A, B, C lose in order, they had originally £$13a/8$, £$7a/8$, £$4a/8$. **(34.)** 3 hours, and 4 hours. **(35.)** £20. **(36.)** 8. **(37.)** 80. **(38.)** 60 quarts. **(39.)** 1, 2, 3, 4; or 5, 6, 7, 8. **(40.)** The duties corresponding to maximum and minimum revenues are $100(p-a)/3a$ % and $100(p-a)/a$ % respectively. **(41.)** 79. **(42.)** 1, 3, 5. **(43.)** $\frac{2}{3}a-\sqrt{\{2(b^2-\frac{1}{3}a^2)\}}$, $\frac{2}{3}a$, $\frac{2}{3}a+\sqrt{\{2(b^2-\frac{1}{3}a^2)\}}$. **(45.)** $r+h[1\pm\sqrt{\{2-(1+2r/h)^2\}}]/2$, $r=(\sqrt{2}-1)h/2$.

XL.

(1.) 495. **(2.)** $307\frac{1}{2}$. **(3.)** 36. **(4.)** $-\frac{64}{3}$. **(5.)** $\frac{1}{2}(n^2-3n+4)$. **(6.)** $n\{a^2+n(n-3)a+n^2\}$. **(7.)** $(3l+l^2+5l^3-l^4)/2(1-l^2)$. **(8.)** $8998148\frac{4}{27}$. **(9.)** 3. **(10.)** 1000. **(11.)** Any A.P. whose first term is a and whose C.D. is $2a/(m+1)$ has the required property. **(12.)** £2131 : 5s. **(13.)** 50,500 yards. **(14.)** 2525 yards. **(15.)** 9. **(16.)** l/c. **(17.)** $\frac{75}{16}$, $\frac{102}{16}$, &c. **(18.)** $-\frac{27}{11}$, $-\frac{21}{11}$, &c. **(19.)** 20. **(20.)** 20. **(22.)** 1, 5, 9, 13. **(24.)** $\frac{3}{2}n(3n+5)$. **(25.)** $b+(a-b-\frac{1}{2}rb)n+\frac{1}{2}rbn^2$. **(26.)** 3, 2. **(28.)** $(2n)^2=4+12+\ .\ .\ .+(8n-4)$, $(2n+1)^2-1$

$=8+16+\ldots+8n$. **(29.)** The first odd number is $n^{p-1}-n+1$. The second part follows from § 7. **(30.)** $T_n=\frac{1}{2}(-1)^{n-1}n(n+1)$; $\Sigma_n=-\frac{1}{4}n(n+2)$ or $\frac{1}{4}(n+1)^2$, according as n is even or odd. **(31.)** $\frac{2}{3}n(n+1)$. **(33.)** n^3. **(34.)** $\frac{1}{2}n(6n^2+15n+11)$. **(35.)** ${}_ns_4-{}_ns_3-{}_ns_1+n$. **(36.)** $(p-q)(p-2q)n+q(2p-3q)_ns_1+q^2{}_ns_2$. **(37.)** $-2n^2-n$. **(38.)** $(a-b)^3n+3b(a-b)^2{}_ns_1+3b^2(a-b)_ns_2+b^3{}_ns_3$. **(39.)** $\frac{1}{4}(n-1)n(n+1)(n+2)$. **(40.)** $\frac{1}{12}n(n+1)(n+2)(3n+5)$. **(41.)** $\frac{1}{6}n(n+1)(6n^2-2n-1)$. **(42.)** $\frac{1}{3}n(6n^3+32n^2+33n-8)$. **(43.)** $\frac{1}{12}n(n+1)^2(n+2)$. **(44.)** 4960. **(45.)** $\frac{1}{6}m(m+1)(2m+1)$. **(46.)** $\frac{1}{6}l(l+1)(3m+2l-2)$.

XLI.

(1.) $3^{13}-3$. **(2.)** $-\frac{1}{2}(3^{13}-3)$. **(3.)** $\frac{1}{3}(1-1/10^n)$. **(4.)** $\frac{4}{5}(1-(-1)^n/4^n)$. **(5.)** $\frac{18}{5}\{1-(2/3)^{10}\}$. **(6.)** $\frac{1}{2}(3+\sqrt{3})\{1-(\sqrt{3}-1)^{20}\}$. **(7.)** $\frac{3}{2}(1-1/3^n)$. **(8.)** $\frac{2}{3}$. **(9.)** $1/(1+x)$. **(10.)** $2\sqrt{2}$. **(11.)** $\frac{3}{2}$. **(12.)** $(a+x)^3/2(a-x)(a^2+x^2)$. **(13.)** $(1+x)/(1+x^2)$. **(14.)** $(x^{2n}-y^{2n})/(x+y)x^{2n-2}$. **(15.)** $x(1-x^n)/(x-y)(1-x)-y(1-y^n)/(x-y)(1-y)$. **(16.)** $\frac{1}{27}\{9n-1/10+1/10^{n+1}\}$. **(17.)** $\frac{18}{5}(6^n-1)$. **(18.)** $x^2(x^{2n}-1)/(x^2-1)-\frac{1}{6}n(n+1)(2n+1)$. **(19.)** $\{(xy)^{n+1}-(xy)^{-n}\}/(xy-1)-\{x(x/y)^n-y(y/x)^n\}/(x-y)$. **(20.)** $p^2(p^n-1)/(p^2-1)-q^2(q^{2n}-1)/(q^2-1)$. **(21.)** $\frac{3}{5}6^n-2^n+\frac{2}{5}$. **(22.)** $-a^3(1-(-1)^na^{3n})/(1+a^3)$. **(23.)** $2n+(r^{2n}+1/r^{4n-2})(r^{2n}-1)/(r^2-1)$. **(24.)** $n+2(1-r^{-n})/(r-1)+(1-r^{-2n})/(r^2-1)$. **(26.)** $a(r^n-1)(r^t-1)/(r-1)^2$. **(28.)** $\Sigma_n=\{1-(-)^n(n+1)x^n-(-)^n nx^{n+1}\}/(1+x)^2$, $\Sigma_\infty=1(1+x)^2$. **(29.)** $\Sigma_n=\frac{2}{9}+(-)^{n-1}(6n+1)/9.2^{n-1}$, $\Sigma_\infty=\frac{2}{9}$. **(30.)** $\Sigma_n=\frac{4}{27}+(-)^{n-1}(9n^2+12n+2)/27.2^{n-1}$, $\Sigma_\infty=\frac{4}{27}$. **(31.)** $\Sigma_n=1+2x^2/(x-1)^3-\{(n^2+n)x^2-(2n^2-2)x+(n^2-n)\}/x^{n-1}(x-1)^3$, $\Sigma_\infty=1+2x^2/(x-1)^3$. **(32.)** $\Sigma_\infty=(1+4x+x^2)/(1-x)^4$. **(33.)** $\Sigma_\infty=11/57$. **(34.)** $(ar^2+br+c)(1-r^{-3n})/(r^3-1)$. **(35.)** £5825 : 8 : $5\frac{1}{4}$. **(36.)** $100(v/V)^n$. **(37.)** $\frac{1}{3}\Delta_1$. **(38.)** $a^2/(a-b)$. **(39.)** $\alpha^nP-(1-\alpha^n)E/(1-\alpha)$, where $\alpha=1-(d-b)/100$. **(41.)** 7. **(42.)** 180/121. **(43.)** $-\frac{5}{2}$ or $\frac{3}{2}$. **(44.)** 5, or -6. **(45.)** 5. **(46.)** $2(5^{10}-1)$, or $\frac{4}{3}(5^{10}-1)$. **(47.)** $\frac{3}{2}$. **(48.)** $\frac{9}{4}$, $\frac{3}{2}$, 1. **(49.)** 4, 8, 16, 32. **(50.)** $12\sqrt[3]{}/45$. **(51.)** 3, 48. **(52.)** 11, 33, 99, 297; or -22, 66, -198, 594. **(53.)** $(ac-b^2)/(2b-a-c)$. **(54.)** 5, 10; or $-25/3$, $-970/3$. **(55.)** If p and q be the given sums, the results are $2pq/(p^2+q)$, $(p^2-q)/(p^2+q)$. **(58.)** $(1+a)\{1-(ac)^n\}/(1-ac)$. **(60.)** $a^3r^3/(1+r)(1-r)^2(1-r^3)$.

XLII.

(1.) $\frac{9}{7}$, $\frac{9}{5}$; $\frac{144}{23}$, $\frac{144}{22}$, &c. **(2.)** $\frac{20}{11}$, $\frac{300}{137}$. **(3.)** $\frac{12}{5}$, $\frac{12}{4}$, $\frac{12}{3}$, $\frac{12}{2}$, $\frac{12}{1}$, $\frac{12}{0}$, $-\frac{12}{1}$, &c. **(4.)** For the corresponding A.P. $a=\frac{13}{45}$, $b=-\frac{2}{45}$. **(5.)** For the corresponding A.P. $a=\{(p-1)P-(q-1)Q\}/(p-q)PQ$, $b=(Q-P)/(p-q)PQ$. **(8.)** $\frac{1}{2}\{p-\sqrt{(p^2-900)}\}$, 15, $\frac{1}{2}\{p+\sqrt{(p^2-900)}\}$, where p is arbitrary.

XLIII.

(1.) 4, 5, $\bar{3}$, 4. **(2.)** 4, $\bar{3}$, 3, $\bar{4}$. **(3.)** $3\frac{1}{3}$. **(4.)** 5·1871 . . . **(5.)** 32617·105. **(6.)** ·00794818. **(7.)** 3114·973. **(8.)** 1·3800812. **(9.)** ·0 . . . (22 cyphers) 433352. **(10.)** ·2674734. **(11.)** ·979467. **(12.)** 7·7794. **(13.)** 38·37. **(14.)** 48·3. **(15.)** 4·81213. **(16.)** 282351500. **(17.)** 3·560625. **(18.)** The common ratio is 1·079188. **(19.)** 1·2921592. **(20.)** 36·833432. **(21.)** 20.

(22.) 35. **(23.)** 1·041393. **(24.)** 2432, number of digits 19. **(25.)** 5. **(26.)** 13·73454. **(27.)** $\frac{5}{2}$. **(28.)** ·98397. **(29.)** ·47320. **(30.)** 2·10372, $-1·10372$. **(31.)** $x=-3·313811$, $y=·000527696$. **(32.)** 1·49947. **(33.)** $x=·76028$, $y=·02060$. **(34.)** 1·24207. **(35.)** $\log(a^2-b^2)/2\log(a+b)$. **(36.)** $x+y=\pm 2a$, $x=y^2$. **(37.)** 2·793925.

XLIV.

(1.) 3 months. **(2.)** £1660 : 12 : 10. **(3.)** £225 : 4 : 10. **(4.)** 12 years. **(5.)** 254 years. **(6.)** 7s., 7s. 6d. **(7.)** 15 years. **(11.)** $P(1+nr)/(1+nr')$ if the surplus interest be not reinvested; otherwise $P\{(1-r/r')/(1+r')^n+r/r'\}$. **(12.)** 4·526 %. **(13.)** 6 %. **(14.)** £41,746, £48,837, £59,417. **(15.)** $\Sigma A_r n_r/\Sigma A_r$, $(\log\Sigma A_r-\log\Sigma A_r R^{-n}{}_r)/\log R$. **(16.)** £1107 : 3 : 7. **(17.)** £10 : 5 : 6. **(19.)** £8078. **(20.)** £11,231. **(21.)** £1801 : 14 : 10. **(22.)** 4 %. **(23.)** £479 : 14 : 11. **(24.)** 10 years. **(25.)** £75 : 12 : 10. **(26.)** $A(1-R^{-2n})/2(R+1)$. **(27.)** £2904 : 2 : 5. **(28.)** AR^n. **(29.)** 20 years. **(30.)** $\{\log 2-\log(1+R^{-m})\}/\log R$ years. **(32.)** £1912 : 8 : 11. **(33.)** Present value $=a/(R-1)+b(1-R^{-n+1})/(R-1)^2-\{a+(n-1)b\}R^{-n}/(R-1)$. **(34.)** Present value $=a\{1-(b/R)^n\}/(R-b)$. **(35.)** $A\{mR^{(m+1)q}-(m+1)R^{mq}+1\}/R^{mq}(R^q-1)(R-1)$.

XLV.

(λ and ρ denote roots of the resolvents of Lagrange and Descartes.)

(1.) $\lambda=2r/p$. **(2.)** $\lambda=\pm 2$. **(3.)** $\lambda=-3$. **(4.)** $\lambda=10$. **(5.)** $\lambda=pq$. **(6.)** 3/2 is a root. **(7.)** 2 and $-3/2$ are roots. **(8.)** The equation reduces to $(x^2+2x+3)(2x^2+x-2)=0$. **(9.)** $\lambda=\frac{1}{2}\sqrt[3]{566}$. **(10.)** $x=1+\sqrt{2}+\sqrt{3}$, etc. **(11.)** $\rho=4$.

END OF PART I.

Printed by R. & R. Clark, Limited, *Edinburgh.*